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ANTIPLANE PROBLEM ON A CRACK, PROPAGATING WITH AN ARBITRARY SPEED IN ANISOTROPIC INHOMOGENEOUS ELASTIC MEDIA[1]

A. G. BAGDOEV*, S. G. SAHAKYAN**

*Institute of Mechanics of NAS of Armenia, Yerevan
**Yerevan Institute of Architecture and Construction, Armenia

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A problem on a crack, propagating with an arbitrary speed in anisotropic inhomogeneous elastic media, is solved. The initial problem is reduced to an isotropic one by the change of variables. First of all, the problem for small inhomogeneity is considered. Its solution is obtained by the iteration method and is expressed by quadratures from the solution of the homogeneous case. The stresses outside the crack and displacements on its faces are obtained. Besides, the solution for an arbitrary value of the inhomogeneity parameter is obtained. It is shown that its first order approximation coincides with the solution obtained by the method of small parameter.

Решена задача о трещине, распространяющейся с произвольной скоростью в анизотропной неоднородной эластичной среде. Исходная задача сведена изотропной с помощью замены переменной. Прежде всего, рассмотрена задача для малой неоднородности. Ее решение получено итерационным методом и записано в квадратурах относительно решения для однородного случая. Найдены напряжения вне трещины и перемещения на ее границе. Кроме того, получено решение для произвольного значения параметра неоднородности. Показано, что аппроксимация первого порядка для него совпадает с решением, полученным методом малого параметра.

Розв'язано задачу про тріщину, яка поширюється з довільною швидкістю в анізотропному неоднорідному пружному середовищі. Вихідну задачу зведено до ізотропної за допомогою заміни змінної. Насамперед, розглянуто задачу для малої неоднорідності. Її розв'язок отримано ітераційним методом і записано в квадратурах відносно рішення для однорідного випадку. Знайдено напруження поза тріщиною й переміщення на її границі. Окрім того, отримано розв'язок для довільного значення параметра неоднорідності. Показано, що апроксимація першого порядку для нього збігається з рішенням, отриманим методом малого параметра.

INTRODUCTION

Propagation of crack with arbitrary velocity is very important problem for application in seismology and the engineering use of metallic details. Because all practical materials, as a rule are inhomogeneous, it is interesting to study the influence of inhomogeneity on the stress intensity coefficient near crack's edge and condition of opening of the crack.

A solution of the plane problem for crack, moving with arbitrary speed in isotropic homogeneous elastic medium, is obtained in [1] by the convolution method. Both the antiplane and plane problems for homogeneous isotropic medium are considered in [2]. For the first time a solution of the antiplane problem for homogeneous isotropic medium was given in [3]. Wide range of questions related with propagation of cracks was considered in [4]. The solutions, presented [2, 3], are based on technique developed in study on flow around wing in [5]. An antiplane anisotropic problem on crack in homogeneous medium is considered in [6]. In this paper a crack propagating with arbitrary speed in an inhomogeneous anisotropic elastic medium for is discussed. In particular, an antiplane problem is considered. Solutions for displacements on crack are obtained under conditions of small and arbitrary inhomogeneities. Besides that, a solution for stresses out of crack is obtained for small inhomogeneity. By the convolution method the alternative symmetric problem on crack with given displacements on its faces is solved.

1. STATEMENT OF PROBLEM

Assume that the crack occupies some region along x-axis in plane (x, y). At that its edges $x=l_1(t)$ and $x=l_2(t)$ move with arbitrary speeds. Under mentioned conditions one component of displacement of medium (u along z-axis) and two components of stresses (τ_{xz}, τ_{yz}) exist. The following relation is valid [6] for the stresses:

$$\frac{\tau_{xz}}{\rho} = a_1^2 \frac{\partial u}{\partial x} + a_{12}^2 \frac{\partial u}{\partial y}, \qquad \frac{\tau_{yz}}{\rho} = a_{12}^2 \frac{\partial u}{\partial x} + a_2^2 \frac{\partial u}{\partial y}, \tag{1}$$

where ρ is a density of the medium; a_1^2, a_2^2, a_{12}^2 are some constants.

In what follows we consider semiplane $y>0$ and

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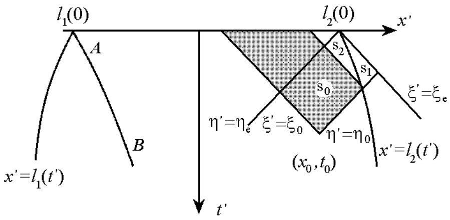

Fig. 1. Characteristics pattern for propagating crack

take that $\rho=\rho(y)$. Equation of motion is written in the form

$$\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}=\rho\,\frac{\partial^2 u}{\partial t^2}. \qquad (2)$$

Substituting Eq. (1) into Eq. (2) and introducing new variables

$$x_1=x-\frac{a_{12}^2}{a_2^2}\,y, \qquad y_1=y-\frac{a}{a_2^2}\,y, \qquad (3)$$

we obtain that

$$\frac{\tau_{yz}}{\rho}=a\,\frac{\partial u}{\partial y_1}, \qquad a^2=a_1^2a_2^2-a_{12}^4, \qquad (4)$$

$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial y_1^2}+\frac{1}{\rho}\frac{\partial \rho}{\partial y_1}\frac{\partial u}{\partial y_1}=\frac{a_2^2}{a^2}\frac{\partial^2 u}{\partial t^2}. \qquad (5)$$

So, the initial problem is reduced to problem for isotropic inhomogeneous medium. Assuming that $\rho(y)=\rho_0 e^{ky_1}$ and introducing function

$$u=Pe^{(-k/2)y_1} \qquad (6)$$

we obtain from Eqs (4), (5) the following relations:

$$\frac{\tau_{yz}}{\rho}=a\left(\frac{\partial P}{\partial y_1}-\frac{k}{2}\,P\right)e^{(-k/2)y_1}, \qquad (7)$$

$$\frac{\partial^2 P}{\partial x_1^2}+\frac{\partial^2 P}{\partial y_1^2}-\frac{k^2}{4}\,P=\frac{a_2^2}{a^2}\frac{\partial^2 P}{\partial t^2}. \qquad (8)$$

To complement the statement we write the boundary conditions in the form

$$\frac{\tau_{yz}}{\rho}=a\,\frac{\partial u}{\partial y_1}=\frac{\tau'(x,t)}{\rho}$$
$$\text{for} \quad y_1=0, \quad l_1(t)<x<l_2(t), \qquad (9)$$
$$u=0 \quad \text{for} \quad x>l_2(t), \quad x>l_1(t).$$

