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# THE PRINCIPLE OF VIRTUAL WORK AND THE THIRD ORDER WAVE FOR CONTINUA WITH SECOND ORDER CONSTITUTIVE EQUATIONS

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**Abstract:** The generalized form of the principle of virtual work is obtained, when the virtual work is considered as a time integral of virtual power. The corresponding this form Euler – Lagrange equation includes the divergence of the Lie derivative of stress. So, the equation of motion on the stress rate field is one of the results of this paper. When bying studied the third order wave, a generalization of the acoustic tensor is obtained. The generalized acoustic tensor seems the most important result of these paper. This one can also be found by investigating the acceleration wave.

**Key words:** motion equation, Lie derivative of stress, generalized principle of virtual work, compatibility conditions, the third order wave, generalized acoustic tensor.

**Introduction**.

The investigation of the third order wave necessitates the knowledge of the dynamical compatibility equation. This equation rises from the first equation of motion in case of the acceleration wave. Now it needs the time derivative of the first equation of motion. The material time derivative isn't simple in the current configuration. Using the principle of virtual power, namely the principle of virtual work, the derivative will be obvious and indisputable. We assume that the integral of the virtual power with respect to time is the virtual work. Hence, from the principle of virtual work the time derivative of the first equation of motion can be obtained and then the dynamical compatibility equation can be calculated. The time derivative of the first equation of motion will be called the equation of motion on the stress rate field. Many authors have dealt with this question when the body was in equilibrium [8, 9, 10]. The third order wave can be investigated by using the compatibility equations (dynamic, kinematic and constitutive). When the constitutive equation is a system of first order nonlinear partial differential equations the investigation of wave propagation is more convenient by use of the third order wave.

**1. The principle of virtual work.**

In continuum mechanics, the principle of virtual power is writing as follows

$$\int_V t^{kl} v^*_{k;l} dV = \int_V q^k v^*_k dV + \int_{A_p} \tilde{p}^k v^*_k dA, \qquad (1)$$

where $t^{kl}, v^*_k, v^*_{k;l}$ and $q^k$ denote the Cauchy stress, the virtual velocity, the virtual velocity gradient and the difference between the body force and the force of inertia in domain $V$, respectively, and $\tilde{p}^k$ is the surface force on boundary surface $A_p$ ( $A = A_v + A_p$ , the velocity $\tilde{v}^k$ is known on $A_v$).

The stress tensor in the satisfies the second Cauchy equation of motion, that is, $t^{kl} = t^{lk}$.

Assume as a starting point that the integral of the power for a given period $\left[t_1, t_2\right]$ means the work during this period. Thus, the integrated with respect to time *t* equation (1) gives

$$\int_{t_1}^{t_2}\int_{V} t^{kl} v^*_{k;l} dV dt = \int_{t_1}^{t_2}\int_{V} q^k v^*_k dV dt + \int_{t_1}^{t_2}\int_{A_p} \tilde{p}^k v^*_k dA dt \,. \tag{2}$$

As it can be seen, the virtual deformation rate $v^*_{kl}$ on the left hand side of the equation has been replaced with virtual velocity gradient $v^*_{k;l}$. This replacement leaves the product $t^{kl} v^*_{kl}$ unchanged since $t^{kl} = t^{lk}$. The material time derivative of the deformation gradient is

$$\dot{x}^k_{\ ,K} = v^k_{\ ;p} x^p_{\ ,K} \,.$$

Then

$$v^k_{\ ;p} = \dot{x}^k_{\ ,K} X^K_{\ ,p} \,. \tag{3}$$

With displacement vector $\underline{u}$ used $\underline{r} = \underline{R} + \underline{u}$ and the derivatives with respect to time and $X^K$ are written in indexed form as

$$v^k = \dot{u}^k \qquad \text{and} \qquad \dot{x}^k_{\ ,K} = \dot{u}^k_{;q} x^q_{,K} \equiv \dot{u}^k_{:K}$$

respectively, the formula (3) becomes

$$v^k_{\ ;p} = \dot{u}^k_{:K} X^K_{\ ,p} = \dot{u}^k_{\ ;p} \,. \tag{4}$$

With the volume integral on the left hand side of (2) transformed to the initial configuration, the integrals with respect to time and over volume $V_o$ can be interchanged

$$\int_{V_o}\int_{t_1}^{t_2} t^l_k \dot{u}^{*k}_{\ \ :K} X^K_{\ ,l} \bar{J}\, dt\, dV_o = \int_{V_o}\int_{t_1}^{t_2} \bar{J}\, q^k \dot{u}^*_k \, dt\, dV_o +$$

$$+ \int_{A^o_p}\int_{t_1}^{t_2} \bar{J}\, t^{kl} \dot{u}^*_k \, X^K_{\ ,l}\, dt\, dA^o_K; \quad \bar{J} = \frac{dV}{dV_o}. \tag{5}$$

