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Exact solutions for some modifications of the nonlinear Cahn–Hilliard equation

(Presented by Academician of the NAS of Ukraine N. F. Shul'ga)

The exact travelling wave solutions for convective, higher-order convective, and convective-viscous Cahn–Hilliard equations are obtained. Without any additional restrictions on the parameters, the solutions with non-zero propagation velocity exist only for an asymmetric potential. However, for an additional constraint on the higher-order convective term or for a special balance between nonlinearity and viscosity, the non-zero velocity exists for a symmetric potential as well. In the latter case, the exact two-wave solution is obtained; asymptotically, it converges to the well-known static kink solution.

Recently, the nonlinear convective Cahn–Hilliard equation in one space dimension for a symmetric double-well potential was introduced in several articles [1–6]. Leung [1] proposed this equation as a continual description of the lattice gas phase separation under the influence of an external field. Similarly, Emmott and Bray [2] proposed this equation as a model of the spinodal decomposition of a binary alloy in an external field. In both cases, the dependence of the mobility on the order parameter is presumed, which introduces, in turn, the connection to the external field. In [3–4], this equation was derived in a model of kinetically controlled evolution of two-dimensional crystals. Several approximate solutions and only two exact static kink and anti-kink solutions were obtained; the "coarsening" of domains separated by kinks and anti-kinks was also discussed [1–6]. Here, we consider the slightly different equation corresponding to an asymmetric potential; we also rescale the variables to get

$$u_t - \alpha u u_x = (u^3 - \delta u^2 - u - u_{xx})_{xx}. \tag{1}$$

In (1), α is the (rescaled) applied field; in accord with [2] α is positive for positive direction of the external field. For $\delta = 0$, the case of symmetric potential is recovered. For the classic Cahn–Hilliard equation [7], the static kink solution corresponding to asymmetric potential was generally discarded [8], because it violates the global conservation of the order parameter. However, there is generally no global conservation for (1). Even more, the very notion of coarsening or "Ostwald ripening, " as considered in the theory of first-order phase transitions [9], relates to the competitive growth of stable-phase domains inserted into the metastable phase. In terms of the quartic free energy for the Cahn–Hilliard equation, this corresponds to unequal depths of two potential wells and $\delta \neq 0$ in (1). The asymmetric potential naturally arises in some applications and generalizations of the convective Cahn–Hilliard equations [3, 10].

Here, we give the exact travelling wave solutions of (1). *Introducing the* travelling wave coordinate $z = x - vt$ and integrating once yield

$$-vu - \frac{\alpha}{2}u^2 + c\frac{\alpha}{2} = (u^3 - \delta u^2 - u - u_{zz})_z, \tag{2}$$

where c is an arbitrary constant. At $z = \pm\infty$, all derivatives equal zero; that is, the left-hand side also equals zero. This yields a quadratic equation; the roots of this equation, i. e., the values of u at $\pm\infty$, are

$$u_{1,2} = -\frac{v}{\alpha} \mp \eta, \qquad \eta = \left|\sqrt{\frac{v^2}{\alpha^2} + c}\right|; \tag{3}$$

The travelling kink solutions connecting u_1 and u_2 are

$$u = \frac{u_1 + u_2 \exp\{2\kappa\eta(x - vt)\}}{1 + \exp\{2\kappa\eta(x - vt)\}}. \tag{4}$$

In addition to (3), there are three relations for κ, v, η:

$$v = -\frac{1}{3}\alpha\delta, \qquad \eta^2 = c + \frac{v^2}{\alpha^2} = 1 + \frac{\alpha}{2\kappa} + \frac{1}{3}\delta^2. \tag{5}$$

If $\kappa > 0$, the solution increases monotonically from a smaller stationary value u_1 at $z = -\infty$ to a larger stationary value u_2 at $z = +\infty$; it is usually called "kink". If $\kappa < 0$, the solution decreases from u_2 at $z = -\infty$ to u_1 at $z = +\infty$; it is usually called "anti-kink". For $\delta = 0$, i. e., $v = 0$, this solution reduces to the well-known static symmetric kink/anti-kink solutions [1, 2]. Emmott and Bray [2] pointed out that the "negative-field-kink" combination is equivalent to the "positive-field-anti-kink" one. They have also shown that, for a negative field, the static kink solution is stable, while the anti-kink solution is unstable. This simple symmetry is broken for $\delta \neq 0$. Depending on the signs of α and δ, there are four cases for the travelling-wave solutions, as shown in Fig. 1. The study of the stability of these solutions is more complicated and will be given elsewhere.

