FAST COMPUTATION OF THE EXACT PLASMA DISPERSION FUNCTIONS

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The paper is concerned with a computation of the exact relativistic plasma dispersion functions for complex argument $z = x + iy$ in the region $Q:\ y \geq 0$ on the base the theory of the continued fractions of Jacobi. It is first observed that these fractions represent those functions asymptotically for $z \to \infty$ in the sector Q.

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1. INTRODUCTION

An evaluation of plasma dielectric tensor is the basic to study theoretically phenomena connected with linear electromagnetic waves in magnetized plasma. In order to reduce such the evaluation to the evaluation of plasma dispersion functions (PDFs) one can write this tensor as a finite Larmor radius expansion in terms of these PDFs. In many numerical applications those PDFs must be computed a large number of times. It is therefore important to search for computation methods which are as fast as possible.

One of such fast methods was developed for calculation of a special function $W(z) = \exp(-z^2)\times\ (1 + \frac{2i}{\sqrt{\pi}}\int_0^z \exp(t^2)dt)$, called Kramp or the complex error function and relating to the non-relativistic PDF as $Z(z) = i\sqrt{\pi}W(z)$, on the base of the theory of continued fractions [1,2]. The continued fractions together with their generalization, Padé approximants, are of great interest in many fields of pure and applied mathematics and in numerical analysis, where they are closely connected with the convergence acceleration techniques and provide, in particulary, efficient computer approximations to special functions [3]. This method is also proved suitable for the fast computation of the weakly relativistic PDFs (Shkarofsky functions) since they are satisfied up to the second order recursion relation and the two PDFs, starting the recursive procedure, are expressed through the non-relativistic PDF.

The exact PDFs, introduced in order to give a recipe to evaluate the fully relativistic plasma dielectric tensor of Trubnikov [4] for arbitrary plasma and wave parameters, were computed on the base of the theory of Cauchy-type integrals from complex analysis [5,6]. This method, being exact and general, is not yet as fast as the method [1,2]. The main scope of the present work is generalization of the last method for the case of the exact PDFs. We review some analytical properties of these PDFs and relevant peculiarities of this continued fraction technique and develop the faster algorithm computing these functions. The main performance characteristics and data on algorithm testing are given as well.

2. MATHEMATICAL PRELIMINARIES

The main idea of the works [1, 2], devoted to the fast computation of the Kramp function $W(z) = \exp(-z^2)\times erfc(-iz)$, is utilizing the continued J-fractions instead of series to accelerate its convergence in the regions, where a module of the function is monotonous. For instance, though in the region, where the condition $|z| >> 1$ is true, one can use, in principle, the asymptotic series

$$W(z) \sim \frac{i}{\pi}\sum_{k=0}^{\infty}\frac{\mu_k}{z^{k+1}},\ \mu_k = \begin{cases} 0, & k = 2n-1 \\ \Gamma\left(\frac{k+1}{2}\right), & k = 2n \end{cases},\quad n = 1,2,... \quad (1)$$

to evaluate the Kramp function, however this series is divergent and that makes difficult such a usage. Really, for accuracy it is necessary to control constantly the number of series terms and, moreover, for not very large in module z values this number can be proved huge. A usage of the continued J-fraction on the base equality

$$\sum_{k=0}^{\infty}\frac{\mu_k}{z^{k+1}} = \frac{|\sqrt{\pi}}{z-|} + \frac{|1/2}{z-|} + \frac{|1}{z-|} + \frac{|3/2}{z-|} + ... = \cfrac{\sqrt{\pi}}{z - \cfrac{1/2}{z - \cfrac{1}{z - \cfrac{3/2}{z - ...}}}}, \quad (2)$$

instead the series (1) removes these difficulties, as it was shown by Gautschi [1]. In consequence of the properties of the continued fraction in (2), for evaluation of $W(z)$ it is enough only some first terms even for not very large in module z values. He has shown also that in the case of moderate and small z values one can use the same approach. At that only coefficients of Teylor series, expanded along positive imagine axis into inverse direction in early calculated large in module z points, and coefficients of relating continued fraction will change.

The exact relativistic PDFs [5,6] are a natural generalization of non-relativistic and weakly relativistic PDFs to the case of arbitrary temperature and, as a consequence, their analytical properties are rather close, though more sophisticated for higher temperatures. In the case $|z| >> 1$ these PDFs have the next asymptotic expansions:

$$Z_{q+3/2}(a, z, \mu) \sim \sum_{k=0}^{\infty}\frac{A_k^q}{z^{k+1}}, \quad (3)$$

where $a = \mu N_{//}^2/2$, $N_{//}$ is longitudinal refractive index, $\mu = (c/V_T)^2$, V_T is thermal speedy of electrons with the rest mass, $A_0^q = K_q(\mu)/K_2(\mu)$, $K_q(x)$ is Macdonald function of order q, $A_1^q = A_0^q - A_0^{q+1}$ and reminder coefficients A_k^q can be obtained from recursive relation of the second order [5]. Obviously, this series is a generalization of the series (1) to the case $\mu_k > 0$ for $k = 2n-1$ ($n = 1,2,...$) and, consequently, one can hope to use continued fractions instead of the expression (3) for more fast computation of functions $Z_{q+3/2}(a,z,\mu)$. For this aim it is only necessary to transform the series of type (3) into continued J-fraction, i.e. to define in the equality