Consider now the integrals with respect to time, one after the other:

$$\int_{t_1}^{t_2} \bar{J}\, t^l_k\, X^K_{\ ,l}\, \dot{u}^{*k}_{\ \ :K}\, dt = \int_{t_1}^{t_2}\left[\left(\bar{J}\, t^l_k\, X^K_{\ ,l} u^{*k}_{\ \ :K}\right)^{\cdot} - \left(\bar{J}\, t^l_k\, X^K_{\ ,l}\right)^{\cdot} u^{*k}_{\ \ :K}\ \right] dt \,.$$

The first integral can be calculated from time $t_1$ to $t_2$ on the right hand side

$$\int_{t_1}^{t_2} \bar{J}\, t^l_k\, X^K_{\ ,l}\, \dot{u}^{*k}_{\ \ :K}\, dt = \left(\bar{J} t^l_k u^{*k}_{\ \ ;l}\right)^{t_2}_{t_1} - \int_{t_1}^{t_2} \bar{J}\left(t_k^{\ p}\, v^q_{\ ;q} + \dot{t}_k^{\ p} - t_k^{\ l}\, v^p_{\ ;l}\right) X^K_{\ ,p} u^{*k}_{\ \ :K}\, dt. \tag{6$_1$}$$

After similar transformations, the first integral with respect to time on the right hand side of (5) is as follows:

$$\int_{t_1}^{t_2} \bar{J}\; q^k \dot{u}_k^* \, dt = \left( \bar{J} q^k u_k^* \right)_{t_1}^{t_2} - \int_{t_1}^{t_2} \bar{J} \left( \dot{q}^k + v^s{}_{;s}\, q^k \right) \; u_k^* \, dt. \qquad (6)_2$$

Here again the virtual displacement are not vanishing $u_k^* \neq 0$ at time $t_1$ and $t_2$ in $(6)_2$ .

Finally, after transformation of the second integral on the right hand side of (5)

$$\int_{t_1}^{t_2} \bar{J}\; t^{kl}\, X^K{}_{,l}\; \dot{u}_k^* \, dt = \left( \bar{J} t^{kl} X^K_{,l} u_k^* \right)_{t_1}^{t_2} - \int_{t_1}^{t_2} \bar{J} \left( t^{kp}\; v^s{}_{;s} + \dot{t}^{kp} - t^{kl}\; v^p{}_{;l} \right) X^K{}_{,p} u_k^* \, dt. \qquad (6)_3$$

With $(6)_{1,2,3}$ substituted into (5) and after proper rearrangement, the equation of virtual work is [1]

$$\int_{t_1}^{t_2} \int_V \left( \dot{t}^{kp} - t^{kq}\, v^p{}_{;q} + t^{kp}\, v^s{}_{;s} \right) u^*{}_{k;p}\, dV\, dt = -\int_V \left[ \left( t^{lk}{}_{;l} + q^k \right) u_k^* \right]_{t_1}^{t_2} dV + \int_{A_p} \left[ \left( \tilde{p}^k - t^{kl} n_l \right) u_k^* \right]_{t_1}^{t_2} dA +$$

$$+ \int_{t_1}^{t_2} \int_V \left( \dot{q}^k + q^k\, v^s{}_{;s} \right) u_k^* \, dV\, dt + \int_{t_1}^{t_2} \int_{A_p} \left( \dot{t}^{kp} + t^{kp}\, v^s{}_{;s} - t^{kl}\, v^p{}_{;l} \right) n_p u^*{}_k \, dA\, dt.$$

Therefore

$t^l{}_{k;l} + q_k = 0$ is the first Cauchy equation of motion, $\tilde{p}^k = t^{kp} n_p$ is the dynamic boundary condition on $A_p$.