Obtained boundary problem (8), (9) in principle can be solved by the convolution method [1]. However, complex quadratures are obtained in the solution due to inhomogeneity. Therefore, at first let us assume that the inhomogeneity is small and in only the first order terms with respect to the inhomogeneity parameter k are retained in the solution. In so doing Eq. (8) can be written as an equation for homogeneous isotropic elastic medium

$$\frac{\partial^2 P}{\partial x_1^2}+\frac{\partial^2 P}{\partial y_1^2}=\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2}, \qquad c^2=\frac{a^2}{a_1^2} \qquad (10)$$

with the boundary conditions at $y=0$, derived from Eq. (9):

$$\frac{\partial P}{\partial y_1}-\frac{k}{2}\,P=\frac{\tau'(x,t)}{\rho a} \quad l_1(t)<x<l_2(t),$$
$$P=0, \quad x>l_2(t), \quad x<l_1(t). \qquad (11)$$

Now let us consider the solution for semiinfinite crack, putting $l_1(t)=-\infty$. From the physical point of view such situation means that the edges of the crack do not affect one another. For the finite crack the obtained solution is valid for the right hand side region of the characteristic curve AB (fig. 1). It is expedient to introduce the functions $v=\partial P/\partial y_1$. At that we can put the following conditions for $y=0$:

$$v(x,0,t)=v(x,t),$$
$$P=P_++P_-, \qquad v=v_++v_-. \qquad (12)$$

Here, index "+" corresponds to the functions being equal to zero for $x<l_2(t)$, while index "−" – to functions being equal to zero for $x>l_2(t)$.

After this substitution conditions (11) yield

$$v_--\frac{k}{2}P_-=f(x,t), f(x,t)=\frac{\tau'(x,t)}{\rho a}, P_+=0. \qquad (13)$$

The boundary conditions (11) and Eq. (8) were obtained for the case of exponential law $\rho(y_1)=\rho_0 e^{ky_1}$. It is interesting to consider more general class of functions $\rho(y_1)$, leading to mentioned equations. Let the displacement take the form

$$u(x,y_1,t)=F(y_1)P(x,y_1,t).$$

Putting it into Eq. (5) we obtain that

$$\frac{2F'}{F}+\frac{\rho'}{\rho}=0, \qquad F(0)=1, \qquad 2F_0'=\frac{\rho_0'}{\rho_0},$$

where primes denote differentiation with respect to y_1; $\rho_0(0)=\rho_0$; $\rho_0'(0)=\rho_0'$.

Originating from the above assumptions one can obtain the following relations.

A. For the law

$$\rho=\rho_0\left(1+\frac{\rho_0'}{2\rho_0}y_1\right)^2, \quad \frac{\rho_0'}{\rho_0}=k \qquad (14)$$

the equation

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y_1^2}=\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} \qquad (15)$$

is valid (which is analogous to Eq. (10)) along with boundary conditions (11) for $y_1=0$. For order of k the obtained problem is reduced to exponential problem considered above.

B. For the law

$$\rho=\rho_0\left(\operatorname{ch}\frac{k}{2}y_1+\frac{\rho_0'}{\rho_0 k}\operatorname{sh}\frac{k}{2}y_1\right)^2 \qquad (16)$$

the equation (8) for P is obtained, and boundary condition yields

$$\frac{\partial P}{\partial y_1}-\frac{\rho_0'}{2\rho_0}P=\frac{\tau'(x,t)}{\rho a} \quad \text{at} \quad y_1=0. \qquad (17)$$

As it is seen, this problem also is reduced to the above exponential law problem with the coefficient ρ_0'/ρ_0 replacing the coefficient k in (11). So, in order of k both statements of the boundary problems coincide. In particular, for $\rho_0'/(\rho_0 k)=1$ Eq. (16) gives $\rho=\rho_0 e^{ky_1}$, i. e. the same as in the previous case.

C. For the law

$$\frac{\rho}{\rho_0}=\left(\cos\frac{k}{2}y_1+\frac{\rho_0'}{\rho_0 k}\sin\frac{k}{2}y_1\right)^2 \qquad (18)$$

we obtain the equation

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y_1^2}+\frac{k^2}{4}P=\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} \qquad (19)$$

and the boundary condition (17). So, in order of k this problem coincides with the problem (10), (11).

This is the reason for considering in further the problem stated by equation (8) in the aggregate with the boundary conditions (11), from which in order of k all mentioned cases of distribution of $\rho(y_1)$ can be obtained.

2. SOLUTION OF THE PROBLEM FOR SMALL *k*

Solution of Eq. (10) for the boundary condition

$$\partial P/\partial y_1=v(x,t) \quad \text{at} \quad y_1=0,$$

yields the Possio integral form [3 – 5]

$$P(x,t)=-\frac{c}{\pi}\iint\frac{v(x',t')\,dx'dt'}{\sqrt{c^2(t-t')^2-(x-x')^2}}. \qquad (20)$$

From formula (20), using the condition $P=0$ for $x>l_2(t)$ and introducing the characteristic coordinates [3, 5]

$$\begin{aligned}\xi'&=ct'-x', & \eta'&=ct'+x',\\ \xi_0&=ct-x, & \eta_0&=ct+x,\end{aligned} \qquad (21)$$

one can write the solution for v_+ in the form

$$v_+=-\frac{1}{\pi}\frac{1}{\sqrt{\eta_0-\eta_2(\xi_0)}}\times \\ \times\int\limits_{-\xi_0}^{\eta_2(\xi_0)} v_-(x',t')\,\frac{\sqrt{\eta_0(\xi_0)-\eta'}}{\eta_0-\eta'}\,d\eta'. \qquad (22)$$

Note, that, as in [3 – 5] it is accepted that $\dot{l}_2(t)<c$ (dot denotes differentiation on t), the following equality becomes valid:

$$\eta_2(\xi_0)=l_2(t_2)+ct_2, \quad ct_2-l_2(t_2)=ct-x. \qquad (23)$$

Here, $(l_2(t_2),t_2$ is a point of intersection of the characteristic $\xi'=\xi_0=\text{const}$ with the curve $x'=l_2(t')$, representing a motion law for the edge of the crack (fig. 1).

Changing the integration variable from η' to x' and taking into account Eqs (21) and (23) we obtain that

$$\begin{aligned}\eta_0-\eta'&=2(x_0-x'),\\ \eta_2(\xi_0)-\eta'&=2[l(t_2)-x'],\\ \eta_0-\eta_2(\xi_0)&=2[x-l_2(t_2)]\end{aligned} \qquad (24)$$

along $\xi'=\xi_0$ [3]. Moreover, from Eq. (22) it follows that

$$v_+=-\frac{1}{\pi}\frac{1}{\sqrt{x-l_2(t_2)}}\times \\ \times\int\limits_{x-ct}^{l_2(t_2)} v_-\left(x',t+\frac{x'}{c}-\frac{x}{c}\right)\frac{\sqrt{l_2(t_2)-x'}}{x-x'}\,dx'. \qquad (25)$$

The above expression coincides with the solution from [3] in physical dimensional coordinates. In the present case v_- is unknown. From Eq. (13) we can write that

$$v_-(x,t)=f(x,t)+\frac{k}{2}P_-(x,t), \qquad (26)$$

where P_- is also unknown.

