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ON STABILITY OF PARAMETRICALLY EXCITED LINEAR STOCHASTIC SYSTEMS

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Abstract. The dynamic stability of a coupled two-degrees-of-freedom system subjected to parametric excitation by a harmonic action superimposed by an ergodic stochastic process is investigated. For the stability analysis, the method of moment functions is used. Explicit expressions for the stability of the second moments are obtained when the frequency of the harmonic excitation lies in the vicinity of the combination sum of the natural frequencies. Good agreement between the analytical and numerical results is obtained. As an application, the example of the flexural-torsional instability of a thin elastic beam under dynamic loading is considered.

Key words: stochastic stability, parametric excitation, characteristic equation, Routh – Hurwitz conditions.

1. Introduction.

Over the years two styles have emerged for the investigation of the temporal behavior of physical systems. The first exploits dynamical equations such as Newton's equations of motion, Schrodinger's equation of quantum theory, and Maxwell's equations. These have been fantastically successful for the description of the behavior of relatively simple systems. In most cases, a real physical system and its loading will differ from the mathematical model used in the analysis. For the physical system these differences are connected to the vast amount of small imperfections and defects present, for the load terms – they are due to perturbation which are stochastic in nature. However, the connection between the basic dynamical equations, describing the behavior of the physical system, and calculated results fades through the uncontrolled approximations and assumptions. In contrast to traditional systems analysis based on deterministic concepts, a second style of analysis, which is frequently called the application of the theory of stochastic process, accounts explicitly for uncertainties that always exist in inputs that act on the system [21].

The theory of stochastic processes and random function analysis has been developed to such a large extent, that it is central to the analysis and design of a wide variety of engineering systems. As stochastic models have come to be more fully understandable to engineers and scientists, the study of rather important stochastic system properties has become possible. Among these properties, we have the property of stability.

The stability of systems has been a subject of numerous studies (see for example Refs. [3, 15]), leading to results of basic importance. Extending the classical theory of stability of motion to stochastic systems became necessary. The mathematical aspects of the theory are treated by Kasminskii [11] and Kushener [14].

The Stability studies are concerned with the qualitative behavior of the solutions to differential equations, which can often be studied without a direct recourse to solving the equations. Stability concepts are usually defined in terms of convergence relative to parameters such as the initial conditions, or the time parameters [13]. Fluctuation phenomena about equilibrium and nonequilibrium states of dynamical systems have important practical and theoretical significance. One of the most interesting effects of fluctuations is the possibility of changing the stability characteristics of parametrically excited dynamical systems [12].

Parametric instability under deterministic periodic excitation has been extensively investigated both theoretically and experimentally and several important instability phenomena have been established. A corresponding investigation when the excitation is stochastic has become necessary. The stochastic stability of parametrically excited linear systems has been the subject of several papers [1,2,7,8,16 – 18]. The moment stability of a damped Mathieu oscillator under the effect of parametric random excitation was previously investigated by Ariaratnam et al. [1], where conditions for stability of the first and second moments of the response were obtained. A coupled two-degrees-of-freedom system of the same class was studied in Ref. [16], where conditions for stability of the first moments were found, while the boundaries of the instability regions of the second moments are obtained numerically from the character of the roots of the characteristic equation, where at least one of the roots has a positive real part.

In this paper, the problem investigated in Refs. [1, 16] is extended to coupled multi-degrees-of-freedom linear systems subjected to parametric excitation by a harmonic action superimposed by an ergodic stochastic process. General expressions for the drift and diffusion coefficients of Itô's equations are obtained using the stochastic averaging method (built upon assumption of weak excitation of wide-band process). For the stability analysis, the method of moment functions is used. In this present study, the boundaries of the instability regions of the second moments of the coupled two dimensional linear systems, investigated in Ref. [16], are obtained numerically and analytically. It is found, that there is a good agreement with the results obtained by the numerical method. As an application, the example of the flexural-torsional instability of a thin elastic beam under dynamic loading is considered [23].

2. Formulation.

We first consider systems that are described by the equations of motion of the form

$$\ddot{q}_i + 2\varepsilon\sum_{j=1}^{n}\beta_{ij}\dot{q}_j + \omega_i^2\left[q_i + \sum_{j=1}^{n}h_{ij}q_j\varepsilon\sin 2\nu t\right] = 0 \qquad \left(i = 1,\ldots,n\right), \tag{1}$$

where the q_i are the generalized normal coordinates, β_{ij} are damping constants, ω_i are the natural frequencies of the system, h_{ij} and ν represent, respectively, the amplitude and the frequency of the harmonic excitation, $\varepsilon \ll 1$ is a small parameter. These equations describe exactly the parametrically motion of a class of discrete mechanical systems with n degrees of freedom about the equilibrium configuration $q_i = 0$.

The stability of the trivial solution $q_i = 0$ has been extensively investigated. A survey of results with application to the stability of several elastic systems has been given in [4]. For a given system, the instability conditions define certain regions which correspond to instability of the equilibrium configuration $q_i = 0$. These regions have peaks at discrete points, called parametric resonances, that arise when certain relations between the frequency of the parametric action, 2ν, and the natural frequencies of a system, ω_k, are satisfied. It includes the cases when $2\nu = 2\omega_k/p$, referred to simple resonances, $2\nu = (\omega_j + \omega_k)/p$, referred to combination sum resonances, and $2\nu = (|\omega_j - \omega_k|)/p$, referred to combination differences resonances (j, k, $p = 1, 2, \ldots, j \neq k$). Here p represents the order of instability. Common examples of mechanical systems exhibiting instability under simple resonances are a simple pendulum whose support is given a vertical sinusoidal oscillation, and an elastic column subjected to a harmonically varying axial thrust.

An example of a coupled two-degree of freedom system that can show combination resonance corresponding to $2\nu=\omega_1+\omega_2$ is a thin elastic beam in lateral bending and tortional vibrations under a transverse harmonic action $P(t)$ having constant direction, i.e., acting in a non-follower fashion (see Fig. 1). While the same load acting in a follower fashion can cause combination resonance for $2\nu=|\omega_1-\omega_2|$ (see Fig. 2) [10, 20, 23].

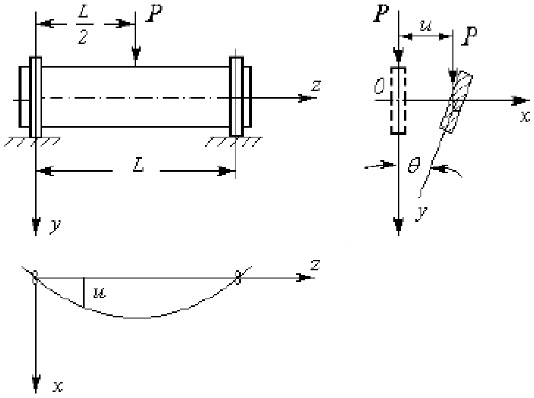

Fig. 1. Flexural-tortional vibration of a rectangular beam (non-follower case).

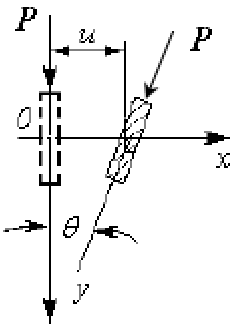

Fig. 2. Follower case.