The principle of virtual work is as follows

$$\int_{t_0}^{t_2} \int_V \left( \dot{t}^{kp} - t^{kq}\, v^p{}_{;q} + t^{kp}\, v^s{}_{;s} \right) u^*{}_{k;p}\, dV\, dt =$$

$$= \int_{t_0}^{t_1} \int_V \left( \dot{q}^k + q^k\, v^s{}_{;s} \right) u_k^* \, dV\, dt + \int_{t_0}^{t_1} \int_{A_p} \left( \dot{t}^{kp} - t^{kh}\, v^p{}_{;h} + t^{kp}\, v^s{}_{;s} \right) n_p u^*{}_k \, dA\, dt. \qquad (7)$$

Equation (7) refers to continua and its any part. Otherwise, on the basis of what has been said above, the equation given below is obtained after suitable mathematical transformation

$$L_V \left( t^{ij} \right)_{;j} + \left( t^{hj} v^i{}_{;h} + t^{ij} v^h{}_{;h} \right)_{;j} + \dot{q}^i + q^i v^h{}_{;h} = \rho\, \ddot{v}^i \qquad (8)$$

supposing that the Cauchy equations of motion and the boundary condition are satisfied. Here $L_V \left( t^{ij} \right)$ denotes the Lie derivative of Cauchy′s stress tensor, that is,

$$L_V \left( t^{ij} \right) = \dot{t}^{ij} - t^{hj} v^i{}_{;h} - t^{ih} v^j{}_{;h}\,; \qquad (9)$$

$q^k$ is the body force density; $\rho$ is the mass density and satisfies the continuity equation.

Equation (8) is the equation of motion on the stress rate (or Lie derivative of stress) field, [8, 9, 10, 12].

**2. The third order wave.**

When the basic quantities $v^k, t^{kl}, a_{kl}$ and the first derivatives of them are continuous, but the second derivatives have a jump by crossing surface $\varphi\left(x^k, t\right) = 0$, we speak about the third order waves [2]. Let us denote the jump of some quantity $v^k_{;p}$ by $\langle v^k_{;p} \rangle$. When the velocity gradient is $v^k_{;p}$, in case of the wave of order three $\langle v^k_{;p} \rangle = 0$, but $\langle v^k_{;pq} \rangle \neq 0$. Thus in (9) $\langle L_V(t^{kp}) \rangle = 0$, but $\langle L_V(t^{kp})_{;q} \rangle \neq 0$ and so on.

Now the dynamic condition of the third order wave is

$$\left\langle \left( L_V \left( t^{kl} \right) \right)_{;l} \right\rangle + t^{pq} \left\langle v^k{}_{;pq} \right\rangle + t^{kj} \left\langle v^l{}_{;lj} \right\rangle = \rho \left\langle \ddot{v}^k \right\rangle . \tag{10}$$

Let the kinematic equation [4, 5] be

$$L_V\left(L_V\left(a_{ij}\right)\right) = \ddot{a}_{ij} + \left(a_{kj} v^k{}_{;i} + a_{ik} v^k{}_{;j}\right)^{\square} + \left(\dot{a}_{lj} + a_{kj} v^k{}_{;i} + a_{lk} v^k{}_{;i}\right) v^l{}_{;i} +$$

$$+ \left(\dot{a}_{il} + a_{ik} v^k{}_{;l} + a_{kl} v^k{}_{;i}\right) v^l{}_{;i}.$$

When the Lie derivative of the velocity field is $L_V$, expression $L_V \equiv £_v + \frac{\partial}{\partial t}$ in (10) is a generalization of the velocity [3]. The Euler strain tensor is $a_{ij}$. As it is well known $L_V\left(a_{ij}\right) = v_{ij}$, for the strain rate , thus the kinematic compatibility condition of the third order wave is

$$\left\langle \dot{v}_{ij} \right\rangle = \left\langle \ddot{a}_{ij} \right\rangle + a_{kj} \left\langle \dot{v}^k{}_{;i} \right\rangle + a_{ik} \left\langle v^k{}_{;j} \right\rangle \tag{11}$$

namely

$$L_V\left(L_V\left(a_{ij}\right)\right) \equiv L_V\left(v_{ij}\right) = \dot{v}_{ij} + v_{lj} v^l{}_{;i} + v_{il} v^l{}_{;j} .$$

It can easily be shown that $\dot{v}^k{}_{;i} \neq \left(v^k{}_{;i}\right)^{\bullet}$ , but $\left\langle \dot{v}^k{}_{;i} \right\rangle = \left\langle \left(v^k{}_{;i}\right)^{\bullet} \right\rangle$ and this property is the same for the second derivatives of all other functions.

