

# Интеллектуальное управление и системы

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## ON THE APPROXIMATION OF VECTOR OPTIMIZATION PROBLEMS

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В работе исследуются условия сходимости приближенного метода решения задач многокритериальной оптимизации, когда целевые функции и допустимое множество заменяются их приближениями. Доказано, что достаточным условием сходимости являются равномерная сходимость приближенных функций к исходной функции и сходимость допустимого множества приближенных задач к допустимому множеству исходной задачи, по крайней мере, в окрестности решения.

**Ключевые слова:** векторная оптимизация, стохастическая многокритериальная оптимизация, парето-оптимальность, дискретная аппроксимация, эpsilon-доминирование.

У роботі досліджено умови збіжності наближеного методу розв'язання задач багатокритеріальної оптимізації у випадку коли цільові функції і допустима область замінюються їх наближеннями. Доведено, що достатньою умовою збіжності є рівномірна збіжність наближених функцій до початкової функції та збіжність допустимої множини наближених задач до допустимої множини початкової задачі хоча б в околі розв'язку.

**Ключові слова:** векторна оптимізація, стохастична багатокритеріальна оптимізація, парето-оптимальність, дискретна аппроксимация, епсилон-домінування.

## INTRODUCTION

This paper studies conditions of convergence of the successive approximations method for solving deterministic and stochastic vector optimization problems. A general form of a vector optimization problem reads as follows:

$$\bar{F}(x) = \{F_1(x), \dots, F_m(x)\} \rightarrow \max_{x \in X \subset \mathbb{R}^n}, \quad (1)$$

where functions  $F_i(x)$ ,  $i = 1, \dots, m$ , are continuous on a compact set  $X \subset \mathbb{R}^n$ . The problem is (a) to find individual elements or (b) the whole set of weakly Pareto optimal points  $X^* \subset X$  such that for any  $x \in X^*$  there is no  $x' \in X$  and  $\bar{F}(x') > \bar{F}(x)$  (component-wise). There are numerous approaches for solving problem (1) [1–3]. The most known of them consists in maximization of some component function  $F_i$  under constraints on other functions or in aggregation of criteria  $F_i$  by some linear or non-linear utility function and solving the resulting

optimization problems by nonlinear programming methods [1]. However, in case of non-convex functions  $F_i$  it is not always possible to find all Pareto-optimal solutions in such a way. So, other approaches, that are not reduced to optimization of a scalar criterion, were developed, in particular, the parameter space investigation method [2] and methods of evolutionary programming [3]. The latter in effect are variants of *controlled iterative random search method*: at each iteration an approximate discrete solution  $X_N^*$  consisting of a finite number of points is constructed and then, using information about the objective function  $\bar{F}(X_N^*)$  at points  $X_N^*$  a new generation of points  $Y^N$  is generated in some way (randomly) and a new discrete approximate solution  $X_{N+1}^*$  is selected from the set  $X^N = (X_N^*, Y^N)$ , and so on. Finding out conditions of evolutionary programming algorithm convergence to the set of Pareto-optimal points is a serious scientific problem and is the subject of active research [4–6]. Even convergence of the simplest algorithms of this type is studied only in case of discrete feasible set  $X$  [6]. Let us, for example, sample points  $Y^N$  in the described approach uniformly in  $X$  and let a new approximation  $X_{N+1}^*$  be a Pareto-optimal subset of the discrete pair  $X^N = (X_N^*, Y^N)$ .

Does the sequence  $\{X_N^*\}$  of approximations converge to the Pareto-optimal set  $X^*$  of problem (1)? This article, in particular, aims at finding answers to such type of questions.

## PROBLEM SETTING

In practice, a formulation of the vector optimization problem may be more complex than (1). For example, in the case of the stochastic vector optimization, functions  $\bar{F}$  are actually expectations,  $\bar{F}(x) = E\bar{f}(x, \omega)$ , where the random variable  $\omega$  is defined on some probability space  $(\Omega, \Sigma, P)$ , symbol E denotes expectation (integral) over measure P [7, 8]. Usually, in practical problems expectations cannot be calculated analytically, so they are estimated numerically using quadratures or Monte Carlo method. In the latter case empirical approximations of functions  $\bar{F}$  have the form  $\bar{F}^N(x, \omega^N) = (1/N) \sum_{k=1}^N \bar{f}(x, \omega_k)$ , where  $\omega^N = (\omega_1, \dots, \omega_N)$  is the set of independent and identically distributed observations  $\omega_k$  of the random variable  $\omega$ . In [9] one can find conditions of uniform convergence of the empirical approximations  $\bar{F}^N(x, \omega^N)$  to the expectation  $\bar{F}(x) = E\bar{f}(x, \omega)$  on a compact set  $X$ . Then, instead of (1), one has to consider a sequence of approximate problems:

$$\bar{F}^N(x) = \{F_1^N(x), \dots, F_m^N(x)\} \rightarrow \max_{x \in X^N \subset \mathbb{R}^n}, \quad N = 1, 2, \dots, \quad (2)$$

where the set of feasible solutions  $X^N$ , in general, can also vary from task to task, for example, the set  $X^N$  can be a discrete approximation of the initial feasible set  $X$  [2]. In the latter case, the approximating functions  $F_1^N(x), \dots, F_m^N(x)$  can be defined only on this discrete set  $X^N$ . Denote  $X_N^*$  the set of Pareto-optimal solutions of (2). The problem is to establish conditions under which the sets  $X_N^*$  approximate the set of solutions  $X^*$ .

In (1)  $\alpha_i$ -quantiles of random variables  $F_i(x, \omega)$  can serve as components of the vector objective function  $\bar{F}(x) = \{EF_1(x, \omega), \dots, EF_m(x, \omega)\}$ . It is known [10] that these quantiles can be found by solving the following auxiliary optimization problem:

$$\Phi_i(x, y_i) = E \max\{(1 - \alpha_i)(F_i(x, \omega) - y_i), \alpha_i(y_i - F_i(x, \omega))\} \rightarrow \min_{y_i \in \mathbb{R}^1}. \quad (3)$$

An approximate solution  $y_i^N(x)$  of (3) for each fixed  $x$  can be found, for example, by  $N$  iterations of the stochastic quasi-gradient method [11], or as a corresponding term of the variational series of the random variable  $F_i(x, \omega)$ . Thus, one again encounters with the approximate problem (2) with functions  $F_i^N(x) = y_i^N(x)$ . Another approach to solving (3) is to use an empirical approximation of components  $\Phi_i(x, y_i)$ :

$$\begin{aligned} \Phi_i^N(x, y_i) = \\ = (1/N) \sum_{k=1}^N \max\{(1 - \alpha_i)(F_i(x, \omega_k) - y_i), \alpha_i(y_i - F_i(x, \omega_k))\} \rightarrow \min_{y_i \in \mathbb{R}^1} \end{aligned} \quad (4)$$

and finding its approximate solution  $y_i^N(x)$  for each  $x$  by the linear programming method.

## MAIN RESULTS

As noted in [2], the question of convergence of the discrete approximation of problem (1) is not easy. In our case the problem is further complicated by the fact that not only feasible set, but also the objective functions are approximated. To study convergence of approximate solutions of problems (2) to the solution of the original task (1), we'll need some more definitions.

**Definition 1** ( $\bar{\varepsilon}$ -nondominated solutions). Point  $x \in X$  is called  $\bar{\varepsilon}$ -nondominated solution of (1), if there is no other point  $z \in X$  such that  $\bar{F}(z) > \bar{F}(x) + \bar{\varepsilon}$ , where  $\bar{\varepsilon}^N \in \mathbb{R}^m$ .

**Definition 2** [12, Section 4A]. For a sequence of sets  $\{Z_N \subset \mathbb{R}^n, N = 1, 2, \dots\}$  let us define the following cluster sets:

$$\limsup_{N \rightarrow \infty} Z_N = \{z : \exists z_{N_k} \in Z_{N_k}, z = \lim_{k \rightarrow \infty} z_{N_k}\},$$

$$\liminf_{N \rightarrow \infty} Z_N = \{z : \exists z_N \in Z_N, z = \lim_{N \rightarrow \infty} z_N\},$$

$$\lim_{N \rightarrow \infty} Z_i = \liminf_{N \rightarrow \infty} Z_N = \limsup_{N \rightarrow \infty} Z_N.$$

Denote  $X^*(\bar{\varepsilon})$  the set of all  $\bar{\varepsilon}$ -nondominated solutions of (1). In the same manner define  $X_N^*(\bar{\varepsilon}^N)$   $\bar{\varepsilon}^N$ -nondominated solutions of (2). Our goal is to establish conditions under which sets  $X_N^*(\bar{\varepsilon}^N)$  approximate  $X^*(\bar{\varepsilon})$ .

Let us note some properties of the multivalued mapping  $\bar{\varepsilon} \rightarrow X^*(\bar{\varepsilon})$ .

**Lemma 1.** The mapping  $\bar{\varepsilon} \rightarrow X^*(\bar{\varepsilon})$  is monotone, i.e. for  $\bar{\varepsilon}^1 \leq \bar{\varepsilon}^2$  one has  $X^*(\bar{\varepsilon}^1) \subseteq X^*(\bar{\varepsilon}^2)$ .