The convective Cahn–Hilliard equation with cubic nonlinearity in the convective term was introduced and approximately solved in [3]. Higher-order polynomials both for the convective term and the potential were also considered in [10]. Here, we give the exact travelling-wave solutions of the equation

$$u_t - \alpha_1 u u_x + 2\alpha_2 u^3 u_x = (u^3 - \delta u^2 - u - u_{xx})_{xx}. \tag{6}$$

The scaling and notations here and below differ from those of [3] and other cited papers, to match our above consideration. Introducing the travelling-wave coordinate $z = x - vt$ and integrating once, we obtain

$$\frac{\alpha_2}{2}\left(u^4 - \frac{\alpha_1}{\alpha_2}u^2 - \frac{2v}{\alpha_2}u - c\right) = (u^3 - \delta u^2 - u - u_{zz})_z, \tag{7}$$

where c is an integration constant. As different from (2), the polynomial on the left-hand side is of the fourth order. For large enough (positive) c, this polynomial has at least two real roots u_1 and u_2; for definiteness, we take $u_1 < u_2$. They will be the stationary values of u at $z = \pm\infty$, where the right-hand side of (7) equals zero. Then the solutions may be again written in the form (4); however, the system of equations for κ, v, η and u_1, u_2 is more complicated and admits several different solutions:

