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A simple proof of the Radó and Král theorems on removability of the zero locus for analytic and harmonic functions

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A unified approach to the proof of the Radó and Král theorems on removability of the zero locus for analytic and harmonic functions is proposed.

Keywords: Radó theorem, analytic function, harmonic function, subharmonic function, zero locus.

In 1924, T. Radó [1] proved the following result.

Theorem 1. *Let $f(z)$ be a complex-valued continuous function in a domain $G \subset \mathbb{C}$, and let $f(z)$ be analytic in $G \setminus f^{-1}(0)$. Then $f(z)$ is analytic in the whole of G.*

Here, $f^{-1}(0) = \{z \in G\colon f(z) = 0\}$. J. Král [2] established the following analogue of the Radó result for harmonic functions.

Theorem 2. *Let $u(x)$ be a real-valued continuously differentiable function in a domain $G \subset \mathbb{R}^n$, $n \geqslant 2$, and let $u(x)$ be harmonic in $G \setminus u^{-1}(0)$. Then $u(x)$ is harmonic in the whole of G.*

The formulations of Theorems 1 and 2 are similar, but all known proofs of these results are completely different. In particular, the analytic functions form an algebra, and this is essentially used in the proofs of the Radó theorem presented in [3] and [4]. On the other hand, it is easy to verify the removability of the set $u^{-1}(0) \bigcap \{x \in G\colon \nabla u(x) \neq 0\}$ in Theorem 2. Indeed, it follows from the implicit function theorem that this set can be locally represented in the form of the graph of a continuously differentiable function of $n-1$ real variables. Consequently, it has a locally finite Hausdorff measure of order $n-1$. But such sets are removable for continuously differentiable harmonic functions (see, e. g., [5]). Hence, Theorem 2 is a corollary of the following result.

Theorem 3. *Let $u(x)$ be a real-valued continuously differentiable function in a domain $G \subset \mathbb{R}^n$, $n \geqslant 2$, and let $u(x)$ be harmonic in $G \setminus |\nabla u|^{-1}(0)$. Then $u(x)$ is harmonic in the whole of G.*

The proof of Theorem 3 in [2] is the most delicate part of this paper and contains the investigation of differentiability properties of subharmonic functions following from the F. Riesz representation theorem. Using the concept of the viscosity solution, P. Juutinen and P. Lindqvist [6] generalized Theorems 2 and 3 for some class of quasilinear elliptic and parabolic equations of the second order. In particular, these theorems hold for p-harmonic functions with $1 < p < \infty$.

In the present paper, we propose a unified approach to the proof of the Radó and Král theorems, which essentially simplifies all known proofs of these results.

1. Notation and auxiliary results. As usual, $\overline{\partial}$ and Δ denote the Cauchy–Riemann operator in the complex plane $\mathbb{C}$ and the Laplace operator in $\mathbb{R}^n$, respectively. If $u(x)$ is a real-valued continuously differentiable function in a domain $G \subset \mathbb{R}^n$, then $\nabla u(x) = (\partial_1 u(x), \ldots, \partial_n u(x))$ denotes the gradient of this function in G. We shall use the following two lemmas.

Lemma 1. *Let $f(z)$ be an analytic function in a domain $G \subset \mathbb{C}$. Then the function $|f(z)|^\alpha$ is subharmonic in G for all $\alpha > 0$.*

Lemma 2. *Let $u(x)$ be a real-valued harmonic function in a domain $G \subset \mathbb{R}^n$, $n \geqslant 2$. Then the function $|\nabla u(x)|^{n/(n+1)}$ is subharmonic in G.*

Lemma 1 is very simple and well-known, while Lemma 2 is a non-trivial and sharp result of E. M. Stein and G. Weiss [7].

2. Proof of theorem 1. Suppose that the conditions of Theorem 1 hold. Let $f(z) = u(z) + iv(z)$, where $u(z)$ and $v(z)$ are real-valued functions in G. For any point $z_0 \in f^{-1}(0)$, we have $|u(z)| = o(|f(z)|^{1/2})$ as $z \to z_0$. Combining this with Lemma 1, we conclude that the function $u(z) + \varepsilon|f(z)|^{1/2}$ $(z \in G)$ is continuous, subharmonic for all $\varepsilon > 0$, and superharmonic for all $\varepsilon < 0$. Hence, the function $u(z)$ is harmonic in G.

Similarly, the function $v(z)$ is harmonic in G. Therefore, the functions $f = u + iv$ and $\overline{\partial} f$ are harmonic in G. If $G \setminus f^{-1}(0) \neq \varnothing$, then $\overline{\partial} f = 0$ in $G \setminus f^{-1}(0)$ and, consequently, everywhere in G.

3. Proof of Theorem 3. Suppose that the conditions of Theorem 3 hold. For any $k \in \{1, \ldots, n\}$ and for any point $x_0 \in |\nabla u|^{-1}(0)$, we have $|\partial_k u(x)| = o(|\nabla u(x)|^{n/(n+1)})$ as $x \to x_0$. Combining this with Lemma 2, we conclude that the function $\partial_k u(x) + \varepsilon|\nabla u(x)|^{n/(n+1)}$ $(x \in G)$ is subharmonic for all $\varepsilon > 0$ and superharmonic for all $\varepsilon < 0$. Consequently, the function $\partial_k u(x)$ is harmonic in G $(k = 1, \ldots, n)$. Hence, $u(x)$ is an infinitely differentiable function in G and $\Delta u = 0$ on the set $E = (G \setminus |\nabla u|^{-1}(0)) \bigcup F$, where F is the interior of $|\nabla u|^{-1}(0)$. Since E is dense in G, it follows that $\Delta u = 0$ everywhere in G.

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Просте доведення теорем Радо та Крала про усувність множини нулів для аналітичних та гармонічних функцій

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Запропоновано єдиний підхід до доведення теорем Радо та Крала про усувність множини нулів для аналітичних та гармонічних функцій.

Ключові слова: теорема Радо, аналітична функція, гармонічна функція, субгармонічна функція, множина нулів.

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Простое доказательство теорем Радо и Крала об устранимости множества нулей для аналитических и гармонических функций

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Предложен единый подход к доказательству теорем Радо и Крала об устранимости множества нулей для аналитических и гармонических функций.

Ключевые слова: теорема Радо, аналитическая функция, гармоническая функция, субгармоническая функция, множество нулей.