Let the constitutive equation be

$$f_\alpha\left(L_V{}^2\left(t^{ij}\right), L_V\left(t^{ij}\right)_{;k}, t^{ij}{}_{;kh}, L_V{}^2\left(a_{pq}\right), L_V\left(a_{pq}\right)_{;r}, a_{pq;rs}, L_V\left(v_{pq}\right), v_{pq;r}, t^{ij}, a_{pq}, v_{pq}\right) = 0 , \tag{12}$$

$$a = 1, 2, 3, 4, 5, 6.$$

where $L_V{}^2 \equiv L_V\left(L_V\left(..\right)\right)$ . The equations contain the second order derivatives with respect to space and time, hence they are called the second order constitutive equations. The constitutive compatibility conditions can be obtained from equation (12) by calculation after and before the wave front.

$$f_\alpha\left(L_V{}^2\left(t^{ij}\right) + \left\langle L_V{}^2\left(t^{ij}\right)\right\rangle, \ldots., L_V\left(a_{pq}\right)_{;r} + \left\langle L_V\left(a_{pq}\right)_{;r}\right\rangle, \ldots, v_{pq}\right) - f_\alpha\left(L_V{}^2\left(t^{ij}\right), \ldots., v_{pq}\right) = 0 . \tag{13}$$

Notation $\langle \ \rangle$ means the jump across the wavefront, for example

$$\left\langle L_V{}^2\left(t^{ij}\right)\right\rangle = \tilde{\gamma}^{ij}\left(\frac{\partial \varphi}{\partial t} + v^p \frac{\partial \varphi}{\partial x^p}\right)^2 - \left(\frac{\partial \varphi}{\partial t} + v^p \frac{\partial \varphi}{\partial x^p}\right)\left(\tilde{\lambda}^i t^{pj} \frac{\partial \varphi}{\partial x^p} + \tilde{\lambda}^j t^{iq} \frac{\partial \varphi}{\partial x^q}\right)$$

or

$$\left\langle a_{pq;rs}\right\rangle = \tilde{\alpha}_{pq}\frac{\partial\varphi}{\partial x^r}\frac{\partial\varphi}{\partial x^s} \text{ and so on.}$$

The system (13) is the system of first order partial differential equations. Using the characteristic equation of (13), the constitutive compatibility condition is obtained in form

$$\frac{\partial f_\alpha}{\partial\varphi_{\hat{i}}}\varphi_{\hat{i}} = 0,..., \quad \hat{i} = 1,2,3,4\,, \tag{14}$$

where $\varphi_{\hat{i}} = \dfrac{\partial\varphi}{\partial x^{\hat{i}}}$ if $\dfrac{\partial\varphi}{\partial t} = \dfrac{\partial\varphi}{\partial x^4}$.

In the following we use Cartesian coordinates. When the jumps in the second derivatives of the stress, strain tensors and the velocity on the surface $\varphi(x^k, t) = 0$ are denoted by $\gamma^{ij}, \alpha_{ij}, \lambda^k$, the unit normal vector of the wavefront is introduced

$$n_k \equiv \frac{\dfrac{\partial\varphi}{\partial x^k}}{\sqrt{g^{pq}\dfrac{\partial\varphi}{\partial x^p}\dfrac{\partial\varphi}{\partial x^q}}}$$

and the wave propagation velocity is denoted

$$C = c - v^k n_k\,,$$

the equations (10), (11) and (14) lead to the dynamic

$$\gamma^{kl} n_l = -\rho\, C\lambda^k\,; \tag{15}$$

kinematic

$$\alpha_{ij} = \frac{1}{2C}\left[n_i\left(2a_{kj} - g_{kj}\right) + n_j\left(2a_{ik} - g_{ik}\right)\right]\lambda^k \tag{16}$$

and the constitutive compatibility equations [2 , 4]

$$\gamma^{ij}\left(S_{\alpha ij4}C^2 - S_{\alpha ij}{}^k n_k C + S_{\alpha ij}{}^{kl} n_k n_l\right) + \alpha_{pq}\left(E_\alpha{}^{pq}{}_4 C^2 - E_\alpha{}^{pqr} n_r C + E_\alpha n_r n_s\right) +$$