We consider now the following coupled multi-degrees-of-freedom linear stochastic system described by the equations of motion of the form

$$\ddot{q}_i+2\varepsilon\sum_{j=1}^{n}\beta_{ij}\dot{q}_j+\omega_i^2 q_i+\omega_i^2\left[\sum_{j=1}^{n}h_{ij}q_j\varepsilon\sin 2\nu t+f(t)\varepsilon^{1/2}\sum_{j=1}^{n}k_{ij}q_j\right]=0 \quad (i=1,\dots,n), \tag{2}$$

where k_{ij} are constants, $f(t)$ is a stationary random process with zero mean value, and the phase vector $(q,\dot{q})$ forms a Markov process.

The moment stability of the solutions of system (2) can be investigated either through the associated Fokker-Planck-Kolmogorov equation or by the use of Ito's differential rule permitting to reduce the analysis of stability of solutions of stochastic differential equations to the analysis of the stability of deterministic differential equations describing the evolution of the moment functions. For the purpose of stability analysis, we consider the effect of the random parametric excitation on the stability of trivial solutions of system (2) when the frequency of the harmonic component falls within the region of combination parametric resonance, i.e. $2\nu\cong\omega_i+\omega_k$, $i, k=1, 2, \dots, n, i\neq k$.

Considering the case of parametric resonance, i.e. when $p_i\cong\omega_i$, and setting

$$\omega_i^2=p_i^2+\varepsilon\Delta_i, \tag{3}$$

where $\varepsilon\Delta$ denotes the amount of detuning; equation (2) may be rewritten as

$$\ddot{q}_i+p_i^2 q_i=-\varepsilon\left[2\sum_{j=1}^{n}\beta_{ij}\dot{q}_j+\Delta_i q_i+\omega_i^2\sum_{j=1}^{n}h_{ij}q_j\sin 2\nu t\right]-\varepsilon^{1/2}\omega_i^2 f(t)\sum_{j=1}^{n}k_{ij}q_j, \quad i=1,\dots,n. \tag{4}$$

Transforming to new variables z_i, and y_i by the relation

$$q_i(t)=z_i\cos p_i t+y_i\sin p_i t, \qquad \dot{q}_i(t)=-p_i\left[z_i\sin p_i t-y_i\cos p_i t\right], \quad i=1,\dots,n, \tag{5}$$

And

$$\dot{z}_i=-\frac{1}{p_i}\left(\ddot{q}_i+p_i^2 q_i\right)\sin p_i t, \qquad \dot{y}_i=\frac{1}{p_i}\left(\ddot{q}_i+p_i^2 q_i\right)\cos p_i t, \quad i=1,\dots,n. \tag{6}$$

We consider that the frequencies of resonance oscillations satisfy the relation $p_i+p_k=2\nu$, i, $k=1, \ldots, n$, $i\neq k$. The system of equations (4) may be replaced by the n-pairs of first-order equations, using the relations (5) and (6):

$$\dot{z}_i=\varepsilon\left[-2\sum_{j=1}^{n}\beta_{ij}\frac{p_j}{p_i}\left(z_j\sin p_jt-y_j\cos p_jt\right)\sin p_it+\frac{\Delta_i}{p_i}\left(z_i\cos p_it+y_i\sin p_it\right)\sin p_it+\frac{\omega_i^2}{p_i}\times\right. \tag{7a}$$

$$\left.\times\sum_{j=1}^{n}h_{ij}\left(z_j\cos p_jt+y_j\sin p_jt\right)\sin p_it\sin 2\nu t\right]+\varepsilon^{1/2}\frac{\omega_i^2}{p_i}f(t)\sum_{j=1}^{n}k_{ij}\left(z_j\cos p_jt+y_j\sin p_jt\right)\sin p_it,$$

$$\dot{y}_i=\varepsilon\left[2\sum_{j=1}^{n}\beta_{ij}\frac{p_j}{p_i}\left(z_j\sin p_jt-y_j\cos p_jt\right)\cos p_it-\frac{\Delta_i}{p_i}\left(z_i\cos p_it+y_i\sin p_it\right)\cos p_it-\frac{\omega_i^2}{p_i}\times\right.$$

$$\left.\times\sum_{j=1}^{n}h_{ij}\left(z_j\cos p_jt+y_j\sin p_jt\right)\cos p_it\sin 2\nu t\right]-$$

$$-\varepsilon^{1/2}\frac{\omega_i^2}{p_i}f(t)\sum_{j=1}^{n}k_{ij}\left(z_j\cos p_jt+y_j\sin p_jt\right)\cos p_it. \tag{7b}$$

We assume that the oscillation frequencies of the n-degrees-of-freedom systems are commensurable, i.e., $n_i\cdot p_i=n_k\cdot p_k$, where n_i and n_k are integers (i, k =1, 2, …, n, $i\neq k$). We can easily show that the fluctuation in two different degrees of freedom have a common period $T= n_i\cdot T_k + n_k\cdot T_i$, where $T_i=2\pi/p_i$, and it is possible to directly apply the Stratonovich–Khasminskii theory to standard systems of equations in view of the periodicity of the deterministic functions [9].

When applying on system (7) the averaging principle of Krelov – Bogolyubov and the Stratonovich – Khasminskii theory [9], it leads to the following homogenous Itô equations:

$$\dot{z}_i=m_{z_i}+\sum_{j=1}^{n}\left(\sigma_{z_iz_j}dw_{2j-1}+\sigma_{z_iy_j}dw_{2j}\right), \tag{8}$$

$$\dot{y}_i=m_{y_i}+\sum_{j=1}^{n}\left(\sigma_{y_iz_j}dw_{2j-1}+\sigma_{y_iy_j}dw_{2j}\right),\quad i=1,2,\ldots,n,$$

where $w_j(t)$ are independent Wiener processes of unit intensity and

$$m_{z_i}=\varepsilon\left\{\left[-\beta_{ii}+\omega_i^2\frac{k_{ii}^2}{8}\left[S(2p_i)-S(0)\right]\right]z_i+\left[\frac{\Delta_i}{2p_i}-\omega_i^2\frac{k_{ii}^2}{8}\Psi(2p_i)\right]y_i+\omega_i\sum_{j\neq i}^{n}\frac{h_{ij}}{4}z_j+\right.$$

$$\left.+\frac{1}{8}\sum_{k\neq i}^{n}\omega_i\omega_k\sum_{j\neq k}^{n}k_{ik}k_{kj}\times\left(\left[S(2\nu)-\delta_{ij}S(\delta p_{ik})\right]z_j+\left[-\Psi(2\nu)+\delta_{ij}\Psi(\delta p_{ik})\right]y_j\right)\right\};$$

$$m_{y_i}=\varepsilon\left\{\left[-\beta_{ii}+\omega_i^2\frac{k_{ii}^2}{8}\left[S(2p_i)-S(0)\right]\right]y_i-\left[\frac{\Delta_i}{2p_i}-\omega_i^2\frac{k_{ii}^2}{8}\Psi(2p_i)\right]z_i-\omega_i\sum_{j\neq i}^{n}\frac{h_{ij}}{4}y_j+\right.$$

$$\left.+\frac{1}{8}\sum_{k\neq i}^{n}\omega_i\omega_k\sum_{j\neq k}^{n}k_{ik}k_{kj}\times\left(\left[\Psi(2\nu)-\delta_{ij}\Psi(\delta p_{ik})\right]z_j+\left[S(2\nu)-\delta_{ij}S(\delta p_{ik})\right]y_j\right)\right\};$$

$$\left[\sigma\sigma^T\right]_{z_iz_i}=\varepsilon\frac{\omega_i^2}{8}\left\{k_{ii}^2\left[S(2p_i)z_i^2+\left[S(2p_i)+2S(0)\right]y_i^2\right]+\right.$$