*Proof.* Assume the contrary that for some  $x' \in X^*(\bar{\varepsilon}^1)$ ,  $x' \notin X^*(\bar{\varepsilon}^2)$  there is  $z' \in X$  such that  $\bar{F}(z') > \bar{F}(x') + \bar{\varepsilon}^2 \geq \bar{F}(x') + \bar{\varepsilon}^1$ . This inequality contradicts the assumption  $x' \in X^*(\bar{\varepsilon}^1)$ .

The lemma is proved.

**Lemma 2.** For upper semi-continuous (component-wise) on the closed set  $X \subset \mathbb{R}^n$  vector-function  $\bar{F}(x)$  the mapping  $\bar{\varepsilon} \rightarrow X^*(\bar{\varepsilon})$  is upper semicontinuous, i.e.  $\limsup_{N \rightarrow \infty} X^*(\bar{\varepsilon}^N) \subseteq X^*(\bar{\varepsilon})$  for any sequence  $\bar{\varepsilon}^N \rightarrow \bar{\varepsilon}$ .

*Proof.* Let  $\bar{\varepsilon}^N \rightarrow \bar{\varepsilon}$ ,  $N \rightarrow \infty$  and  $X^*(\bar{\varepsilon}^{N_k}) \ni x^{N_k} \rightarrow x'$ ,  $k \rightarrow \infty$ . We must show that  $x' \in X^*(\bar{\varepsilon})$ . Assume the contrary, that  $x' \notin X^*(\bar{\varepsilon})$ . Then there is  $z' \in X$  such that  $\bar{F}(z') > \bar{F}(x') + \bar{\varepsilon}$ . From upper semicontinuity of  $\bar{F}$  it follows  $\bar{F}(z') > \bar{F}(x') + \bar{\varepsilon} \geq \limsup_{k \rightarrow \infty} (\bar{F}(x^{N_k}) + \bar{\varepsilon}^{N_k})$  and thus for sufficiently large  $k$  relation  $\bar{F}(z') > \bar{F}(x^{N_k}) + \bar{\varepsilon}^{N_k}$  is fulfilled. This contradicts the initial assumption  $x^{N_k} \in X^*(\bar{\varepsilon}^{N_k})$ .

The lemma is proved.

Let us make the following assumptions on relations between problems (1) and (2)

**A1.** For any sequence  $\{X^N \ni x^N \rightarrow x\}$  it holds true  $\bar{F}^N(x^N) \rightarrow \bar{F}(x)$ ,  $N \rightarrow \infty$ .

**A2.** The sequence of feasible sets  $\{X^N, N=1,2,\dots\}$  of (2) satisfies conditions:  $X^N \subseteq X$  and for some  $\bar{\varepsilon} \in \mathbb{R}^m$  it holds  $X^*(\bar{\varepsilon}) \subseteq \liminf_{N \rightarrow \infty} X^N$ , i.e. for each point  $x \in X^*(\bar{\varepsilon})$  there is a sequence of feasible points  $x^N \in X^N$  convergent to this point  $x \in X^*(\bar{\varepsilon})$ .

Condition A1 is satisfied in particular if functions  $\bar{F}^N$  are defined on the set  $X \supseteq X^N$  and uniformly converge to  $\bar{F}$  on  $X$ . Other possibilities are considered in [13]. Assumption A2 is automatically satisfied if  $\lim_{N \rightarrow \infty} X^N = X$ , for example, if  $X^N$  discretely approximates, with increasing accuracy, the feasible set  $X$ .

**Theorem 1** (on convergence of solutions of approximate tasks (2) to the solutions of the original problem (1)). Suppose that the vector function  $\bar{F}(x)$  is continuous on a compact set  $X$ , conditions A1–A2 are fulfilled and  $\lim_{N \rightarrow \infty} \bar{\varepsilon}^N = \bar{\varepsilon} > 0$ . Then

- 1)  $\limsup_{N \rightarrow \infty} X_N^*(\bar{\varepsilon}^N) \subseteq X^*(\bar{\varepsilon})$ ,
- 2)  $X^*(\bar{\varepsilon}') \subseteq \liminf_{N \rightarrow \infty} X_N^*(\bar{\varepsilon}^N)$  for all  $\bar{\varepsilon}' < \bar{\varepsilon}$ .