$$+\lambda.^r\left(\left(S_{\alpha pq4}T^{pq}{}_r - E_\alpha{}^{pq}{}_4 A_{rpq} - \frac{1}{2}W_\alpha{}^{pq}{}_4 G_{rpq}\right)C - S_{\alpha ij}{}^k n_k T^{ij}{}_r + \right.$$

$$\left. + E_\alpha{}^{pqs} A_{rpqs} + \frac{1}{2}W_\alpha{}^{pqk} n_k G_{rpqs}\right) = 0. \tag{17}$$

Here the notations are used:

$$S_{\alpha ij4} = \frac{\partial f_\alpha}{\partial {L_V}^2\left(t^{ij}\right)}; \quad S_{\alpha ij}{}^k = \frac{\partial f_\alpha}{\partial L_V\left(t^{ij}\right)_{;k}}\;; \quad S_{\alpha ijkl} = \frac{\partial f_\alpha}{\partial t^{ij}{}_{;kl}}\;;$$

$$E_\alpha{}^{pq}{}_4 = \frac{\partial f_\alpha}{\partial {L_V}^2\left(a_{pq}\right)}; \quad E_\alpha{}^{pqr} = \frac{\partial f_\alpha}{\partial L_V\left(a_{pq}\right)_{;r}}\;; \quad E_\alpha{}^{pqrs} = \frac{\partial f_\alpha}{\partial a_{pqrs}}\,;$$

$$\mathrm{W}_\alpha{}^{pq}{}_4 = \frac{\partial f_\alpha}{\partial L_V\left(v_{pq}\right)};\quad W_\alpha{}^{pqk} = \frac{\partial f_\alpha}{\partial v_{pq;k}};$$

$$A_{pijq} = a_{pi}n_j n_q + a_{iq}n_j n_p;\quad G_{pijq} = g_{pi}n_j n_q + g_{iq}n_j n_p;$$

$$T^{pq}{}_{ij} = t^{ps}n_s g^q{}_i n_j + t^{sq}n_s g^p{}_i n_j,$$

where $g_{pi}$ and $g^p_i$ are the metric tensors and $L_V{}^2$ is the second order Lie derivative.

Substituting (15) and (16) into (17) we can write the wave propagation equation for the stress amplitude $\gamma^{ij} = \gamma^{ji} \sim \gamma^\beta$

$$\left\{2\rho S_{\alpha\beta}C^4 - 2\rho\overline{S}_{\alpha\beta}C^3 + B_{\alpha\beta}C^2 + D_{\alpha\beta}C + H_{\alpha\beta}\right\}\gamma^\beta = 0, \tag{18}$$

where the following notations were used:

$$S_{\alpha\beta} = S_{\alpha\beta(ij)4};\ \overline{S}_{\alpha\beta} = S_{\alpha\beta(ij)}{}^k n_k;$$

$$B_{\alpha\beta} = 2\rho\quad S_{\alpha\beta}{}^{kh}n_k n_h + E_\alpha{}^{pq}{}_4 G_{p\beta(ij)q} - 2S_{\alpha pq4}T^{pq}{}_\beta + W_\alpha{}^{pq}{}_4 G_{p\beta(ij)q};$$

$$D_{\alpha\beta} = -E_\alpha{}^{pqr}n_r G_{p\beta(ij)q} + 2S_{\alpha ps}{}^k n_k T^{pss}{}_{\beta(ij)} - W_\alpha{}^{pqk}n_k G_{p\beta(ij)q},$$

$$H_{\alpha\beta} = E_\alpha{}^{pqrs}n_r n_s\left(2A_{p\beta(ij)q} - G_{p\beta(ij)q}\right).$$

The interpretation of index β is: $(\dots)_{\beta(ij)} = \begin{cases} (\dots)_i & if\ \ i = j; \\ 2(\dots)_{i+j+1} & if\ \ i \neq j. \end{cases}$

The determinant of the matrix in bracket $\{\ \}$ in (18) is zero, because $\gamma^\beta$ is not zero $\det\{\ \} = 0.$

This is the equation of the propagation of the third order wave, being the 24-th order algebraical equation for the propagation velocity $C$. Matrix $\{\}$ can be considered as the characteristic matrix (6x6) of a generalization of the acoustic tensor.