$$+\sum_{j\neq i}^{n}\sum_{l\neq i}^{n}k_{ij}k_{il}\left(\left[S(2\nu)+\delta_{jl}S\left(\delta p_{ij}\right)\right]\cdot\left[z_j z_l+y_j y_l\right]\right)\Bigg\};$$

$$\left[\sigma\sigma^T\right]_{y_i y_i}=\varepsilon\frac{\omega_i^2}{8}\left\{k_{ii}^2\left[\left[S(2p_i)+2S(0)\right]z_i^2+S(2p_i)\,y_i^2\right]+\right.$$

$$+\sum_{j\neq i}^{n}\sum_{l\neq i}^{n}k_{ij}k_{il}\left(\left[S(2\nu)+\delta_{jl}S\left(\delta p_{ij}\right)\right]\cdot\left[z_j z_l+y_j y_l\right]\right)\Bigg\};$$

$$\left[\sigma\sigma^T\right]_{z_i y_i}=\left[\sigma\sigma^T\right]_{y_i z_i}=-\varepsilon\frac{\omega_i^2}{4}k_{ii}^2 S(0)\,z_i y_i\,;$$

$$\left[\sigma\sigma^T\right]_{z_i z_k}=\left[\sigma\sigma^T\right]_{z_k z_i}=\varepsilon\frac{\omega_i\omega_k}{8}\left\{2k_{ii}k_{kk}S(0)\,y_i y_k+\right.$$

$$+\sum_{j\neq i}^{n}\sum_{l\neq k}^{n}k_{ij}k_{kl}\left[\left\{S(2\nu)-\delta_{jk}\delta_{li}S\left(\delta p_{ik}\right)\right\}z_j z_l+\left\{S(2\nu)+\delta_{jk}\delta_{li}S\left(\delta p_{ik}\right)\right\}y_j y_l\right]\Bigg\};$$

$$\left[\sigma\sigma^T\right]_{y_i y_k}=\left[\sigma\sigma^T\right]_{y_k y_i}=\varepsilon\frac{\omega_i\omega_k}{8}\left\{2k_{ii}k_{kk}S(0)\,z_i z_k+\right.$$

$$+\sum_{j\neq i}^{n}\sum_{l\neq k}^{n}k_{ij}k_{kl}\left[\left\{S(2\nu)+\delta_{jk}\delta_{li}S\left(\delta p_{ik}\right)\right\}z_j z_l+\left\{S(2\nu)-\delta_{jk}\delta_{li}S\left(\delta p_{ik}\right)\right\}y_j y_l\right]\Bigg\};$$

$$\left[\sigma\sigma^T\right]_{z_i y_k}=\left[\sigma\sigma^T\right]_{y_k z_i}=\varepsilon\frac{\omega_i\omega_k}{8}\left\{-2k_{ii}k_{kk}S(0)\,z_k y_i+\right.$$

$$+\sum_{j\neq i}^{n}\sum_{l\neq k}^{n}k_{ij}k_{kl}\left[\left\{S(2\nu)-\delta_{jk}\delta_{li}S\left(\delta p_{ik}\right)\right\}z_l y_j-\left\{S(2\nu)+\delta_{jk}\delta_{li}S\left(\delta p_{ik}\right)\right\}z_j y_l\right]\Bigg\};$$

$$\left[\sigma\sigma^T\right]_{y_i z_k}=\left[\sigma\sigma^T\right]_{z_k y_i}=\varepsilon\frac{\omega_i\omega_k}{8}\left\{-2k_{ii}k_{kk}S(0)\,z_i y_k+\right.$$

$$+\sum_{j\neq i}^{n}\sum_{l\neq k}^{n}k_{ij}k_{kl}\left[\left\{S(2\nu)-\delta_{jk}\delta_{li}S\left(\delta p_{ik}\right)\right\}z_j y_l-\left\{S(2\nu)+\delta_{jk}\delta_{li}S\left(\delta p_{ik}\right)\right\}z_l y_j\right]\Bigg\};$$

$$\delta p_{ik}=p_i-p_k\,,\qquad \underset{i\neq k}{i,k}=1,2,\ldots n,$$

and δ_{ij} is the Kronecker Delta defined by

$$\delta_{ij}=\begin{cases}0 & \text{for } i\neq j;\\ 1 & \text{for } i=j.\end{cases}$$

Here $S(\omega)$ and $\Psi(\omega)$ denote, respectively, the cosine and sine power spectral densities of the stochastic process $f(t)$ defined by

$$S(\omega) + i\psi(\omega) = 2\int_0^\infty E[f(t)f(t+\tau)]e^{i\omega\tau}d\tau .$$

3. Stability analysis.

The general procedure of solving stochastic stability problems are given in Ref. [5]. The mathematical aspects of the theory of stochastic stability are introduced in Ref. [11]. In this paper the boundaries of instability regions of system (2), for n=2, are investigated using the method of moment functions [22]. The later method permits one to reduce the analysis of stability of solutions of stochastic differential equations to the analysis of the stability of deterministic differential equations describing the evolution of the moment functions. The basic equations of the method of moments are deduced by a termwise integration of the Fokker – Planck – Kolmogorov equation [5].

The coupled two-degrees-of-freedom system considered in this section has the form

$$\ddot{q}_1 + 2\varepsilon(\beta_{11}\dot{q}_1 + \beta_{12}\dot{q}_2) + \omega_1^2 q_1 + \omega_1^2(h_{11}q_1 + h_{12}q_2)\varepsilon\sin 2\nu t + \varepsilon^{1/2}\omega_1^2(k_{11}q_1 + k_{12}q_2)f(t) = 0,$$
$$\ddot{q}_2 + 2\varepsilon(\beta_{21}\dot{q}_1 + \beta_{22}\dot{q}_2) + \omega_2^2 q_2 + \omega_2^2(h_{21}q_1 + h_{22}q_2)\varepsilon\sin 2\nu t + \varepsilon^{1/2}\omega_2^2(k_{21}q_1 + k_{22}q_2)f(t) = 0. \tag{9}$$

The stability of system of equations (9) in probability was investigated in Ref. [18], where explicit expressions for the stability boundaries were deduced using Khasminkii's formulation. System (9) in the extended phase space is equivalent to Itô's system of stochastic equations for the four-dimensional process $X(t) = (z_1, y_1, z_2, y_2)$ with the following components deduced from the system of equations (8)

$$\dot{z}_i = m_{z_i} + \sum_{j=1}^{2}\left(\sigma_{z_i z_j}dw_{2j-1} + \sigma_{z_i y_j}dw_{2j}\right), \tag{10}$$

$$\dot{y}_i = m_{y_i} + \sum_{j=1}^{2}\left(\sigma_{y_i z_j}dw_{2j-1} + \sigma_{y_i y_j}dw_{2j}\right), \quad i = 1,2,$$

where w_j – are independent Wiener processes of unit intensity and

$$m_{z_1} = \varepsilon\left[(-\beta_{11} + d_1)z_1 + \left(\frac{\Delta_1}{2p_1} - d_2\right)y_1 + \omega_1\frac{h_{12}}{4}z_2\right],$$

$$m_{z_2} = \varepsilon\left[\omega_2\frac{h_{21}}{4}z_1 + (-\beta_{22} + d_3)z_2 + \left(\frac{\Delta_2}{2p_2} - d_4\right)y_2\right],$$

$$m_{y_1} = \varepsilon\left[-\left(\frac{\Delta_1}{2p_1} - d_2\right)z_1 + (-\beta_{11} + d_1)y_1 - \omega_1\frac{h_{12}}{4}y_2\right],$$