$$\sum_{k=0}^{\infty}(-1)^k \frac{m_k}{z^{k+1}} = \frac{|\, a_0^2}{z-b_0-|} - \frac{|\, a_1^2}{z-b_1-|} - ... - \frac{|\, a_k^2}{z-b_k-|} - ..., \quad (4)$$

the connection between coefficients m_k and coefficients a_k, b_k. Using the method of mathematical induction one can prove the next formulas

$$a_0^2 = m_0,\ a_0^2 a_1^2 = \begin{vmatrix} m_0 & m_1 \\ m_1 & m_2 \end{vmatrix} m_0^{-1},\ a_0^2 a_1^2 a_2^2 = \begin{vmatrix} m_0 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{vmatrix} \begin{vmatrix} m_0 & m_1 \\ m_1 & m_2 \end{vmatrix}^{-1}, ...$$

$$b_0 = -\frac{m_1}{m_0}, \quad b_0 + b_1 = -\begin{vmatrix} m_0 & m_1 \\ m_2 & m_3 \end{vmatrix} \begin{vmatrix} m_0 & m_1 \\ m_1 & m_2 \end{vmatrix}^{-1}, \quad (5)$$

$$b_0 + b_1 + b_2 = -\begin{vmatrix} m_0 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_3 & m_4 & m_5 \end{vmatrix} \begin{vmatrix} m_0 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{vmatrix}^{-1}, \ ...$$

where were used so named determinants of Hankel. These formulas can be reduced by the change $z = 1/x$ to the formulas of Heilermann for connection coefficients of Teylor expansion near zero with coefficients of related continued J-fraction [7]. Now on the base (4), (5) one can obtain the equality (2). Thus, we can evaluate the functions $Z_{q+3/2}(a,z,\mu)$ for the case $|z| >> 1$ on the base continued J-fractions also using (4), (5). For reminder z values one can use the technique of Cauchy-type integrals [6] or the way of Gautschi for small and moderate in module z values.

3. COMPUTATIONAL PROCEDURE, PERFORMANCE CHARACTERISTICS AND TESTING DATA

Our objective is to devise an efficient procedure for computing the main branch of the function $Z_{5/2}(a,z,\mu)$ to a given number d of correct decimal digits after the decimal point, i.e., to within an (absolute) error of 0.5×10^{-d}. We shall assume $z = x + iy$ to lie in the region Q: $y \geq 0$ of the complex plane. This is no restriction of generality, since the formulas [6]

$$Z_q^+(a,z,\mu) = \begin{cases} Z_q^+(a,z^*,\mu) - 2\pi i f(a,a_r - z,\mu), & \mathrm{Re}\, z < a_r, y < 0 \\ Z_q^+(a,z^*,\mu) & \mathrm{Re}\, z > a_r, y < 0, \end{cases}$$

$$f(a,t,\mu) = \begin{cases} \dfrac{\sqrt{\pi\beta} e^{-\mu\sqrt{\beta}}}{\sqrt{2\mu} K_2(\mu) a^{(q-1)/2}} \left(\sqrt{t\left(\dfrac{t}{2\mu} + \beta^{-1/2}\right)} \right)^{q-1} \times \\ I_{q-1}\left(2\beta\sqrt{a}\sqrt{t\left(\dfrac{t}{2\mu} + \beta^{-1/2}\right)} \right) e^{-\beta t}, & t > 0 \\ 0, & t \leq 0. \end{cases}$$

can be used to continue $Z_{5/2}(a,z,\mu)$ into the remaining region.

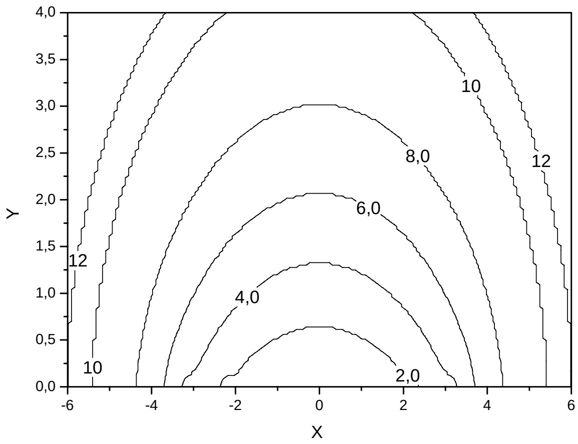

Fig.1 Altitude map of the function $Z_{5/2}(a,z,\mu)$ (Cauchy)-$Z_{5/2}(a,z,\mu)$ (chain fraction, n=9) for $N_{//} = 1.1$ and T_i =40keV in ion plasma