The matrix of acoustic tensor can be obtained from (18). Let us denote the coefficients of $C$ in the form of 6x6 matrices by $\mathbf{S}, \overline{\mathbf{S}}, \mathbf{B}, \mathbf{D}, \mathbf{H}$. The the acoustic matrix

$$\mathbf{S}C^4 + \overline{\mathbf{S}}C^3 + \mathbf{B}C^2 + \mathbf{D}C + \mathbf{H} \tag{19}$$

gives the wave propagation, if it is multiplied by $\boldsymbol{\gamma}$ and set equal to zero

$$\left(\mathbf{S}C^4 + \overline{\mathbf{S}}C^3 + \mathbf{B}C^2 + \mathbf{D}C + \mathbf{H}\right)\boldsymbol{\gamma} = \mathbf{0}.$$

The most general acoustic tensor can be obtained from (19), when the coefficients of $C^4, C^3, C^2, C$ and $C^o$ have been denoted in form of 6x6 matrices $\mathbf{S}, \overline{\mathbf{S}}, \mathbf{B}, \mathbf{D}, \mathbf{H}$. By introducting the inverse $\mathbf{S}^{-1}$ and unit matrices $\mathbf{I}$ , the wave propagation equation [6, 7] is

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{S}^{-1}\mathbf{H} & \mathbf{S}^{-1}\mathbf{D} & \mathbf{S}^{-1}\mathbf{B} & \mathbf{S}^{-1}\overline{\mathbf{S}} \end{bmatrix} \begin{bmatrix} \gamma \\ \gamma^{*} \\ \gamma^{**} \\ \gamma^{***} \end{bmatrix} = C \begin{bmatrix} \gamma \\ \gamma^{*} \\ \gamma^{**} \\ \gamma^{***} \end{bmatrix}.$$

Now the generalized acoustic matrix can be written [6] in the following form

$$\begin{bmatrix} \mathbf{-IC} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-IC} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-IC} & \mathbf{I} \\ \mathbf{S}^{-1}\mathbf{H} & \mathbf{S}^{-1}\mathbf{D} & \mathbf{S}^{-1}\mathbf{B} & \mathbf{S}^{-1}\overline{\mathbf{S}}\mathbf{-IC} \end{bmatrix}. \tag{20}$$

The elements of the acoustic matrix are 6x6 matrices and the final matrix is 24x24.

**3. Two special cases.**

i). Let $\mathbf{H}$ and $\mathbf{D}$ be identical to zero; then equation (18) is

$$C^{12}\left\{2\rho\mathbf{S}C^2 - 2\rho\overline{\mathbf{S}}\,C + \mathbf{B}\right\}\gamma = 0.$$

We can designate such bodies as the quasi-viscoelastic bodies.

ii). Let us write the one dimensional form of equation (18)

$$\Big[2\rho\left(f_{\sigma_{tt}}C^4 - f_{\sigma_{tx}}C^3\right) + \left(2\rho f_{\sigma_{xx}} + f_{\varepsilon_{tt}} - 2\sigma f_{\sigma_{tt}} + f_{v_t}\right)C^2 +$$
$$+\left(-f_{\varepsilon_{tx}} + 2\sigma f_{\sigma_{tx}} - f_{v_x}\right)C + f_{\varepsilon_{xx}}\Big]\gamma = 0, \tag{21}$$

where notations are as usual: stress $\sigma$, strain $\varepsilon$, velocity of strain $v$ and subscripts denote the partial derivatives.

The algebraic equation (21) has at least two positive and negative real roots. The coefficients of that equation satisfy this condition. Similar conditions can also be supposed to the equation (18).

These conditions for the material coefficients enable us to approximate the constitutive equations, when the suitable experiments are performed.

Р Е З Ю М Е . Отримано узагальнений принцип віртуальних зміщень, коли віртуальні зміщення розглядаються як інтеграл по часу від віртуальної енергії. Рівняння Ейлера-Лагранжа дають рівняння для дивергенції похідної Лі по напруженнях. Рівняння руху в термінах поля швидкості напружень є одним з нових результатів цієї статті. При вивченні хвилі третього порядку отримано узагальнення акустичного тензора, яке можна вважати найбільшим досягненням у проведеному дослідженні. Цей результат може бути отриманий також при дослідженні хвилі прискорення.

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