It should be noted that integration in Eq. (20) is carried out over the domains $(s_0+s_1+s_2)$ (see fig. 1) [5]. At the same time, originating from the boundary condition $P=0$ for $x>l_2(t)$, we obtain from Eq. (20), written with respect to the characteristic variables [2], that

$$\int\limits_{-\xi'}^{\eta_2(\xi')} \frac{\tau_1(\xi',\eta')}{\sqrt{\eta_0-\eta'}}\,d\eta' + \int\limits_{\eta_2(\xi')}^{\eta_0} \frac{\tau_1(\xi',\eta')}{\sqrt{\eta_0-\eta'}}\,d\eta' = 0, \qquad (27)$$

where $\tau_1(\xi',\eta')=v(x',t')$; $\eta_2(\xi')=l_2(t'_2)+ct'_2$; $\xi'=l_2(t'_2)-ct'_2$. Therefore, it can be shown that the integrals over s_1 and s_2 in Eq. (20) are cancelled. So, only the integral over s_0 remains [5] (filled domain in the figure). Performing integration in Eq. (20), as it was made in [2, 4, 5], one can obtain that

$$P_- = -\frac{1}{2\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0} \frac{d\xi'}{\sqrt{\xi_0-\xi'}} \int\limits_{-\xi'}^{\eta_0} \frac{v_1(\xi',\eta')}{\sqrt{\eta_0-\eta'}}\,d\eta', \qquad (28)$$

where

$$v_1(x',t') = f(x',t') + \frac{k}{2}\,\Phi_-(\xi',\eta'); \qquad \Phi_-(\xi',\eta') = P_-(x',t'). \qquad (29)$$

Thus, for $\Phi_-(\xi',\eta')$ one can obtain the integral equation

$$f_1(\xi',\eta') = f(x',t'),$$

$$\Phi_-(\xi_0,\eta_0) = -\frac{1}{2\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0} \frac{d\xi'}{\sqrt{\xi_0-\xi'}} \times \int\limits_{-\xi'}^{\eta_0} \frac{f_1(\xi',\eta') + (k/2)\,\Phi_-(\xi',\eta')}{\sqrt{\eta_0-\eta'}}\,d\eta', \qquad (30)$$

where

$$\xi_a(\eta_0) = ct_3 - l_2(t_3); \quad ct_3 + l_2(t_3) = ct + x \qquad (31)$$

along the characteristic $\eta'=\eta_0$. In homogeneous case ($k=0$) the formula

$$\Phi^0_-(\xi_0,\eta_0) = -\frac{1}{2\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0} \frac{d\xi'}{\sqrt{\xi_0-\xi'}} \int\limits_{-\xi'}^{\eta_0} \frac{f_1(\xi',\eta')}{\sqrt{\eta_0-\eta'}}\,d\eta' \qquad (32)$$

is valid. Passing to the edge of the crack $x\approx l_2(t)$ and accounting for narrowness of the domain of integration in ξ'-direction and Eq. (24), one can obtain the following expression in physical coordinate x:

$$\Phi^0_-(\xi_0,\eta_0) = -\frac{\sqrt{2}}{\pi}\sqrt{\xi_0-\xi_a(\eta_0)} \times \int\limits_{x-ct}^{x} f\left(x', t+\frac{x'}{c}-\frac{x}{c}\right)\frac{dx'}{\sqrt{x-x'}}, \qquad (33)$$

where due to Eq. (31)

$$\xi_0 - \xi_a(\eta_0) \approx \frac{2(l_2(t)-x)}{1+\dot{l}_2(t)/c}. \qquad (34)$$

Finally, near the crack's edge one can obtain for $k=0$ (zero approximation) that

$$\Phi^0_- = P^0_- = -\frac{1}{\pi}\sqrt{\frac{l_2(t)-x}{1+\dot{l}_2(t)/c}} \times \int\limits_{x-ct}^{x} f\left(x', t+\frac{x'}{c}-\frac{x}{c}\right)\frac{dx'}{\sqrt{x-x'}}. \qquad (35)$$

Hence, from Eq. (30) in the first order of k a closed solution can be obtained, where $\Phi_-(\xi',\eta')$ in Eq. (30) should be taken from Eq. (32). Also, and in determination of $\Phi^0_-(\xi',\eta')$ the domain of integration should be chosen between the characteristics $\eta''=\eta'$; $\xi''=\xi'$; $\xi''=\xi_a(\xi')$. Finally, we obtain that

$$\Phi^0_-(\xi',\eta') = -\frac{1}{2\pi}\int\limits_{\xi_a(\eta')}^{\xi'} \frac{d\xi''}{\sqrt{\xi'-\xi''}} \int\limits_{-\xi''}^{\eta'} \frac{f_1(\xi'',\eta'')}{\sqrt{\eta'-\eta''}}\,d\eta'', \qquad (36)$$

where

$$\xi_a(\eta') = ct'_3 - l_2(t'_3), \qquad ct'_3 + l_2(t'_3) = \eta'.$$

Formula (36) is valid for the points of integration (ξ',η') in Eq. (30), for which the characteristic $\eta''=\eta'$ intersects with the curve $x'=l_2(t')$ (i. e., for $\eta'>\eta_e=l_2(0)$, see fig. 1). For the points (ξ',η'), in which it intersects with x'-axis within region $x'<l_2(0)$ (i. e. for $\eta'<\eta_e$) one should integrate in (36) within the limits

$$\xi_1(\eta') < \xi'' < \xi', \qquad \xi_1(\eta') = -\eta'.$$

So, formula (30) gives for arbitrary point (x,t) the value of $P_-(x,t)$ equal to $\Phi_-(\xi_0,\eta_0)$. Therefore, for small k the value $\Phi^0_-(\xi',\eta')$ should be be taken from Eq. (36). For $x\approx l_2(t)$ we obtain

$$P_- = -\frac{2}{\pi}\sqrt{\frac{l_2(t)-x}{1+\dot{l}_2(t)/c}} \times \int\limits_{x-ct}^{x}\left\{ f\left(x', t+\frac{x'}{c}-\frac{x}{c}\right) + \frac{k}{2}\,\Phi^0_-(\xi_0,\xi_0+2x')\right\}\frac{dx'}{\sqrt{x-x'}}. \qquad (37)$$

Eqs (30) and (37) give the displacement on crack $u=P_-$. It is evident that the inhomogeneity essentially effects the solution.