$$m_{y_2} = \varepsilon\left[-\omega_2\frac{h_{21}}{4}y_1 - \left(\frac{\Delta_2}{2p_2} - d_4\right)z_2 + (-\beta_{22} + d_3)y_2\right],$$

$$b_{z_1z_1} = \left[\sigma\sigma^T\right]_{z_1z_1} = \varepsilon\frac{\omega_1^2}{8}\left[k_{11}^2\left(S(2p_1)z_1^2 + c_2 y_1^2\right) + k_{12}^2 c_1\left(z_2^2 + y_2^2\right)\right],$$

$$b_{z_1 y_1} = \left[\sigma\sigma^T\right]_{z_1 y_1} = \left[\sigma\sigma^T\right]_{y_1 z_1} = -\varepsilon \frac{\omega_1^2}{4} k_{11}^2 S(0)\, z_1 y_1,$$

$$b_{y_1 y_1} = \left[\sigma\sigma^T\right]_{y_1 y_1} = \varepsilon \frac{\omega_1^2}{8}\left[k_{11}^2\left(c_2\, z_1^2 + S(2p_1)\, y_1^2\right) + k_{12}^2 c_1\left(z_2^2 + y_2^2\right)\right],$$

$$b_{z_1 z_2} = \left[\sigma\sigma^T\right]_{z_1 z_2} = \left[\sigma\sigma^T\right]_{z_2 z_1} = \varepsilon \frac{\omega_1 \omega_2}{8}\left(c_4 z_1 z_2 + c_5 y_1 y_2\right),$$

$$b_{z_2 z_2} = \left[\sigma\sigma^T\right]_{z_2 z_2} = \varepsilon \frac{\omega_2^2}{8}\left[k_{21}^2 c_1\left(z_1^2 + y_1^2\right) + k_{22}^2\left(S(2p_2)\, z_2^2 + c_3\, y_2^2\right)\right],$$

$$b_{z_1 y_2} = \left[\sigma\sigma^T\right]_{z_1 y_2} = \left[\sigma\sigma^T\right]_{y_2 z_1} = \varepsilon \frac{\omega_1 \omega_2}{8}\left(c_4 z_1 y_2 - c_5 y_1 z_2\right),$$

$$b_{y_2 y_2} = \left[\sigma\sigma^T\right]_{y_2 y_2} = \varepsilon \frac{\omega_2^2}{8}\left[k_{21}^2 c_1\left(z_1^2 + y_1^2\right) + k_{22}^2\left(c_3\, z_2^2 + S(2p_2)\, y_2^2\right)\right],$$

$$b_{z_2 y_2} = \left[\sigma\sigma^T\right]_{z_2 y_2} = \left[\sigma\sigma^T\right]_{y_2 z_2} = -\varepsilon \frac{\omega_2^2}{4} k_{22}^2 S(0)\, z_2 y_2,$$

$$b_{y_1 z_2} = \left[\sigma\sigma^T\right]_{y_1 z_2} = \left[\sigma\sigma^T\right]_{z_2 y_1} = \varepsilon \frac{\omega_1 \omega_2}{8}\left(c_4 z_2 y_1 - c_5 y_2 z_1\right),$$

$$b_{y_1 y_2} = \left[\sigma\sigma^T\right]_{y_1 y_2} = \left[\sigma\sigma^T\right]_{y_2 y_1} = \varepsilon \frac{\omega_1 \omega_2}{8}\left(c_5 z_1 z_2 + c_4 y_1 y_2\right),$$

$$d_1 = \frac{\omega_1}{8}\left\{\omega_1 k_{11}^2\left[S(2p_1) - S(0)\right] + \omega_2 k_{12} k_{21}\left[S(2v) - S(\delta p_{12})\right]\right\},$$

$$d_2 = \frac{\omega_1}{8}\left\{\omega_1 k_{11}^2 \Psi(2p_1) + \omega_2 k_{12} k_{21}\left[\Psi(2v) - \Psi(\delta p_{12})\right]\right\},$$

$$d_3 = \frac{\omega_2}{8}\left\{\omega_1 k_{12} k_{21}\left[S(2v) - S(\delta p_{12})\right] + \omega_2 k_{22}^2\left[S(2p_2) - S(0)\right]\right\},$$

$$d_4 = \frac{\omega_2}{8}\left\{\omega_1 k_{12} k_{21}\left[\Psi(2v) + \Psi(\delta p_{12})\right] + \omega_2 k_{22}^2 \Psi(2p_2)\right\},$$

$$c_1 = S(2v) + S(\delta p_{12}), \quad c_2 = S(2p_1) + 2S(0), \quad c_3 = S(2p_2) + 2S(0),$$

$$c_4 = k_{12} k_{21}\left[S(2v) - S(\delta p_{12})\right],$$

$$c_5 = k_{12} k_{21}\left[S(2v) + S(\delta p_{12})\right] + 2k_{11} k_{22} S(0), \quad \delta p_{12} = p_1 - p_2.$$

For convenience, we denoted the amplitudes z_1, y_1, z_2, y_2 by x_1, x_2, x_3, x_4, respectively.

The vector $X(t)=(x_1, x_2, x_3, x_4)$ describes the diffusion Markov process. Its joint probability density $p(X, t)$ is defined as the solution of the Fokker – Planck – Kolmogorov equation [22]. The problem is reduced to the analysis of the trivial solutions of system (10) of Markov type.

Not going beyond the scope of the correlation theory, we require, that the second moments of the coordinates and velocities remain restricted at $t \longrightarrow \infty$.

Let's consider a set of moment functions of the second order of vector process $X(t)$

$$m_{jk}(t)=\langle x_j(t)x_k(t)\rangle \quad (j,k=1,2,3,4). \tag{11}$$

By formulating the equation with respect to m_{jk}, we obtain a system of deterministic differential equations

$$\frac{dm_{jk}}{dt}=F_{jk}(m_{11}, m_{12}, ..., m_{44}) \quad (j,k=1,2,3,4). \tag{12}$$

Here

$$F_{jk}=\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\frac{\partial p(X,t)}{\partial t}x_j x_k dx_1 dx_2 dx_3 dx_4, \quad (j,k=1,2,3,4). \tag{13}$$

The joint probability density $p(X, t)$ of the process $X(t)$ satisfies the Fokker – Planck – Kolmogorov equation

$$\begin{aligned}\frac{\partial p}{\partial t}&=2(\beta_{11}+\beta_{22})p+\left\{(\beta_{11}+d_1)x_1-\left(\frac{\Delta_1}{2p_1}-d_2\right)x_2-\omega_1\frac{h_{12}}{4}x_3\right\}\frac{\partial p}{\partial x_1}+ \\ &+\left\{\left(\frac{\Delta_1}{2p_1}-d_2\right)x_1+(\beta_{11}+d_1)x_2+\omega_1\frac{h_{12}}{4}x_4\right\}\frac{\partial p}{\partial x_2}+ \\ &+\left\{-\omega_2\frac{h_{21}}{4}x_1+(\beta_{22}+d_3)x_3-\left(\frac{\Delta_2}{2p_2}-d_4\right)x_4\right\}\frac{\partial p}{\partial x_3}+ \\ &+\left\{\omega_2\frac{h_{21}}{4}x_2+\left(\frac{\Delta_2}{2p_2}-d_4\right)x_3+(\beta_{22}+d_3)x_4\right\}\frac{\partial p}{\partial x_4}+ \\ &+\frac{1}{2}\left(b_{z_1z_1}\frac{\partial^2 p}{\partial x_1^2}+b_{y_1y_1}\frac{\partial^2 p}{\partial x_2^2}+b_{z_2z_2}\frac{\partial^2 p}{\partial x_3^2}+b_{y_2y_2}\frac{\partial^2 p}{\partial x_4^2}\right)+ \\ &+b_{z_1y_1}\frac{\partial^2 p}{\partial x_1\partial x_2}+b_{z_1z_2}\frac{\partial^2 p}{\partial x_1\partial x_3}+b_{z_1y_2}\frac{\partial^2 p}{\partial x_1\partial x_4}+b_{y_1z_2}\frac{\partial^2 p}{\partial x_2\partial x_3}+b_{y_1y_2}\frac{\partial^2 p}{\partial x_2\partial x_4}+b_{z_2y_2}\frac{\partial^2 p}{\partial x_3\partial x_4}.\end{aligned} \tag{14}$$