$$2\kappa^2 - 1 - \frac{\alpha_2}{6\kappa} = 0, \tag{8}$$

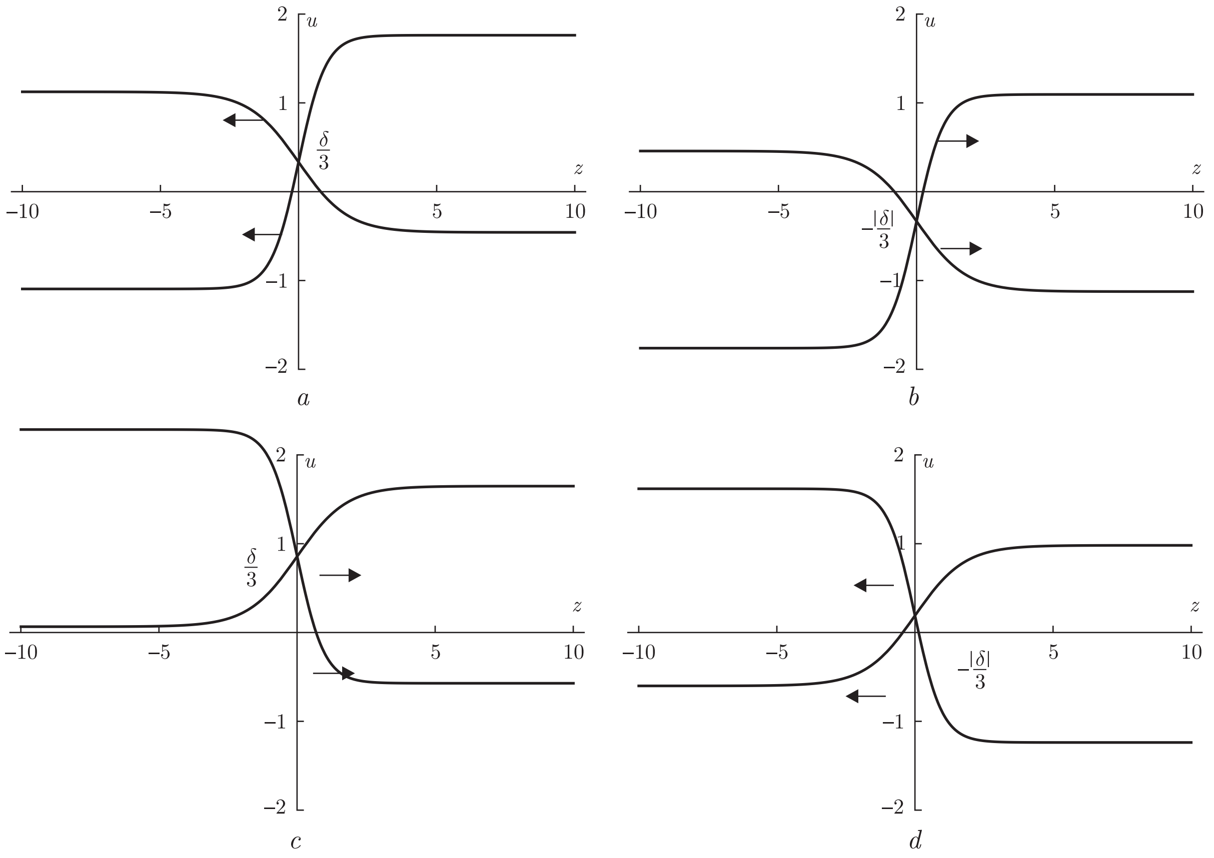

Fig. 1

$$v = \frac{u_1 + u_2}{2}[\alpha_2(u_1^2 + u_2^2) - \alpha_1], \tag{9}$$

$$(u_1 + u_2)\left(\frac{\alpha_2}{4\kappa} + 3\kappa^2\right) = \delta, \tag{10}$$

$$(u_1 + u_2)^2\left(\frac{\alpha_2}{2\kappa} - \kappa^2\right) - u_1u_2\left(\frac{\alpha_2}{2\kappa} - 2\kappa^2\right) = 1 + \frac{\alpha_1}{2\kappa}, \tag{11}$$

$$2\eta = u_2 - u_1. \tag{12}$$

For $\alpha_2 = 0$, (8) reduces to the first constraint in (5). For arbitrary non-zero α_2, (8) is a cubic equation; it has three different real roots if $\alpha_2^2 < 8/3$, and only a single real root if $\alpha_2^2 > 8/3$. It is evident from (9) and (10) that, for arbitrary α_2, the non-zero velocity of a kink is possible if $\delta \neq 0$ only. Solving system (10), (11), we find the corresponding stationary values u_1, u_2 for each root of (8):

$$u_{1,2} = \frac{2\kappa\delta}{\alpha_2 + 3\kappa} \mp \eta, \tag{13}$$

$$\eta = \left|\left\{\frac{3}{2\alpha_2 + 3\kappa}\left[\frac{\kappa^2\delta^2(3\kappa - \alpha_2)}{(\alpha_2 + 3\kappa)^2} + \frac{\alpha_1}{2} + \kappa\right]\right\}^{1/2}\right|. \tag{14}$$

The substitution of (13) into (9) determines the kink velocity v. If $\delta = 0$, we still have either a single or three static kink solutions, depending on the number of the roots of (8). On the other hand, for $\delta = 0$ and the additional constraint imposed on α_2 and κ,

$$\frac{\alpha_2}{4\kappa} + 3\kappa^2 = 0, \tag{15}$$

relation (10) is satisfied for arbitrary values of $(u_1 + u_2)$. It is possible, however, only for the special value of $\alpha_2^2 = 9/4$ and the corresponding $\kappa = -\alpha_2/3$. In other words, even for $\delta = 0$, there is a travelling kink solution for $\alpha_2 = -3/2$ and the anti-kink one for $\alpha_2 = 3/2$; and there is only the single constraint (11) imposed on u_1, u_2.