Simultaneously, at arbitrary (x,t) for $x>l_2(t)$ the value v_+ is given by Eqs (22), (25), where

$$v_-\left(x',t+\frac{x'}{c}-\frac{x}{c}\right)= = f\left(x',t+\frac{x'}{c}-\frac{x}{c}\right)+\frac{k}{2}\,\Phi^0_-\left(\xi_0,\eta'\right). \qquad (38)$$

Due to Eq. (36), etc. the following relations hold true:

$$\Phi^0_-\left(\xi_0,\eta'\right)=-\frac{1}{2\pi}\int\limits_{\xi_a(\eta')}^{\xi_0}\frac{d\xi''}{\sqrt{\xi_0-\xi''}}\times \times\int\limits_{-\xi''}^{\eta'}\frac{f_1\left(\xi'',\eta''\right)}{\sqrt{\eta'-\eta''}}\,d\eta'',\quad \eta'>\eta_e,$$
$$\Phi^0_-\left(\xi_0,\eta'\right)=-\frac{1}{2\pi}\int\limits_{-\eta'}^{\xi_0}\frac{d\xi''}{\sqrt{\xi_0-\xi''}}\times \times\int\limits_{-\xi''}^{\eta'}\frac{f_1\left(\xi'',\eta''\right)}{\sqrt{\eta'-\eta''}}\,d\eta'',\quad \eta'<\eta_e. \qquad (39)$$

It should be mentioned that for $x\approx l_2(t)$ value $\xi_a(\eta')$, in contrary to $\xi_a(\eta_0)$, is not close to ξ_0 and inhomogeneity significantly changes the stress v_+ near the crack's edge.

3. CONDITION OF CRACK'S PROPAGATION AND SOLUTION FOR A CONSTANT TRACTION

For $x\approx l_2(t)$ $(x>l_2(t))$, originating from (25) we can write the following:

$$\tau_{yz}=v_+=\frac{K}{\pi}\,\frac{1}{\sqrt{x-l_2(t)}},$$

$$K=-\sqrt{1-\frac{\dot{l}_2(t)}{c}}\times \times\int\limits_{x-ct}^{l_2(t)}\frac{f\left(x',t+\frac{x'}{c}-\frac{x}{c}\right)+\frac{k}{2}\,\Phi^0_-\left(\xi_0,\xi_0+2x'\right)}{\sqrt{l_2\left(t'\right)-x'}}dx', \qquad (40)$$

where $\Phi^0_-(\xi_0,\xi_0+2x')$ is given by (39). From (37) we obtain that

$$u_-=P_-=\frac{2K}{\pi}\sqrt{\frac{l_2(t)-x}{1-\dot{l}_2^2(t)/c^2}},\qquad x<l_2(t). \qquad (41)$$

The Irvin's condition [2] gives

$$\frac{K^2}{\pi}\,\frac{\rho a}{\sqrt{1-\dot{l}_2^2(t)/c^2}}=2\gamma', \qquad (42)$$

where γ' is the surface energy of opening of the crack.

In the problem for a constant traction on the crack $f=$const the Eq. (25) leads to the following expression:

$$v_+=-\frac{1}{\pi}\,\frac{1}{\sqrt{x-l_2\left(t_2\right)}}\int\limits_{x-ct}^{l_2(t_2)}\frac{\sqrt{l_2\left(t_2\right)-x'}}{x-x'}dx'- -\frac{k}{2\pi}\,\frac{1}{\sqrt{x-l_2\left(t_2\right)}}\int\limits_{x-ct}^{l_2(t)}\Phi^0_-\left(\xi_0,\eta'\right)\frac{\sqrt{l_2\left(t_2\right)-x'}}{x-x'}dx'. \qquad (43)$$

Due to Eq. (39) $\Phi^0_-(\xi_0,\eta')$ can be calculated in form

$$\Phi^0_-\left(\xi_0,\eta'\right)=\frac{1}{\pi}\,f\left\{\sqrt{\xi_0-\xi_a(\eta')}\sqrt{\eta'+\xi_a\left(\eta'\right)}+ +\left(\xi_0+\eta'\right)\mathrm{arctg}\sqrt{\frac{\xi_0-\xi_a\left(\eta'\right)}{\eta'+\xi_a\left(\eta'\right)}}\right\}\quad \text{for}\quad \eta'>\eta_e \qquad (44)$$

and

$$\Phi^0_-\left(\xi_0,\eta'\right)=-\frac{f}{2}\left(\xi_0+\eta'\right)\quad \text{for}\quad \eta'<\eta_e. \qquad (45)$$

Here,

$$\eta'=ct'+x',\quad \xi_0=ct'-x'=ct-x,$$
$$\xi_a\left(\eta'\right)=ct'_3-l_2\left(t'_3\right),\quad ct'_3+l_2\left(t'_3\right)=ct'+x'. \qquad (46)$$

Using Eq. (43) and passing to physical variables we obtain for the stress distribution out from crack

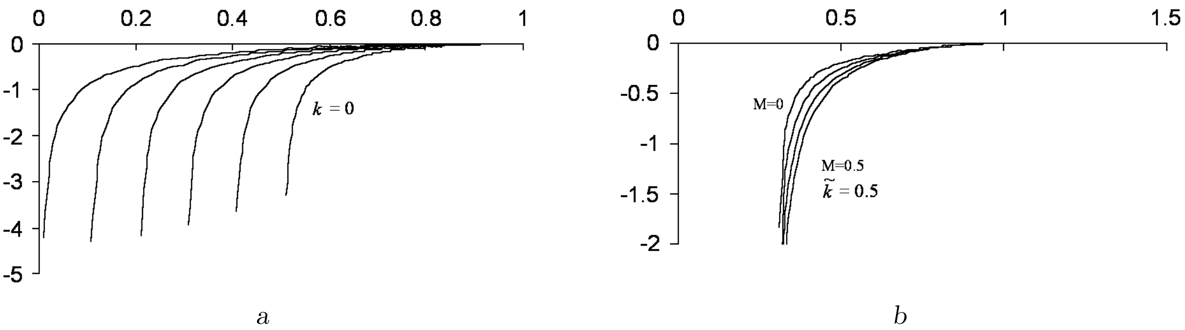

Fig. 2. Calculations after Eq. (48), giving the stresses out of the crack for constant velocity of the crack (a) $\dot{l}(t)=v$ and constant traction f on the crack (b); $x_0=x/ct$, $M=v_0/c$: $M<1$, $\tilde{k}=kct$, $x'=ct\xi$

that

$$v_+ = -\frac{2}{\pi}\frac{f}{\sqrt{x-l_2(t_2)}}\left\{\sqrt{l_2(t_2)-x+ct}-\right.$$
$$\left.-\sqrt{x-l_2(t_2)}\,\mathrm{arctg}\sqrt{\frac{l_2(t_2)-x+ct}{x-l_2(t_2)}}\right\}+$$
$$+\frac{k}{\pi^2}\frac{f}{\sqrt{x-l_2(t_2)}}\int\limits_{\alpha}^{l_2(t_2)}\left\{\sqrt{l_2(t_3')-x'}\times\right.$$
$$\times\sqrt{ct-x+2x'-l_2(t_3')}+(ct-x+x')\times \qquad (47)$$
$$\left.\times\,\mathrm{arctg}\sqrt{\frac{l_2(t_3')-x'}{ct-x+2x'-l_2(t_3')}}\right\}\times$$
$$\times\frac{\sqrt{l_2(t_2)-x'}}{x-x'}dx'+\frac{k}{2\pi}\frac{f}{\sqrt{x-l_2(t_2)}}\times$$
$$\times\int\limits_{x-ct}^{\alpha}\left(ct-x+x'\right)\frac{\sqrt{l_2(t_2)-x'}}{x-x'}\,dx',$$

where $\alpha=(\eta_e+x-ct)/2$.

