INHOMOGENEOUS RELATIVISTIC PLASMA DIELECTRIC TENSOR

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The paper is concerned with the method to derive the inhomogeneous relativistic plasma dielectric tensor on the base principle of analytic continuation for Cauchy-type integrals and the exact plasma dispersion functions concept.

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1. INTRODUCTION

Deriving of the inhomogeneous fully relativistic plasma dielectric tensor, taking into account the spatial dispersion of plasma, is of current importance for the theoretical study of electromagnetic waves in thermonuclear plasma in the electron cyclotron (EC) frequency range. An importance of the exact taking into account relativistic effects follows from the fact that those effects can even arise in laboratory plasmas with moderate temperatures and especially in quasi-perpendicular, in respect to magnetic field, propagation regime. An importance of accounting the spatial dispersion is defined by more and more deep development of idea to use the electron Bernstein waves for plasma heating and current drive.

The fully relativistic plasma dielectric tensor for homogeneous plasma [1] in the form, suitable for analytical and numerical applications, was derived on the base of principle of analytic continuation and concept of the exact plasma dispersion functions (PDFs), generating the weakly relativistic PDFs to the case of arbitrary plasma temperatures [2, 3].

The main purpose of the present work is an extension of this method to the case of plasma, inhomogeneous in transverse to magnetic field direction.

2. INHOMOGENEOUS RELATIVISTIC PLASMA DIELECTRIC TENSOR

In the relativistic limit the Vlasov kinetic equation, being the start point for deriving of plasma dielectric tensor, has the form [1]

$$\gamma\frac{\partial f}{\partial t}+\frac{\boldsymbol{p}}{m}\frac{\partial f}{\partial \boldsymbol{r}}-e\left(\gamma\boldsymbol{E}+\left[\frac{\boldsymbol{p}}{mc},\boldsymbol{B_0}+\boldsymbol{B}\right]\right)\frac{\partial f}{\partial \boldsymbol{p}}=0, \quad (1)$$

where $\gamma=\sqrt{1+(p/mc)^2}$, $\boldsymbol{p}=m\mathbf{V}/\sqrt{1-(V/c)^2}$ and m is momentum and the rest mass of the electron. Linearization of the equation (1) leads to equation for small perturbations of electron distribution function $\tilde{f}(\boldsymbol{r},\boldsymbol{p},t)$ with respect to the relativistic Maxwellian distribution $f_M(p)$

$$\gamma\frac{\partial \tilde{f}}{\partial t}+\frac{\boldsymbol{p}}{m}\frac{\partial \tilde{f}}{\partial \boldsymbol{r}}+\omega_B\frac{\partial \tilde{f}}{\partial \varphi}-e\gamma\left(\boldsymbol{E}\cdot\frac{\partial f_M}{\partial \boldsymbol{p}}\right)=0, \quad (2)$$

where $\omega_B=eB_0/mc$, $f_M=A\exp(-\varepsilon/T)=A\exp(-\mu\gamma)$, $\varepsilon=\sqrt{c^2p^2+m\,c^4}$ is the energy of the free electron, $A^{-1}=4\pi(mc)^3K_2(\mu)/\mu$, $K_2(\mu)$ is the MacDonald function, $\mu=(c/V_T)^2$ is the main relativistic parameter, $V_T=\sqrt{T/m}$ is thermal speed of electrons, φ is the azimuth angle in the momentum space with the polar axis along the base magnetic field $\boldsymbol{B_0}$. It's worth note that the equation (2) is true for the case of sharp changing of magnetic field in comparison with changing of electron density and temperature, what is typifying for plasma in the depth of plasma column.

Let us search a decision of the equation (2) for small perturbations of electron distribution function and electric field in the form $\tilde{f}(\boldsymbol{r},\boldsymbol{p},t)=\tilde{f}(x,k_{//},\omega)\,exp[i(k_{//}z-\omega t)]$, $\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}(x,k_{//},\omega)\,exp[i(k_{//}z-\omega t)]$ by means of the perturbation method in the finite electron Larmor radius. In accordance with this technique a decision is looking for in the form of series

$$\tilde{f}(x,k_{//},\omega)=f^{(0)}+f^{(1)}+f^{(2)}+\ldots, \quad (3)$$

where the perturbation $f^{(n)}$ is defined from equations

$$i\left(\gamma\omega-\frac{p_{//}k_{//}}{m}\right)f^{(0)}-\omega_B\frac{\partial f^{(0)}}{\partial\varphi}=-\frac{e\mu}{(mc)^2}f_M(E\cdot p),$$
$$i\left(\gamma\omega-\frac{p_{//}k_{//}}{m}\right)f^{(n)}-\omega_B\frac{\partial f^{(n)}}{\partial\varphi}=\frac{p_x}{m}f_M\frac{\partial f^{(n-1)}}{\partial x},\ (n\geq 1). \quad (4)$$