The ***viscous*** Cahn–Hilliard (VCH) equation was introduced by Novick-Cohen [11] to include some viscous effects, which are neglected in the derivation of the classic Cahn–Hilliard equation [7]. The VCH equation could also be derived [12] as a certain limit of the classic phase-field model (as for the phase-field model and its relations to other phase-separation models, see [13, 14] and references therein). To study the joint effects of nonlinear convection and viscosity, Witelski [15] introduced the *convective-viscous-Cahn–Hilliard* equation with a general symmetric double-well potential. With an additional constraint on nonlinearity and viscosity, the approximate travelling-wave solution was obtained in [15]. Here, we consider a polynomial generally asymmetric potential, i. e., the equation

$$u_t - \alpha u u_x = (u^3 - \delta u^2 - u - u_{xx} + \mu u_t)_{xx} \tag{16}$$

Equation (16) has exact travelling kink solutions of the form (4), where the five parameters κ, v, η and u_1, u_2 are given as the solutions of the system

$$2\kappa^2 = 1, \tag{17}$$

$$v\left(\kappa\mu - \frac{3}{\alpha}\right) = \delta, \tag{18}$$

$$\eta^2 + (2\kappa\mu\alpha - 3)\frac{v^2}{\alpha^2} - \left(1 + \frac{\alpha}{2\kappa}\right) = 0, \tag{19}$$

$$u_1 + u_2 = -2\frac{v}{\alpha}, \tag{20}$$

$$u_2 - u_1 = 2\eta. \tag{21}$$

If the "driving force" α and the viscosity μ are given independently, the travelling kinks are possible for $\delta \neq 0$ only, i. e., for an asymmetric potential (see (18)). However, relation (18) with $\delta = 0$ is satisfied for arbitrary non-zero v if $\kappa\alpha\mu = 3$, i. e., for a special balance between the driving force and the viscosity, which is in accord with the result in [15]. Even more, in this case, (16) has the exact two-wave solution

$$u = \frac{\sigma}{\kappa}\frac{\exp\{\sigma(x + vt) + \varphi\} - \exp\{-\sigma(x - vt) + \varphi\}}{1 + \exp\{\sigma(x + vt) + \varphi\} + \exp\{-\sigma(x - vt) + \varphi\}}, \tag{22}$$

where $2\kappa^2 = 1$, $v = (\alpha\sigma)/(2\kappa)$, $\sigma = (1/2 + \alpha/(4\kappa))^{1/2}$, and φ is an arbitrary constant. This solution is obtained using the bilinear Hirota method. For $\kappa > 0$, it consists of two kinks (or anti-kinks for $\kappa < 0$) moving toward each other and merging asymptotically (as $t \to \infty$) into the well-known symmetric static kink/anti-kink solution [1, 2]. The discussion on the stability and applications of the above solutions will be given in the following communications.

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Точні розв'язки для деяких модифікацій нелінійного рівняння Кана–Хилларда

Отримано точні розв'язки у вигляді біжучої хвилі для конвективного, конвективного з більшим ступенем нелінійності та конвективно-в'язкого рівняння Кана–Хілларда. Без будь-яких додаткових обмежень на параметри розв'язки з ненульовою швидкістю існують тільки для асиметричного потенціалу. Однак при додатковому обмеженні на конвективний член старшого ступеня або у випадку спеціального балансу між нелінійністю та в'язкістю розв'язки з ненульовою швидкістю існують і для симетричного потенціалу. Для останнього випадку отримано і точний двохвильовий роз'язок; асимптотично він збігається до відомого статичного кінк-роз'язку.

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Точные решения для некоторых модификаций нелинейного уравнения Кана–Хилларда

Получены точные решения в виде бегущей волны для конвективного, конвективного с более высокой степенью нелинейности и конвективно-вязкого уравнений Кана–Хилларда. Без каких-либо дополнительных ограничений на параметры решения с ненулевой скоростью распространения существуют только для асимметричного потенциала. Однако при дополнительном ограничении на конвективный член старшего порядка или для случая специального баланса между нелинейностью и вязкостью решения с ненулевой скоростью существуют и для симметричного потенциала. Для последнего случая получено и точное двухволновое решение; асимптотически оно сходится к известному статическому кинк-решению.