As it is seen from Eq. (47), the stress out of crack has the same singularity near the crack's edge $x\approx l_2(t)$, as in homogeneous case, but with the stress intensity coefficient essentially depending from inhomogeneity.

4. CASE OF ARBITRARY VALUE OF INHOMOGENEITY

In the case of non-small k Eq. (5) can be solved in the form of the Laplace and the Fourier transformations:

$$U_L=\int\limits_0^{\infty}e^{-st'}u\,dt';$$
$$U_l(x,y_1,s)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}U_{LF}e^{-i(\alpha_1x+\beta y_1)}d\alpha_1. \qquad (48)$$

Here U_{LF} is the Fourier transform for $U_L(x,0,s)$; $(1/\rho)\partial\rho/\partial y_1=k$.

Substituting Eq. (48) into Eq. (5) we obtain

$$\beta=-i\,\frac{k}{2}-i\sqrt{\frac{k^2}{4}+\alpha_1^2+\frac{s^2}{c^2}}. \qquad (49)$$

Introducing the function $\psi=\partial u/\partial y_1$ we follow to the relation between the integral transformants of ψ and u:

$$\psi_{LF}=-i\beta U_{LF},$$
$$U_{LF}=S_{LF}\psi_{LF}, \qquad (50)$$
$$S_{LF}=-\left(\frac{k}{2}+\sqrt{\frac{k^2}{4}+\alpha_1^2+\frac{s^2}{c^2}}\right)^{-1}.$$

The originals are written as

$$S(x,t)=\frac{1}{4\pi^2 i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}e^{st}ds\int\limits_{-\infty}^{\infty}e^{-i\alpha_1x}S_{LF}d\alpha_1. \qquad (51)$$

On the boundary $y_1=0$ the following relation between $u(x,t)$ and derivative $(\partial u/\partial y_1)|_{y_1=0}=\psi(x,t)$ is valid:

$$u(x,t)=\iint\psi(x',t')S(x-x',t-t')dx'dt', \qquad (52)$$

and the integration procedure should be carried out over the complete domain $s_0+s_1+s_2$ (see fig. 1). Now for $x<l_2(t)$ it can be written that

$$u(x,t)= \\ =\frac{1}{2c}\int\limits_{\xi_e}^{\xi_a(\eta_0)}d\xi'\int\limits_{-\xi'}^{\eta_2(\xi')}f_1(\xi',\eta')S_1(\xi_0-\xi',\eta_0-\eta')d\eta'+ \\ +\frac{1}{2c}\int\limits_{\xi_e}^{\xi_a(\eta_0)}d\xi'\int\limits_{\eta_2(\xi')}^{\eta_0}\psi_1(\xi',\eta')S_1(\xi_0-\xi',\eta_0-\eta')d\eta'+ \\ +\frac{1}{2c}\int\limits_{\xi_a(\eta_0)}^{\xi_0}d\xi'\int\limits_{-\xi'}^{\eta_0}f_1(\xi',\eta')S_1(\xi_0-\xi',\eta_0-\eta')d\eta', \tag{53}$$

where $f_1(\xi',\eta')=f(x',t')$ is given in s_0, s_2 behind the crack's edge; $\psi_1(\xi',\eta')=\psi(x',t')$ is a value of $(\partial u/\partial y_1)|_{y_1=0}$ for $x'>l_2(t')$, i.e., out of the crack, which in the order of k is given by Eq. (25) and $S_1(\xi',\eta')=S(x',t')$.

So, the solution for u in the domain $x<l_2(t)$ is reduced to determination of the function $S(x,t)$, given by Eqs (50) and (51).

From the boundary condition $u=0$ for $x>l_2(t)$ and Eq. (52), as in Eq. (8), for $|\xi_0|>|\xi_a(\eta_0)|$ we have the following:

$$\int\limits_{\xi_e}^{\xi_0}d\xi'\int\limits_{-\xi'}^{\eta_2(\xi')}f_1(\xi',\eta')S_1(\xi_0-\xi',\eta_0-\eta')d\eta'+ \\ +\int\limits_{\xi_e}^{\xi_0}d\xi'\int\limits_{\eta_2(\xi')}^{\eta_0}\psi_1(\xi',\eta')S_1(\xi_0-\xi',\eta_0-\eta')d\eta'=0. \tag{54}$$

However, in contrary to Eq. (27), in which

$$S_1(\xi_0-\xi',\eta_0-\eta')=-\frac{c}{\pi\sqrt{(\xi_0-\xi')(\eta_0-\eta')}},$$

as it will be shown later, for arbitrary $k\neq 0$ the multiplies $(\xi_0-\xi')^{-1/2}$ and $(\eta_0-\eta')^{-1/2}$ are not separated. Therefore, the integral over the domain s_2, which contains complex solution ψ_1 given by Eq. (25), can not be excluded. This fact distinguishes the problem with $k\neq 0$, from the problem for homogeneous medium [2,5] (its solution is given by Eqs (32), (35)).

It should be noted that Eq. (53) is valid behind the crack's edge, i.e., for $|\xi_0|<|\xi_a(\eta_0)|$, while Eq. (54) – ahead the crack's edge, i.e., for $|\xi_0|>|\xi_a(\eta_0)|$. That is why this last expression can not be used for simplification of Eq. (53). However, later it will be shown that for the points close to the edge ($x\approx l_2(t)$) Eq. (54) still can be used for simplification of Eq. (53).

To calculate $S(x,t)$ from Eqs (50) and (51) the complex $s=pc\sqrt{k^2/4+\alpha_1}$ should be introduced instead of s. Moreover, integration with respect to s should be replaced by the Laplace integral with respect to p, multiplied by $c\sqrt{k^2/4+\alpha_1}$.