Decisions for first three equations (4), using for shortness symbols $f^{(n)}=f_+^{(n)}+f_-^{(n)}$ and $E_\pm=E_x\pm iE_y$, can be written

$$f_\pm^{(0)}=\frac{iep_\perp}{2(mc)^2}(\mu f_M D_{\mp1}E_\pm)e^{\mp i\varphi},\ D_{\mp1}=\left(\gamma\omega-\frac{p_{//}k_{//}}{m}\pm\omega_B\right)^{-1},$$
$$f_\pm^{(1)}=\frac{ep_\perp^2}{4m(mc)^2}(\mu f_M D_{\mp1}E_\pm)'\left(D_{\mp2}e^{\mp i2\varphi}+D_0\right),$$
$$f_\pm^{(2)}=-\frac{iep_\perp^3}{8m^2(mc)^2}[\ (\mu f_M D_{\mp1}E_\pm)'\,D'_{\mp2}\left(D_{\mp1}e^{\mp i\varphi}+D_{\mp3}e^{\mp i3\varphi}\right)+$$
$$(\mu f_M D_{\mp1}E_\pm)''\left(D_0D_{\pm1}e^{\pm i\varphi}+(D_{\mp2}+D_0)D_{\mp1}e^{\mp i\varphi}+D_{\mp2}D_{\mp3}e^{\mp i3\varphi}\right)\ ] \quad (5)$$

The plasma current additions $\boldsymbol{j}^{(k)}$ ($k=0,1,2$) can be found on the ground of expressions (5) in line with

$$\boldsymbol{j}^{(k)} = \frac{en(r)}{m}\int f^{(k)}(\boldsymbol{p})\frac{\boldsymbol{p}}{\gamma}d^3\boldsymbol{p}$$

For example, in zero approximation for the perturbation $f_-^{(0)}$ and for the $j_x^{(0)}$ component of plasma current one has

$$j_x^{(0)} = \frac{ie^2 n(r)\mu^2}{8m\pi(mc)^5 K_2(\mu)}\int_0^{2\pi} d\varphi \int_0^{+\infty} p_\perp^2 dp_\perp \times$$
$$\int_{-\infty}^{+\infty}\frac{\exp(-\mu\gamma)}{\gamma}p_x D_1 E_- e^{i\varphi}dp_{//}.$$

Here it is convenient to utilize the normalization $\bar{p} = p/mc$

$$j_x^{(0)} = \frac{ie^2 n(r)\mu^2}{8mK_2(\mu)\omega}\int_{-\infty}^{+\infty} d\bar{p}_{//}\int_0^{+\infty}\bar{p}_\perp^3 d\bar{p}_\perp \frac{\exp(-\mu\gamma)}{\gamma}\times$$
$$\frac{E_-}{(\gamma - N_{//}\bar{p}_{//} - \omega_B/\omega)}.$$

Now we make the change of variables of integration $\bar{p}_{//}, \bar{p}_\perp$ into $\bar{p}_{//}, \gamma$ with transform Jacobian $\gamma, \bar{p}_\perp$.

$$j_x^{(0)} = \frac{ie^2 n(r)\mu^2}{8mK_2(\mu)\omega}\int_{-\infty}^{+\infty} d\bar{p}_{//}\int_{\sqrt{1+\bar{p}_{//}^2}}^{+\infty} d\gamma \frac{\exp(-\mu\gamma)\left[\gamma^2 - \left(1+\bar{p}_{//}^2\right)\right]E_-}{(\gamma - N_{//}\bar{p}_{//} - \omega_B/\omega)}.$$

After the new changing of variables $\bar{p}_{//}, \gamma$ into $\bar{p}_{//}, x = \gamma - \sqrt{1+\bar{p}_{//}^2}$ one has

$$j_x^{(0)} = C\int_{-\infty}^{+\infty} d\bar{p}_{//}\int_0^{+\infty} dx \frac{\exp(-\mu(x+\sqrt{1+\bar{p}_{//}^2}))x(x+2\sqrt{1+\bar{p}_{//}^2})E_-}{(x+\sqrt{1+\bar{p}_{//}^2} - N_{//}\bar{p}_{//} - \omega_B/\omega)},$$
$$C = \frac{ie^2 n(r)\mu^2}{8mK_2(\mu)\omega}. \qquad (6)$$

Now on the base of the principle of analytic continuation, one can only leave the anti-Hermitian part of (6) and then one can evaluate the integral in x analytically.

$$j_x^{(0)} = C\int_{\bar{p}_{//}^-}^{\bar{p}_{//}^+} d\bar{p}_{//}\exp(\mu(-N_{//}\bar{p}_{//} - \omega_B/\omega))\times$$
$$\left[(N_{//}\bar{p}_{//} + \omega_B/\omega)^2 - \bar{p}_{//}^2 - 1\right](E_x - iE_y),$$

where the integration limits $\bar{p}_{//}^\pm$ follow from the condition of pole appearing in the expression (6). Hence from the anti-Hermitian part, principle of analytic continuation and definition of exact relativistic PDFs [2], finite expressions for components of plasma dielectric tenor $\varepsilon_{xx}^{(0)}$ and $\varepsilon_{xy}^{(0)}$ can be obtained:

$$\varepsilon_{xx}^{(0)} = 1 - \mu\frac{\omega_p^2}{\omega^2}\frac{1}{2}\left[Z_{5/2}(a,z_1,\mu) + Z_{5/2}(a,z_{-1},\mu)\right],$$
$$\varepsilon_{xy}^{(0)} = \mu\frac{\omega_p^2}{\omega^2}\frac{i}{2}\left[Z_{5/2}(a,z_1,\mu) - Z_{5/2}(a,z_{-1},\mu)\right],$$

where $Z_{5/2}(a,z,\mu)$ is the exact relativistic PDF with index $5/2$ which corresponds to the fundamental electron cyclotron resonance, $z_{\pm 1} = (\omega \pm \omega_B)/(\sqrt{2}k_{//}V_T)$ and longitudinal wave number $k_{//} = N_{//}\omega/c$. Integrating the perturbation $f^{(2)}$ in the same manner as $f^{(0)}$ after more bulky calculations one can obtain expressions for components of plasma dielectric tensor после $\varepsilon_{i,k}^{(2)}$ (i,k =1,2). Here we give the finite result:

$$\varepsilon = \varepsilon^{(0)}\boldsymbol{E} + \frac{d}{dx}\left(\varepsilon^{(2)}\frac{d}{dx}\boldsymbol{E}\right),$$

where

$$\varepsilon_{yx}^{(0)} = -\varepsilon_{xy}^{(0)}, \quad \varepsilon_{yy}^{(0)} = \varepsilon_{xx}^{(0)},$$
$$\varepsilon_{xx}^{(2)} = \mu\left(\frac{\omega_p v_T}{\omega\omega_B}\right)^2[(Z_{7/2}(a,z_2,\mu) + Z_{7/2}(a,z_{-2},\mu)) -$$
$$(Z_{7/2}(a,z_1,\mu) + Z_{7/2}(a,z_{-1},\mu))],$$
$$\varepsilon_{xy}^{(2)} = -i\mu\left(\frac{\omega_p v_T}{\omega\omega_B}\right)^2[(Z_{7/2}(a,z_2,\mu) - Z_{7/2}(a,z_{-2},\mu)) -$$
$$2(Z_{7/2}(a,z_1,\mu) + Z_{7/2}(a,z_{-1},\mu))],$$
$$\varepsilon_{yx}^{(2)} = -\varepsilon_{xy}^{(2)},$$
$$\varepsilon_{yy}^{(2)} = \mu\left(\frac{\omega_p v_T}{\omega\omega_B}\right)^2[(Z_{7/2}(a,z_2,\mu) + Z_{7/2}(a,z_{-2},\mu)) -$$
$$3(Z_{7/2}(a,z_1,\mu) + Z_{7/2}(a,z_{-1},\mu)) + 4Z_{7/2}(a.z_0,\mu)].$$

It can also be demonstrated that present approach to derive inhomogeneous plasma dielectric tensor leaves in force and for the general case of taking also into account in equation (2) the drift terms in background distribution, i.e. without limitations for gradients of inhomogeneous plasma parameters.

3. CONCLUSIONS

The next conclusions can be drawn from the previous text.

1. The present technique, based on principle of analytical continuation for Cauchy-type integrals and on the exact plasma dispersion function concept, provides the exact account of relativistic effects for plasma that is inhomogeneous across direction of magnetic field.
2. In the case of relativistic plasma, inhomogeneous across magnetic field direction likely to the case of homogeneous plasma, instead nonrelativistic PDF $W(x)$, in the plasma dielectric tensor there appeared the exact relativistic PDFs with coefficient $N_{//}\sqrt{2\mu}$. Moreover an index of those PDFs is defined by the order of Larmor

expansion in line with the rule of homogeneous case [2]. Hence a structure of differential operator for this tensor corresponds exactly to non-relativistic case with the same type of inhomogeneity.

3. Present results can be used as for development of EC wave numerical models (for example, of the type [4]), which take into account the spatial plasma dispersion, including the strong one, so and of ICR wave models in the case of hot relativistic plasma.

4. This way can be also used to clarify the exact limits of the weakly and mild relativistic approximations in the numerical wave calculations.

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ТЕНЗОР ДИЭЛЕКТРИЧЕСКОЙ ПРОНИЦАЕМОСТИ НЕОДНОРОДНОЙ РЕЛЯТИВИСТСКОЙ ПЛАЗМЫ

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На основе принципа аналитического продолжения и понятия точных релятивистских плазменных дисперсионных функций представляется метод получения тензора диэлектрической проницаемости неоднородной релятивистской плазмы.

ТЕНЗОР ДІЕЛЕКТРІЧНОЇ ПРОНИКНОСТІ НЕОДНОРІДНОЇ РЕЛЯТИВІСТСЬКОЇ ПЛАЗМИ

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На основі принципу аналітичного продовження та поняття точних релятивістських плазмових дисперсійних функцій представляється метод одержання тензору діелектричної проникності неоднорідної релятивістської плазми.