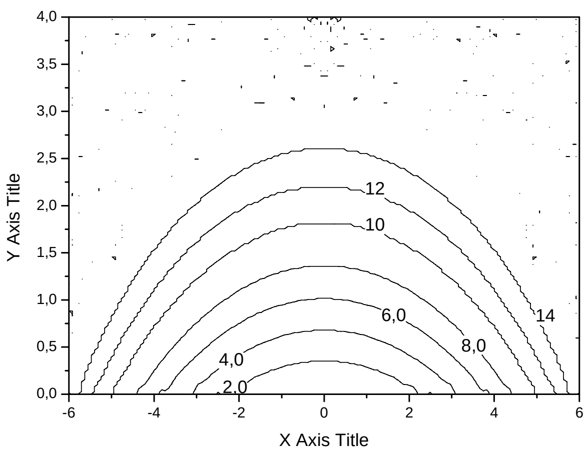

Fig. 2 The same as in Fig. 1 excepting n=29

In computational procedure we will use the asymptotic expansion (3) and the continued J-fraction (4) for the case $|z| >> 1$ and the exact technique of Cauchy-type integrals [6] for reminder z values. First let us plot the line of transition from the method of Cauchy-type integrals to method of continued fraction to a different prescribed accuracy for the function $Z_{5/2}(a,z,\mu)$ for two values of the length of continued fraction $n = 9$ (Fig. 1) and $n = 29$ (Fig. 2) for longitudinal refractive index $N_{//}$=1.1 and ion temperature T_i=40keV of the ion plasma. This case is very close to the non-relativistic one. From these pictures it follows, that continued fraction represents this function asymptotically in essentially wider region for the case $n = 29$ and for $z \to \infty$ in the region Q: $y \geq 0$ than for the case $n = 9$ and this fact can be used by different ways (for example, by Gauschi way) for computation $Z_{5/2}(a,z,\mu)$ for small and moderate in module z values.

The Fig. 3 presents the function $Z_{5/2}(a,z,\mu)$ for the relativistic case with longitudinal refractive index $N_{//}$=0.6 and $n = 9$ and temperature T_e=40keV of the electron plasma. We can conclude that in the relativistic case continued fraction as well represents the function $Z_{5/2}(a,z,\mu)$ asymptotically in the same region Q: $y \ge 0$ for $z \to \infty$.

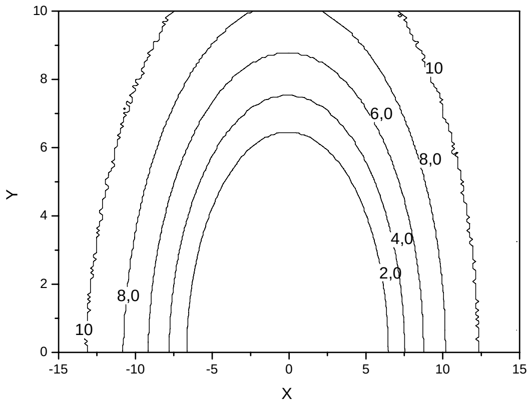

Fig.3 Altitude map of the function $Z_{5/2}(a,z,\mu)$ *(Cauchy)-* $Z_{5/2}(a,z,\mu)$ *(chain fraction, n=9), for* $N_{//} = 0.6$ *and* $T_e =$ *40keV in electron plasma*

Moreover the calculations show that application of continued fraction gives acceleration of convergence (and consequently a gain in computer time) in comparison with an evaluation on the base of asymptotic expansion approximately with the factor 2–3. Such an application gives also a gain in time in comparison with the technique of Cauchy-type integrals approximately with the factor 100.

Thus, the application of continued J-fraction gives the acceleration in comparison with the technique of Cauchy-type integrals of the factor 100 and allows developing essentially faster computational procedure.

4. CONCLUSIONS

The next conclusions can be drawn from the present work with the account of results of [1,2].

1. If the function $f(z)$ can be represented by some Cauchy-type integral with its continues real density, defined on the real axis and tending to 0 in the infinity, then asymptotic series of this function approximates uniformly this function. in the vicinity of the point $z = \infty$ in the sector where $f(z) \to 0$ ($z \to \infty$ along the ray).
2. The continued J-fraction, corresponding to this asymptotic series, extends essentially the present vicinity.
3. The present vicinity is proved to be so wide that the continued J-fraction describes not only the hermitian part of $f(z)$ but the finite anti-hermitian one as well. This fact allows one to revivify $f(z)$ outside of the vicinity by different ways.
4. Utilizing of chain fractions for given class of functions allows one to revivify the whole analytical function on the base only the asymptotic expansion.

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БЫСТРОЕ ВЫЧИСЛЕНИЕ ТОЧНЫХ ПЛАЗМЕННЫХ ДИСПЕРСИОННЫХ ФУНКЦИЙ

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На основе теории цепных дробей развивается метод быстрого вычисления точных релятивистских плазменных дисперсионных функций в комплексной области

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На основі теорії ланцюгових дробів розвивається метод швидкого обчислення точних релятивістських плазмових дисперсійних функцій у комплексній області.