Note that the inverse Laplace transform from

$$g(p)=-\left(\frac{k}{2}+\sqrt{\frac{k^2}{4}+\alpha_1^2 p}\right)^{-1}$$

is

$$f(t')=-c\exp\left(-\frac{k\,c\,t'}{2\sqrt{\frac{k^2}{4}+\alpha_1^2}}\right).$$

Additionally, in view of [8, Eq. (38) on page 123]

$$g\left(\sqrt{p^2+1}\right)=f(t')-\int\limits_0^{t'}J_1(u)f\left(t'^2-u^2\right)du'; \\ t'=c\sqrt{\frac{k^2}{4}+\alpha_1^2}\;t$$

the inverse Laplace transform for S_{LF} gives the following:

$$S_F=-ce^{-\frac{kc}{2}t}+ \\ +c\int\limits_0^{c\sqrt{\frac{k^2}{4}+\alpha_1^2}\,t}J_1(u)\exp\left(-\frac{\frac{k}{2}\sqrt{t'^2-u^2}}{\sqrt{\frac{k^2}{4}+\alpha_1^2}}\right)du. \tag{55}$$

After substitution $u=v\sqrt{k^2/4+\alpha_1^2}$ we have

$$S_F=-ce^{-\frac{kc}{2}t}+c\sqrt{\frac{k^2}{4}+\alpha_1^2}\times \\ \times\int\limits_0^{ct}J_1\left(v\sqrt{\frac{k^2}{4}+\alpha_1}\right)\exp\left(-\frac{k}{2}\sqrt{c^2t^2-v^2}\right)dv. \tag{56}$$

Accounting the well known property of the Bessel's functions $J_1(x)=-J_0'(x)$, and integrating by parts, one can obtain

$$S_F=-cJ_0\left(ct\sqrt{\frac{k^2}{4}+\alpha_1^2}\right)+ \\ +\frac{kc}{2}\int\limits_0^{ct}J_0\left(v\sqrt{\frac{k^2}{4}+\alpha_1^2}\right)\times \\ \times\frac{v}{\sqrt{c^2t^2-v^2}}\exp\left(-\frac{k}{2}\sqrt{c^2t^2-v^2}\right)dv. \tag{57}$$

Again, replacing $\sqrt{c^2t^2-v^2}=u$ gives the following relation:

$$S_F = -cJ_0\left(ct\sqrt{\frac{k^2}{4}+\alpha_1^2}\right) + \\ + \frac{kc}{2}\int\limits_0^{ct} J_0\left(\sqrt{c^2t^2-v^2}\sqrt{\frac{k^2}{4}+\alpha_1^2}\right)e^{-\frac{k}{2}u}du. \tag{58}$$

The function S_F is even on α_1. This allows to replace the exponential Fourier transform on the cosine transform. So, in view of [8, Eq. (35) on page 57], one can obtain the final formula for the function in physical domain:

$$S(x,t) = -\frac{cH(ct-x)}{\pi}\times \\ \times\left\{\frac{\cos\left(k/2\sqrt{c^2t^2-x^2}\right)}{\sqrt{c^2t^2-x^2}}- \right. \\ \left. -\frac{k}{2}\int\limits_0^{\sqrt{c^2t^2-x^2}}\frac{\cos\left(k/2\sqrt{c^2t^2-u^2}\right)}{\sqrt{c^2t^2-x^2-u^2}}e^{-\frac{k}{2}u}du\right\}. \tag{59}$$

Moreover, the function $S_1(\xi_0-\xi',\eta_0-\eta')$ in Eq. (53) is given by the expression:

$$S_1(\xi_0-\xi',\eta_0-\eta') = -\frac{c}{\pi}\left(\frac{1}{T}\cos\frac{kT}{2}- \right. \\ \left. -\frac{k}{2}\int\limits_0^T\frac{\cos\left(k/2\sqrt{T^2-u^2}\right)}{\sqrt{T^2-u^2}}e^{-\frac{k}{2}u}du\right), \tag{60}$$

where

$$T=\sqrt{(\xi_0-\xi')(\eta_0-\eta')}. \tag{61}$$

For small k the the first order approximation from Eqs (60) and (61) gives that

$$S_1(\xi_0-\xi',\eta_0-\eta') = -\frac{c}{\pi}\left(\frac{1}{T}-\frac{k\pi}{4}\right). \tag{62}$$

Let us put in relation (54) that

$$\psi_1 = v_+^0 + v_+^1,$$

where v_+^0 is the solution for homogeneous medium ($k=0$) given by (22), in which v_- is replaced for $f_1(\xi_0,\eta')$. For this addendum Eq. (27) is fulfilled. Retaining in Eq. (54) terms of the k-th order we obtain the following:

$$-\frac{k\pi}{4}\int\limits_{\xi_e}^{\xi_0}d\xi'\int\limits_{-\xi'}^{\eta_2(\xi')}f_1 d\eta'- \\ -\frac{k\pi}{4}\int\limits_{\xi_e}^{\xi_0}d\xi'\int\limits_{\eta_2(\xi')}^{\eta_0}v_+^0(\xi',\eta')\,d\eta'+ \\ +\int\limits_{\xi_e}^{\xi_0}d\xi'\int\limits_{\eta_2(\xi')}^{\eta_0}\frac{v_+^1(\xi',\eta')}{T}\,d\eta'=0. \tag{63}$$

So, from Eqs (53) and (62) the solution precise to the first order of k can be written:

$$u=-\frac{1}{2\pi}\int\limits_{\xi_e}^{\xi_a(\eta_0)}d\xi'\int\limits_{-\xi'}^{\eta_2(\xi')}\frac{f_1(\xi',\eta')}{T}d\eta'+ \\ +\frac{k}{8}\int\limits_{\xi_e}^{\xi_a(\eta_0)}d\xi'\int\limits_{-\xi'}^{\eta_2(\xi')}f_1(\xi',\eta')d\eta'- \\ -\frac{1}{2\pi}\int\limits_{\xi_e}^{\xi_a(\eta_0)}d\xi'\int\limits_{\eta_2(\xi')}^{\eta_0}(v_+^0+v_+^1)\left(\frac{1}{T}-\frac{k\pi}{4}\right)d\eta'+ \\ +\frac{1}{2c}\int\limits_{\xi_a(\eta_0)}^{\xi_0}d\xi'\int\limits_{-\xi'}^{\eta_0}f_1S_1 d\eta'. \tag{64}$$

It worth noting that the first addendum in right-hand side of Eq. (64) and the addendum with v_+^0, both having the zeroth order of k, are cancelled (see (27)). From Eq. (64), accounting for Eq. (63), we obtain for the terms of order of k:

$$u\approx\frac{k}{8}\int\limits_{\xi_0}^{\xi_a(\eta_0)}d\xi'\int\limits_{-\xi'}^{\eta_2(\xi')}f_1 d\eta'+\frac{k}{8}\int\limits_{\xi_0}^{\xi_a(\eta_0)}d\xi'\int\limits_{\eta_2(\xi')}^{\eta_0}v_+^0 d\eta'- \\ -\frac{1}{2\pi}\int\limits_{\xi_0}^{\xi_a(\eta_0)}d\xi'\int\limits_{\eta_2(\xi')}^{\eta_0}\frac{v_+^1}{T}d\eta'+\frac{1}{2c}\int\limits_{\xi_a(\eta_0)}^{\xi_0}d\xi'\int\limits_{-\xi'}^{\eta_0}f_1S_1 d\eta'. \tag{65}$$