The substitution of (14) and (13) into (12) leads to a set of linear differential equations, that may be expressed in a matrix form as

$$\frac{d\mathbf{m}}{dt}=\varepsilon\mathbf{A}\mathbf{m}, \tag{15}$$

where

$$\mathbf{m}=[m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}]^T.$$

The nonzero elements a_{ij} of the matrix A are:

$$a_{11}=a_{55}=2\left(-\beta_{11}+E_1\right),\quad a_{22}=2\left(-\beta_{11}+E_2\right),$$

$$a_{33}=a_{44}=a_{66}=a_{77}=-\left(\beta_{11}+\beta_{22}\right)+2E_3,$$

$$a_{88}=a_{1010}=2\left(-\beta_{22}+E_5\right),\quad a_{99}=2\left(-\beta_{22}+E_4\right),$$

$$\frac{1}{2}a_{12}=-a_{21}=a_{25}=a_{36}=a_{47}=-\frac{1}{2}a_{52}=-a_{63}=-a_{74}=\frac{\Delta_1}{2p_1}-d_2,$$

$$a_{34}=-a_{43}=a_{67}=-a_{76}=\frac{1}{2}a_{89}=-a_{98}=a_{910}=-\frac{1}{2}a_{109}=\frac{\Delta_2}{2p_2}-d_4,$$

$$\frac{1}{2}a_{13}=-a_{24}=a_{26}=a_{38}=a_{49}=-\frac{1}{2}a_{57}=-a_{69}=-a_{710}=\omega_1\frac{h_{12}}{4},$$

$$a_{31}=-a_{42}=a_{62}=-a_{75}=\frac{1}{2}a_{83}=a_{94}=-a_{96}=-\frac{1}{2}a_{107}=\omega_2\frac{h_{21}}{4},$$

$$a_{15}=a_{51}=\frac{\omega_1^2}{8}k_{11}^2c_2,\quad a_{810}=a_{108}=\frac{\omega_2^2}{8}k_{22}^2c_3,$$

$$a_{18}=a_{110}=a_{58}=a_{510}=\frac{\omega_1^2}{8}k_{12}^2c_1,\quad a_{81}=a_{85}=a_{101}=a_{105}=\frac{\omega_2^2}{8}k_{21}^2c_1,$$

$$a_{37}=-a_{46}=-a_{64}=a_{73}=\frac{\omega_1\omega_2}{8}c_5,$$

where

$$E_1=\frac{\omega_1}{8}\left(\omega_1k_{11}^2\left[\frac{3}{2}S\left(2p_1\right)-S\left(0\right)\right]+\omega_2c_4\right),\quad E_2=\frac{\omega_1}{8}\left(\omega_1k_{11}^2\left[S\left(2p_1\right)-2S\left(0\right)\right]+\omega_2c_4\right),$$

$$E_3=\frac{1}{16}\left(\omega_1^2k_{11}^2\left[S\left(2p_1\right)-S\left(0\right)\right]+\omega_2^2k_{22}^2\left[S\left(2p_2\right)-S\left(0\right)\right]+3\omega_1\omega_2c_4\right),$$

$$E_4=\frac{\omega_2}{8}\left(\omega_2k_{22}^2\left[S\left(2p_2\right)-2S\left(0\right)\right]+\omega_1c_4\right),\quad E_5=\frac{\omega_2}{8}\left(\omega_2k_{22}^2\left[\frac{3}{2}S\left(2p_2\right)-S\left(0\right)\right]+\omega_1c_4\right).$$

The system of differential equations (15) can be found by applying Itô's differential rule [16] to the quantities $x_j{\cdot}x_k$ (j, k=1, 2, 3, 4). The system of differential equations (9) in Ref. [16], should have the form $\frac{d}{dt}\overline{W}=2\varepsilon\cdot A\cdot\overline{W}$, which is the same as the system of equations (15) presented above, and the correct expressions of the following elements of the matrix A are: $a_{74}=-a_{65}=-a_{56}=a_{47}=\frac{\omega_1\omega_2}{16}c_5$, $a_{74}=-\omega_2\frac{h_{21}}{8}$; also the first stability condition in Eq. (8) and in Eq. (17), in Ref. [16], should be $\beta_{11}+\beta_{22}>d_1+d_3$ and $\beta_{11}+\beta_{22}>0$, respectively.

In the following we consider the case, when the function *f*(*t*) is a white noise process of small intensity, then

$$S(2p_i) = S(\delta p_i) = S(0) = S_0 = \text{constant, and } \Psi(2p_i) = \Psi(\delta p_i) = 0 \quad (i = 1, 2).$$

3.1. Asymptotic stability criteria. The asymptotic stability of the system of differential equations (15) is completely determined by the character of the roots of the characteristic equation

$$p(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = 0, \tag{16}$$

where the matrix **A** is defined in (15), **I** – the matrix unit, and λ_1, λ_2, …, λ_{10}, are the roots of the characteristic equation (16).

According to Liapunov’s theorems (the stability and instability theorems of the first approximation) [6, 19], the equilibrium of system (15) is asymptotically stable, if all roots of the characteristic equation (16) have negative real parts; and the equilibrium of system (15) is instable, if amongst the roots of the characteristic equation (16) there is at least one with a positive real part.

We consider the algebraic polynomial (16) with constant coefficients. Let the left hand-side of Eq. (16) be given in a polynomial form

$$p(\lambda) = \bar{p}_0\lambda^{10} + \bar{p}_1\lambda^9 + \bar{p}_2\lambda^8 + \bar{p}_3\lambda^7 + \bar{p}_4\lambda^6 + \bar{p}_5\lambda^5 + \bar{p}_6\lambda^4 + \bar{p}_7\lambda^3 + \bar{p}_8\lambda^2 + \bar{p}_9\lambda + \bar{p}_{10}. \tag{17}$$

The necessary condition for negative real parts of all roots of the polynomial equation (17) is that all its coefficients be positive [6], i.e.

$$\bar{p}_k > 0, \quad (k = 0, 1, 2, ..., 10). \tag{18}$$

From the coefficients of the polynomial $p(\lambda)$ we construct Hurwitz matrix $\mathbf{H}_{10}$ of order 10

$$\mathbf{H}_{10} = \begin{bmatrix} \bar{p}_1 & \bar{p}_3 & \bar{p}_5 & \bar{p}_7 & \bar{p}_9 & 0 & 0 & 0 & 0 & 0 \\ \bar{p}_0 & \bar{p}_2 & \bar{p}_4 & \bar{p}_6 & \bar{p}_8 & \bar{p}_{10} & 0 & 0 & 0 & 0 \\ 0 & \bar{p}_1 & \bar{p}_3 & \bar{p}_5 & \bar{p}_7 & \bar{p}_9 & 0 & 0 & 0 & 0 \\ 0 & \bar{p}_0 & \bar{p}_2 & \bar{p}_4 & \bar{p}_6 & \bar{p}_8 & \bar{p}_{10} & 0 & 0 & 0 \\ 0 & 0 & \bar{p}_1 & \bar{p}_3 & \bar{p}_5 & \bar{p}_7 & \bar{p}_9 & 0 & 0 & 0 \\ 0 & 0 & \bar{p}_0 & \bar{p}_2 & \bar{p}_4 & \bar{p}_6 & \bar{p}_8 & \bar{p}_{10} & 0 & 0 \\ 0 & 0 & 0 & \bar{p}_1 & \bar{p}_3 & \bar{p}_5 & \bar{p}_7 & \bar{p}_9 & 0 & 0 \\ 0 & 0 & 0 & \bar{p}_0 & \bar{p}_2 & \bar{p}_4 & \bar{p}_6 & \bar{p}_8 & \bar{p}_{10} & 0 \\ 0 & 0 & 0 & 0 & \bar{p}_1 & \bar{p}_3 & \bar{p}_5 & \bar{p}_7 & \bar{p}_9 & 0 \\ 0 & 0 & 0 & 0 & \bar{p}_0 & \bar{p}_2 & \bar{p}_4 & \bar{p}_6 & \bar{p}_8 & \bar{p}_{10} \end{bmatrix}. \tag{19}$$

The necessary and sufficient condition for the polynomial $p(\lambda)$ to have all roots with negative real parts is that all the principal diagonal minors of the matrix H_{10} be positive (Routh – Hurwitz conditions), i.e.,

$$\begin{aligned} &\bar{\Delta}_1 = \bar{p}_1 > 0, \\ &\bar{\Delta}_2 = \begin{vmatrix} \bar{p}_1 & \bar{p}_3 \\ \bar{p}_0 & \bar{p}_2 \end{vmatrix} > 0, \\ &\vdots \\ &\bar{\Delta}_2 = \bar{p}_{10}\bar{\Delta}_9 > 0. \end{aligned} \tag{20}$$

3.2. Expressions for the boundaries of the instability regions. Making use of the aforementioned instability theorem, which says, that the equilibrium of system (15) is instable if amongst the roots of the characteristic equation (16) there is at least one with a positive real part, boundaries of the instability regions (in different planes) are constructed numerically. In addition, verifications of the signs of the principal diagonal minors of (20) and the sign of the secular term $\overline{p}_{10}$ of (18) are carried out. It is found, that the boundaries of the instability regions, computed numerically, correspond to the case, when the secular term $\overline{p}_{10}$ of the polynomial equation (16) equals to zero, i.e.

$$\overline{p}_{10} = 0 \,. \tag{21}$$

From condition (21), analytical expressions for the boundaries of the instability regions are obtained.

The analytical expression, relating h_{12} to h_{21}, $\beta=\beta_{11}=\beta_{22}$ and the system parameters, has the form

$$h_{12} = \frac{1}{2S_0 k_{21}^2 \omega_1 \omega_2}\left[\overline{a}_0 h_{21} + \sqrt{\overline{b}_0 h_{21}^2 + \overline{b}_1}\,\right], \tag{22}$$

where

$$\overline{a}_0 = \left(k_{11}^2\omega_1^2 + k_{22}^2\omega_2^2\right)S_0 - 8\beta;$$

$$\overline{b}_0 = \left\{\left(k_{11}^2\omega_1^2 + 2\omega_1\omega_2 k_{12}k_{21} + k_{22}^2\omega_2^2\right)S_0 - 8\beta\right\}\left\{\left(k_{11}^2\omega_1^2 - 2\omega_1\omega_2 k_{12}k_{21} + k_{22}^2\omega_2^2\right)S_0 - 8\beta\right\};$$

$$\overline{b}_1 = 4S_0 k_{21}^2\left(\omega_1\omega_2 k_{21}^2\left(k_{11}k_{22} + k_{12}k_{21}\right)S_0 + 8\beta\right)\times$$

$$\times\left\{\left(k_{11}^2 k_{22}^2 - k_{12}^2 k_{21}^2\right)\left(\omega_1\omega_2 S_0\right)^2 - 4\beta\left(k_{11}^2\omega_1^2 + k_{22}^2\omega_2^2\right)S_0 + 16\beta^2\right\}.$$

The boundaries of the instability regions in (h_{21}, h_{12})-plane, computed analytically using relation (22) and numerically, are shown in Figs. 3 and 4 for different values of the parameter β.

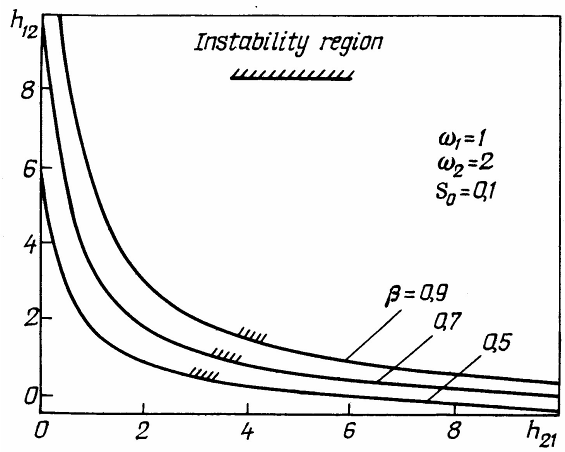

Fig. 3. Boundaries of the instability regions for $k_{11}= k_{22}=1$, $k_{12}= k_{21}=1$.

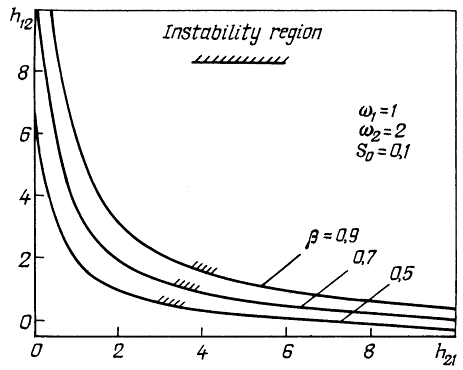

Fig. 4. Boundaries of the instability regions for $k_{11}= k_{22}=0$, $k_{12}= k_{21}=1$.

From the above presented figures, we see that there is a good agreement between the analytical and numerical results.

The analytical expression, relating $h=h_{12}=h_{21}$ to $\beta=\beta_{11}=\beta_{22}$ and the system parameters, has the form

$$h = 2\sqrt{\frac{(8\beta+g_1)\,(\beta-g_2)\,(\beta-g_3)}{2\omega_1\omega_2\,(\beta-g_4)}}\,, \tag{23}$$

which is valid for $\beta \geq g_2$. Here

$$g_1 = \omega_1\omega_2(k_{11}k_{22}+k_{12}k_{21})S_0,$$

$$g_2 = \frac{S_0}{8}\left((\omega_1k_{11})^2+(\omega_2k_{22})^2+\sqrt{\left[(\omega_1k_{11})^2-(\omega_2k_{22})^2\right]^2+\left[2\omega_1\omega_2k_{12}k_{21}\right]^2}\right),$$

$$g_3 = \frac{S_0}{8}\left((\omega_1k_{11})^2+(\omega_2k_{22})^2-\sqrt{\left[(\omega_1k_{11})^2-(\omega_2k_{22})^2\right]^2+\left[2\omega_1\omega_2k_{12}k_{21}\right]^2}\right),$$

$$g_4 = \frac{S_0}{8}\left[(\omega_1k_{11})^2+(\omega_2k_{22})^2-\omega_1\omega_2\left(k_{12}^2+k_{21}^2\right)\right],$$

and it may be remarked from the above presented expressions that $g_2 > g_4$, $g_2 > g_3$, then $\beta > g_4$. The boundaries of the instability regions in (β, h)-plane, computed analytically using relation (23) and numerically, are shown in Figs. 5 and 6 for different values of the parameter S_0.