Near the crack's edge $x\approx l_2(t)$, $\xi_0\approx\xi_a(\eta_0)$ only the terms of order $\sqrt{\xi_0-\xi_a(\eta_0)}$ should be retained. Therefore, the first two terms in the right-hand side of Eq. (65) can be dropped out. This gives

$$u\approx\frac{1}{2\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0}d\xi'\int\limits_{\eta_2(\xi')}^{\eta_0}\frac{v_+^1(\xi',\eta')}{T}d\eta'+ \\ +\frac{1}{2c}\int\limits_{\xi_a(\eta_0)}^{\xi_0}d\xi'\int\limits_{\xi'}^{\eta_0}f_1S_1 d\eta'. \tag{66}$$

Due to Eqs (22) and (26) we find, keeping the terms of order of k, that

$$v_+^1(\xi',\eta') = -\frac{k}{2\pi\sqrt{\eta'-\eta_2(\xi')}} \times$$
$$\times \int\limits_{-\xi'}^{\eta_2(\xi')} \Phi_-^0(\xi',\eta'')\frac{\sqrt{\eta_2(\xi')-\eta''}}{\eta'-\eta''}\,d\eta''.$$

Substituting this relation into Eq. (66), interchanging the integration order on η' and η'' and, finally, accounting that $\xi'\approx\xi_0$, $\eta_2(\xi')\approx\eta_0$, we obtain that near the crack's edge

$$\begin{aligned} u=&-\frac{k}{4\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0}\frac{d\xi'}{\sqrt{\xi_0-\xi'}}\int\limits_{-\xi_0}^{\eta_0}\Phi_-^0(\xi_0,\eta')\frac{d\eta'}{\sqrt{\eta_0-\eta'}}-\\ &-\frac{1}{2\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0}\frac{d\xi'}{\sqrt{\xi_0-\xi'}}\int\limits_{-\xi_0}^{\eta_0}\frac{f_1(\xi_0,\eta')}{\sqrt{\eta_0-\eta'}}\,d\eta'. \end{aligned} \qquad (67)$$

In the last term of Eq. (67) it is used that $\xi'\approx\xi_0$. Moreover, in accordance with Eq. (61) the second addendum in Eq. (66) is the same that for homogeneous medium.

The same expression is obtained from the solution of the first order of k given by Eq. (30). So values of $u=\Phi_-(\xi_0,\eta_0)$ obtained by two methods coincide near the crack's edge.

For arbitrary k the relations (53) and (54) are valid. They keep true for different ξ_0, but for (ξ_0,η_0), taken near the crack's edge, one can believe that both (53) and (54) relations hold. This gives the following:

$$\begin{aligned} u(x,t) = &-\frac{1}{2\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0}d\xi'\int\limits_{-\xi'}^{\eta_2(\xi')}f_1S_1d\eta'-\\ -\frac{1}{2c}\int\limits_{\xi_a(\eta_0)}^{\xi_0}d\xi'\int\limits_{\eta_2(\xi')}^{\eta_0}\psi_1S_1d\eta' &+ \frac{1}{2\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0}d\xi'\int\limits_{-\xi'}^{\eta_0}f_1S_1d\eta'. \end{aligned}$$

Let us denote the value of v_+ for $k=0$ as v_+^0. Then, accounting for equalities $\psi_1=v_+$, $v_+=v_+^0+v_+^1$ and using the approximation $S_1(\xi_0-\xi',\eta_0-\eta')$ at $\xi_0\approx\xi_a(\eta_0)$ (see Eq. (62) under $T\approx 0$), we can drop the small quantities of order of $(\xi_0-\xi_a(\eta_0))$. This procedure leads to Eq. (66). Therefore, the solution for $u(x,t)$ near the crack's edge formally coincide with the solution for the first order of k. Although, it should be mentioned that $v_+^1(\xi',\eta')$ may differ from the similar value for the case of the small inhomogeneity.

Besides calculation of the displacement on the crack, the function $\psi(x,t)=(\partial u/\partial y_1)|_{y_1=0}=\psi_+$ out of the crack should be found. Following the Eq. [1,4] and using the expressions using denotions (12) one can write the solution for u, ψ in the form

$$\begin{aligned} u_-(x,t) = S_- **\big[(S_+ **\psi_- - P'_- ** u_+)\times\\ \times H(l_2-x)\big]; \end{aligned} \qquad (68)$$

$$\begin{aligned} \psi_+(x,t) = -P_+ **\big[(S_+ **\psi_- - P'_- ** u_+)\times\\ \times H(x-l_2)\big], \end{aligned} \qquad (69)$$

where $l_2=l_2(t)$; $H(x)$ is a step function; asterisks denote a convolution on x and t; $S_\pm(x,t)$ and $P'_\pm(x,t)$ are respectively the originals of the functions $S_{LF\pm}$ and $P'_{LF\pm}$, which represent factorization of the functions S_{LF} and $P'_{LF}=1/S_{LF}$.

Finally, according to [7,9] we obtain the following expressions:

$$\begin{aligned} S_{LF} &= S_{LF\pm}S_{LF-};\\ S_{LF\pm} &= \pm\frac{1}{\sqrt{\alpha_1\pm\sqrt{\gamma}}}\times\\ \times\exp&\left\{\frac{1}{2\pi i}\int\limits_{\mp\sqrt{\gamma}}^{\mp\infty}\ln\frac{2\sqrt{\gamma+\alpha'^2_1}+k}{2\sqrt{\gamma+\alpha'^2_1}-k}\,\frac{d\alpha'_1}{\alpha'_1-\alpha_1}\right\}, \end{aligned} \qquad (70)$$

where $\gamma=k^2/4+s^2/c^2$.

The next step is to find the originals $S_\pm(x,t)$ according to Eq. (51). By the same way the originals $P'_\pm(x,t)$ can be determined.

The obtained solution for $\psi_+(x,t)$ see Eqs (51), (69), (70)) is very complex. Therefore, it is more suitable to use the small parameter solution (25) for the first approximation.

As it is seen from the solutions (25), (30) and (30), influence of the inhomogeneity on the solution as near the crack's edge, as along the whole x-axis is essential.

5. SOLUTION OF THE PROBLEM FOR DISPLACEMENTS GIVEN ON THE CRACK'S FACES

It should be noted that the problem considered in §1 is the antisymmetric one, for which same tractions τ_{yz}^- are given on the crack's faces and the displacements have opposite signs. Corresponding symmetric problem implies the stresses having opposite signs and the same displacements imposed on the crack's faces. General case of different stresses and displacements on the crack's faces can be represented as a sum of these two solutions. So, instead of Eq. (9) we can put the following:

$$y_1=0;\quad u_-=\Phi_1(t,x),\quad x<l_2(t),\qquad \psi_+=0,\qquad x>l_2(t). \tag{71}$$

Because of

$$\psi_+=\left.\frac{\partial u}{\partial y_1}\right|_{y_1}=v_+-\frac{k}{2}u_+$$

one can rewrite Eq. (71) in the form

$$y_1=0;\quad u=P_-=\Phi_1(x,t),\quad v_+=\frac{k}{2}u_+. \tag{72}$$

In contrast to Eqs (68), (69), the convolution method [1] can be applied to the functions P, v. This gives

$$P_+=\tilde{S}_+**\left[\left(\tilde{S}_-**v_+-\tilde{P}_+**P_-\right)\times\times H(x-l_2)\right], \tag{73}$$

$$v_-=-\tilde{P}_-**\left[\left(\tilde{S}_-**v_+-\tilde{P}_+**P_-\right)\times\times H(l_2-x)\right]. \tag{74}$$

Taking that $v=\partial P/\partial y_1$ one can obtain from Eq. (10) that

$$P_{LF}=\tilde{S}_{LF}v_{LF};\ \tilde{S}_{LF}=-\left(\sqrt{\alpha_1^2+\frac{s^2}{c^2}}\right)^{-1}, \tag{75}$$

instead of Eq. (50).