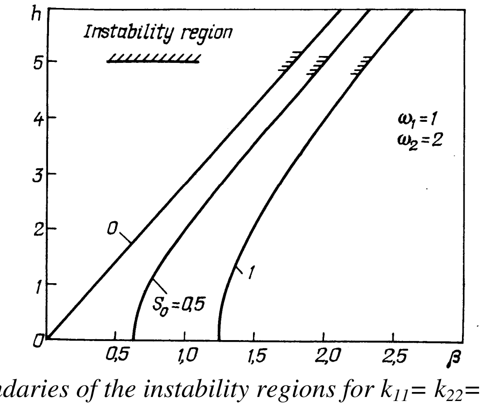

Fig. 5. Boundaries of the instability regions for $k_{11}= k_{22}=1$, $k_{12}= k_{21}=1$.

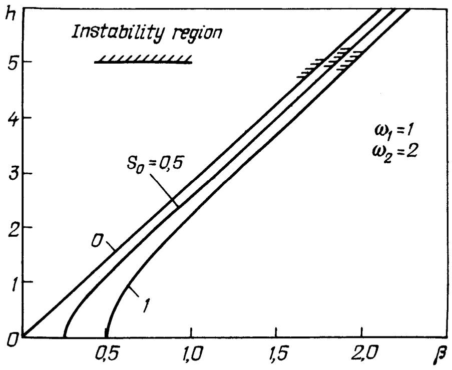

Fig. 6. Boundaries of the instability regions for $k_{11}= k_{22}=0$, $k_{12}= k_{21}=1$.

From the above presented figures, we also see that there is a good agreement between the analytical and numerical results.

Finally, the analytical expression, relating β_{11} to β_{22}, $h=h_{12}=h_{21}$ and the system parameters, has the form

$$\beta_1 = \frac{1}{32\beta_2}\left[a_0\beta_2^2 + a_1\beta_2 + a_2 + \sqrt{b_0\beta_2^4 + b_1\beta_2^3 + b_2\beta_2^2 + b_3\beta_2 + b_4}\,\right], \tag{24}$$

where

$$\beta_1 = \beta_{11},\ \ \beta_2 = \beta_{22},\ \ a_0 = -16,\ \ a_1 = 4\omega_1\omega_2 k_{12}k_{21}S_0,\ \ a_2 = \omega_1\omega_2 h^2,$$

$$b_0 = 256,\ \ b_1 = -128\omega_1\omega_2 k_{12}k_{21}S_0,\ \ b_2 = 16\omega_1\omega_2\left(2h^2 + \omega_1\omega_2\left(k_{12}k_{21}S_0\right)^2\right),$$

$$b_3 = 2\left(2\omega_1\omega_2 h\right)^2 k_{12}k_{21}S_0,\ \ b_4 = \left(\omega_1\omega_2 h^2\right)^2.$$

We note that relation (24) is valid for the cases, when the coefficients $k_{11}=k_{22}=0$.

The boundaries of the instability regions in (β_2, β_1)-plane, computed analytically using relation (24), and numerically, are illustrated in an example in the following section.

4. Application.

As an application the flexural-tortional vibration of a simply supported, uniform, narrow, rectangular, elastic beam of length L is considered. The beam is subjected to a dynamical concentrated load $P(t)$ acting at the centre of the beam cross-section in a non-follower fashion as shown in Fig. 1. The differential equations describing the motion [23] of the lateral deflection $u(t)$ and the angle of twist $\theta(t)$ are:

$$EI_y\frac{\partial^4 u}{\partial z^4} + \frac{\partial^2\left(M_x\theta\right)}{\partial z^2} + m\frac{\partial^2 u}{\partial t^2} + D_u\frac{\partial u}{\partial t} = 0, \tag{25}$$

$$-GJ\frac{\partial^2\theta}{\partial z^2} + \frac{\partial M_x}{\partial z}\frac{\partial u}{\partial z} + M_x\frac{\partial^2 u}{\partial z^2} + \frac{\partial M_z}{\partial z} + mr^2\frac{\partial^2\theta}{\partial t^2} + D_\theta\frac{\partial\theta}{\partial t} = 0,$$

where

$$M_x = \begin{cases} \frac{1}{2}Pz, & 0 \le z \le \frac{1}{2}L; \\ \frac{1}{2}P\left(L-z\right), & \frac{1}{2}L \le z \le L; \end{cases} \qquad M_z = \begin{cases} \frac{1}{2}Pu_m - \frac{1}{2}Pu, & 0 \le z \le \frac{1}{2}L; \\ -\frac{1}{2}Pu_m + \frac{1}{2}Pu, & \frac{1}{2}L \le z \le L; \end{cases}$$

EI_y and GJ denote, respectively, the flexural and tortional rigidities of the cross section, D_u and D_θ are the viscous damping coefficients, m is the mass per unit length, and r is the polar radius of gyration of the cross-section. Here the subscript "m" denotes the value of the midspan $z{=}L/2$.

The conditions of simply supported at the ends imply the following boundary conditions:

$$u(0,t)=u(L,t)=\frac{\partial^2 u}{\partial z^2}(0,t)=\frac{\partial^2 u}{\partial z^2}(L,t)=0, \qquad \theta(0,t)=\theta(L,t)=0\,. \tag{26}$$

For simply supported ends, the mode shapes may be assumed as

$$u(z,t)=\overline{q}_1\sin\frac{\pi}{L}z,\;\; \theta(z,t)=\overline{q}_2\sin\frac{\pi}{L}z, \tag{27}$$

in which $\overline{q}_1=u_{\mathrm{m}}=u(L/2,t)$, $\overline{q}_2=\theta_{\mathrm{m}}=\theta(L/2,t)$. Substituting (27) into equations (25), then multiplying by $\sin(\pi z/2)$, and integrating with respect to z from 0 to L we obtain

$$\int_0^L\left[EI_y\left(\frac{\pi}{L}\right)^4\overline{q}_1+m\ddot{\overline{q}}_1+D_u\dot{\overline{q}}_1\right]\sin^2\frac{\pi}{L}z\,\mathrm{d}z+\int_0^L\frac{\partial^2(M_x\theta)}{\partial z^2}\sin\frac{\pi}{L}z\mathrm{d}z=0,$$

$$\int_0^L\left[GJ\left(\frac{\pi}{L}\right)^2\overline{q}_2+mr^2\ddot{\overline{q}}_2+D_\theta\dot{\overline{q}}_2\right]\sin^2\frac{\pi}{L}z\,\mathrm{d}z+\int_0^L\left(\frac{\partial M_x}{\partial z}\frac{\partial u}{\partial z}+M_x\frac{\partial^2 u}{\partial z^2}+\frac{\partial M_z}{\partial z}\right)\sin\frac{\pi}{L}z\mathrm{d}z=0. \tag{28}$$

Taking into account the boundary conditions (26) and Eqs. (27), the following terms of (28) are evaluated using integration by parts as follows:

$$\int_0^L\frac{\partial^2(M_x\theta)}{\partial z^2}\sin\frac{\pi}{L}z\mathrm{d}z=-\left(\frac{\pi}{L}\right)^2\int_0^L(M_x\theta)\sin\frac{\pi}{L}z\mathrm{d}z=$$

$$=-\left(\frac{\pi}{L}\right)^2\frac{P}{2}\overline{q}_2\left[\int_0^{L/2}z\sin^2\frac{\pi}{L}z\mathrm{d}z+\int_{L/2}^L(L-z)\sin^2\frac{\pi}{L}z\,\mathrm{d}z\right]=-\frac{1}{16}\left(4+\pi^2\right)P\overline{q}_2;$$

$$\int_0^L\left(\frac{\partial M_x}{\partial z}\frac{\partial u}{\partial z}+M_x\frac{\partial^2 u}{\partial z^2}\right)\sin\frac{\pi}{L}z\mathrm{d}z=-\left(\frac{\pi}{L}\right)\int_0^L M_x\frac{\partial u}{\partial z}\cos\frac{\pi}{L}z\mathrm{d}z=$$

$$=-\left(\frac{\pi}{L}\right)^2\frac{P}{2}\overline{q}_1\left[\int_0^{L/2}z\cos^2\frac{\pi}{L}z\mathrm{d}z+\int_{L/2}^L(L-z)\cos^2\frac{\pi}{L}z\,\mathrm{d}z\right]=-\frac{1}{16}\left(-4+\pi^2\right)P\overline{q}_1;$$

$$\int_0^L\frac{\partial M_z}{\partial z}\sin\frac{\pi z}{L}\mathrm{d}z=-\left(\frac{\pi}{L}\right)\int_0^L M_z\cos\frac{\pi z}{L}\mathrm{d}z=$$

$$=-\left(\frac{\pi}{L}\right)\frac{P}{2}\overline{q}_1\left[\int_0^{L/2}\left(1-\sin\frac{\pi z}{L}\right)\cos\frac{\pi z}{L}\mathrm{d}z+\int_{L/2}^L\left(-1+\sin\frac{\pi z}{L}\right)\cos\frac{\pi z}{L}\mathrm{d}z\right]=-\frac{1}{2}P\overline{q}_1.$$

Hence, equations (28) become [23]

$$\begin{aligned}&m\ddot{\overline{q}}_1+D_u\dot{\overline{q}}_1+EI_y\left(\frac{\pi}{L}\right)^4\overline{q}_1-\frac{1}{8L}\left(4+\pi^2\right)P\overline{q}_2=0,\\&mr^2\ddot{\overline{q}}_2+D_\theta\dot{\overline{q}}_2+GJ\left(\frac{\pi}{L}\right)^2\overline{q}_2-\frac{1}{8L}\left(4+\pi^2\right)P\overline{q}_1=0.\end{aligned} \tag{29}$$

Let $\overline{q}_1=-rq_1$, $\overline{q}_2=\gamma q_2$, Eq. (29) may be written in the form

$$\ddot{q}_1 + 2\varepsilon\beta_1\dot{q}_1 + \omega_1^2\left[q_1 + g(t)q_2\right] = 0,\ \ \ddot{q}_2 + 2\varepsilon\beta_2\dot{q}_2 + \omega_2^2\left[q_2 + g(t)q_1\right] = 0, \tag{30}$$

where

$$g(t) = \frac{P(t)}{P_{cr}},\quad P_{cr} = \frac{8mrL\omega_1\omega_2}{\left(4+\pi^2\right)},\quad 2\varepsilon\beta_1 = \frac{D_u}{m},\quad 2\varepsilon\beta_2 = \frac{D_\theta}{mr^2},$$

$$\omega_1^2 = \frac{EI_y}{m}\left(\frac{\pi}{L}\right)^4,\quad \omega_2^2 = \frac{GJ}{mr^2}\left(\frac{\pi}{L}\right)^2,\quad \gamma = \frac{\omega_1}{\omega_2}.$$

In most cases, the excitation g(t) consists of a harmonic term on which a stochastic fluctuation is superposed. Hence the function $g(t)$ may have the form $g(t) = \varepsilon h\,\sin 2vt + +\varepsilon^{1/2}kf(t)$. Thus, the system of equations (30) becomes

$$\begin{aligned} &\ddot{q}_1 + 2\varepsilon\beta_1\dot{q}_1 + \omega_1^2\left[q_1 + \varepsilon h q_2\,\sin 2vt + \varepsilon^{1/2}kf(t)\,q_2\right] = 0;\\ &\ddot{q}_2 + 2\varepsilon\beta_2\dot{q}_2 + \omega_2^2\left[q_2 + \varepsilon h q_1\,\sin 2vt + \varepsilon^{1/2}kf(t)\,q_1\right] = 0. \end{aligned} \tag{31}$$

For system (9), the coefficients of (31) take the form

$$\beta_{11} = \beta_1,\quad \beta_{22} = \beta_2,\quad \beta_{12} = \beta_{21} = h_{11} = h_{22} = k_{11} = k_{22} = 0,\ h_{12} = h_{21} = h,\quad k_{12} = k_{21} = k\,.$$

The boundaries of the instability regions of system (31) in the (β_2, β_1)-plane, computed analytically using Eq. (24) and numerically, are shown in Figs. 7 and 8 for different values of the parameter h.

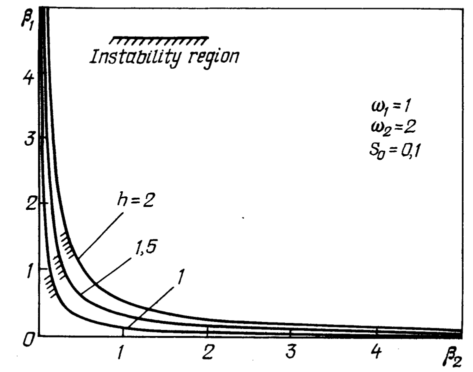

Fig. 7. Boundaries of the instability regions for k= 1.

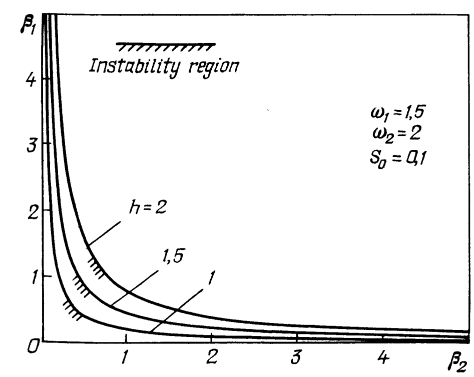

Fig. 8. Boundaries of the instability regions for k= 0.5.

We conclude from the above presented figures, that there is a good agreement between the analytical and numerical results.

5. Conclusion.

A method for investigating the stability of a class of coupled two-degrees-of-freedom systems, subjected to parametric excitation by a harmonic action superimposed by an ergodic stochastic process, has been presented. Explicit expressions for the stability of the second moments are obtained from the secular term of the characteristic equation. There is good agreement between the analytical and numerical results. The method has been successfully applied to an elastic structural element, showing the feasibility of this approach, valid in the first approximation, to realistic engineering structures.

Р Е З Ю М Е . Досліджено динамічну стійкість зв'язаної системи з двома степенями свободи, збудженої параметрично гармонічною дією, накладеною на ергодичний стохастичний процес. В аналізі стійкості використано метод моментних функцій. Отримано явні вирази щодо стійкості других моментів, коли частота гармонічного збудження лежить в околі комбінаційної суми власних частот. Отримано добре узгодження аналітичних і числових результатів. Як приклад, розглянуто стійкість поперечних прогинів резинової балки при динамічному збудженні.

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