Besides, Eq. (75) there is another condition: $\tilde{P}_{LF}=1/\tilde{S}_{LF}$.

Note that for homogeneous medium the original functions $\tilde{S}(x,t)$ and $\tilde{P}(x,t)$, corresponding to $\tilde{S}_{LF}$ and $\tilde{P}_{LF}$, are found from Eq. (4), and their factorization yields

$$\tilde{S}_+(x,t)=-\frac{H(x)\delta(t-x/c)}{\sqrt{\pi}\sqrt{x}};\qquad \tilde{P}_+(x,t)=\frac{\delta(t-x/c)}{2\sqrt{\pi}}\frac{H(x)}{x^{3/2}}. \tag{76}$$

Since $\tilde{S}_{LF+}\tilde{S}_{LF-}=\tilde{S}_{LF}$ corresponds to the original, given by Eq. (59) for $k=0$, the first addendum in (73) is as follows:

$$\tilde{S}_+**\left(\tilde{S}_-**v_+\right)=-\frac{c}{\pi}\iint\frac{v_+dx'dt'}{\sqrt{T}}= =-\frac{ck}{2\pi}\iint\frac{u_+dx'dt'}{\sqrt{T}}. \tag{77}$$

Substituting Eq. (76) into Eq. (73) and carrying out lengthy calculations one can obtain the displacements out of the crack, corresponding to the first order of k:

$$P_+(t,x)=-\frac{ck}{2\pi}\iint\limits_{S_0}\frac{P_+^1(t',x')}{\sqrt{T}}dx'dt'+P_+^1(t,x), \tag{78}$$

where

$$P_+^1(t,x)=\frac{\sqrt{x-l_2(t_0)}}{\pi\sqrt{c}}\int\limits_0^{t_0}\frac{\Phi_1[\tau,x-c(t-\tau)]}{(t-\tau)\sqrt{t_0-\tau}}\,d\tau \tag{79}$$

and $l_2(t_0)-x+ct=ct_0$.

Formula (79) gives known behavior of the displacement near the crack's edge $\sim\sqrt{x-l_2(t)}$ for hte boundary function $\Phi_1(t,x)$, being equal to zero for $x=l_2(t)$. Moreover, the integral in Eq. (79) is convergent for $x=l_2(t)$, $t_0\approx t$. The stresses on the crack can be found from Eq. (74). However, more simple way is to obtain $V_-(t,x)=\partial P/\partial y_1-(k/2)P$ accounting that V satisfies Eq. (10) and the boundary conditions, due to Eq. (72), are of the order of k:

$$y_1=0,\quad \frac{\partial V}{\partial y_1}=f(t,x)-\frac{k}{2}V,\quad x<l_2(t),\qquad V=0,\qquad x>l_2(t), \tag{80}$$

where

$$f(t,x)=\frac{1}{c^2}\frac{\partial^2\Phi_1}{\partial t^2}-\frac{\partial^2\Phi_1}{\partial x^2}.$$

Then from Eq. (20), written for $V(t,x)$, one can obtain that

$$V(t,x)=-\frac{c}{\pi}\iint\frac{f(t',x')-(k/2)V(t',x')}{\sqrt{T}}dx'dt'. \tag{81}$$

Putting $k=0$ in Eq. (81) leads to homogeneous solution $V^0(t,x)$, which can be substituted in Eq. (81) instead of $V(t',x')$. Because of $V=0$ for $x>l(t_2)$ the integrand in Eq. (81) is the same as that in Eq. (30). Hence, after introducing of characteristic variables we find that

$$V_1(\xi_0,\eta_0)=-\frac{1}{2\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0}\frac{d\xi'}{\sqrt{\xi_0-\xi'}}\times \times\int\limits_{-\xi'}^{\eta_0}\frac{f_1(\xi',\eta')-(k/2)V_-^0(\xi',\eta')}{\sqrt{\eta_0-\eta'}}\,d\eta', \tag{82}$$

where $f_1(\xi',\eta')=f(t',x')$. Following Eq. (39) for $\eta'>\eta_e$ we obtain

$$V_-^0(\xi',\eta')=-\frac{1}{2\pi}\int\limits_{\xi_a(\eta')}^{\xi'}\frac{d\xi''}{\sqrt{\xi'-\xi''}}\int\limits_{-\xi''}^{\eta'}\frac{f_1(\xi'',\eta'')}{\sqrt{\eta'-\eta''}}d\eta''. \tag{83}$$

As Eq. (39) shows, for $\eta' < \eta_e$ the integration with respect to ξ'' should be performed within the limits $(-\eta', \xi')$. Using the formula

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\frac{1}{\sqrt{T}} = T^{-3/2}$$

one can obtain

$$V_-(\xi_0, \eta_0) =$$

$$= -\frac{1}{2\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0} \frac{d\xi'}{(\xi_0-\xi')^{3/2}} \int\limits_{-\xi'}^{\eta_0} \frac{\Phi_1(\xi', \eta')}{(\eta_0-\eta')^{3/2}}\,d\eta' +$$

$$+\frac{k}{4\pi}\int\limits_{\xi_a(\eta_0)}^{\xi_0} \frac{d\xi'}{\sqrt{\xi_0-\xi'}} \int\limits_{-\xi'}^{\eta_0} \frac{V_-^0(\xi', \eta')}{\sqrt{\eta_0-\eta'}}\,d\eta',$$

where only finite part of the integral with respect to ξ' is retained.

For the function $\Phi_1(t', x')$, which behaves at the edge of the crack as

$$\Phi_1(t', x') = B(t', x')\left[l_2(t') - x'\right),$$

the integral with respect to η' is finite for $x' = l_2(t')$, so, usual singularity for stresses is observed:

$$V_-(t', x') \sim \frac{1}{\sqrt{l_2(t) - x}}.$$

CONCLUSION

The problem on the antiplane crack propagating with arbitrary velocity in anisotropic inhomogeneous elastic media is considered. The solution is obtained by method of integral equations developed in the theory of wing. For wide class of inhomogeneities the solution is obtained in closed analytical form. The displacements on the crack's faces and the stresses out of the crack are obtained. It is shown that the inhomogeneity essentially effects the solution.

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