*Proof.* Let us prove the first assertion of the theorem. Assume the contrary, that there are  $X_{N_k}^*(\bar{\varepsilon}^{N_k}) \ni x^{N_k} \rightarrow x' \notin X^*(\bar{\varepsilon})$ . Since the vector function  $\bar{F}(x)$  is bounded from above on a compact set  $X$ , then every point  $x'$  outside of  $X^*(\bar{\varepsilon})$  is  $\bar{\varepsilon}$ -dominated by points from  $X^*(\bar{\varepsilon})$ . Indeed, if it is not true, then there is an infinite sequence of points  $z^s \in X$  such that  $\bar{F}(x') + \bar{\varepsilon} < \bar{F}(z^1)$ ,  $\bar{F}(z^1) + \bar{\varepsilon} < \bar{F}(z^2)$ , ..., i.e.  $\infty \leftarrow \bar{F}(x') + s\bar{\varepsilon} < \bar{F}(z^s)$ , that contradicts boundedness of the vector function  $\bar{F}$  on  $X$ . So, there is a point  $z' \in X^*(\bar{\varepsilon})$  such that  $\bar{F}(z') > \bar{F}(x') + \bar{\varepsilon}$ . By virtue of the condition A2 there is a sequence  $z^N \in X^N$  such that  $z^N \rightarrow z'$ ,  $N \rightarrow \infty$ .

Thus, it holds true

$$\bar{F}(z') = \lim_{k \rightarrow \infty} \bar{F}^{N_k}(z^{N_k}) > \bar{F}(x') + \bar{\varepsilon} = \lim_{k \rightarrow \infty} (\bar{F}^{N_k}(x^{N_k}) + \bar{\varepsilon}^{N_k}).$$

Then  $\bar{F}^{N_k}(z^{N_k}) > \bar{F}^{N_k}(x^{N_k}) + \bar{\varepsilon}^{N_k}$  for all sufficiently large  $k$ , that contradicts the assumption  $x^{N_k} \in X_{N_k}^*(\bar{\varepsilon}^{N_k})$ . The first assertion is proved.

Now let us prove the second statement of the theorem. Assume the contrary, that there exists  $x' \in X^*(\bar{\varepsilon}') \subseteq X^*(\bar{\varepsilon})$  (see Lemma 1) such that  $x' \notin \liminf_{N \rightarrow \infty} X_N^*(\bar{\varepsilon}^N)$ . By condition A2 there exists a sequence  $X^N \ni x^N \rightarrow x'$ . Then there exist its subsequence  $\{x^{N_k}, k=1,2,\dots\}$  such that  $x^{N_k} \notin X_{N_k}^*(\bar{\varepsilon}^{N_k})$  for all sufficiently large  $k$ , so there are points  $z^{N_k} \in X_{N_k}$  such that  $\bar{F}^{N_k}(z^{N_k}) > \bar{F}^{N_k}(x^{N_k}) + \bar{\varepsilon}^{N_k}$ . By compactness of  $X \supseteq X_{N_k} \ni z^{N_k}$ , without loss of generality, we can consider that  $z^{N_k} \rightarrow z'$  and  $x^{N_k} \rightarrow x'$ , thus, by assumption A1,

$$\bar{F}(z') = \lim_{k \rightarrow \infty} \bar{F}^{N_k}(z^{N_k}) \geq \lim_{k \rightarrow \infty} (\bar{F}^{N_k}(x^{N_k}) + \bar{\varepsilon}^{N_k}) = \bar{F}(x') + \bar{\varepsilon} > \bar{F}(x') + \bar{\varepsilon}'.$$

Thus, the point  $x'$  is  $\bar{\varepsilon}'$ -dominated, that contradicts the assumption  $x' \in X^*(\bar{\varepsilon}')$ .

The second statement is proved.

**Remark 1.** In [13], an analog of Theorem 1 was proved under a stronger assumption than A2:  $\lim_{N \rightarrow \infty} X^N = X$ . If the set  $X^N$  is a discrete approximation of the feasible set  $X$ , than condition A2 shows that for the validity of the theorem on convergence of solutions  $X_N^*(\bar{\varepsilon}^N)$  to  $X^*(\bar{\varepsilon})$  it is enough to improve approximations of the feasible set only in the vicinity of approximate Pareto-optimal points  $X^*(\bar{\varepsilon})$ .

**Remark 2.** Theorem 1, in particular, means that

$$X^* = X^*(\bar{0}) \subseteq \liminf_{N \rightarrow \infty} X_N^*(\bar{\varepsilon}^N) \subseteq \limsup_{N \rightarrow \infty} X_N^*(\bar{\varepsilon}^N) \subseteq X^*(\bar{\varepsilon}), \quad (5)$$

where  $X^* = X^*(\bar{0})$  is the set of weakly Pareto-optimal solutions of problem (1). And since the mapping  $\bar{\varepsilon} \rightarrow X^*(\bar{\varepsilon})$  is upper semicontinuous, also at  $\bar{\varepsilon} = 0$ , then relation (5) means that  $\bar{\varepsilon}$ -approximate solutions of problem (2) for sufficiently small  $\bar{\varepsilon}$  approximate the set of weakly Pareto optimal solutions of problem (1).

## CONCLUSIONS

The paper studies a general approximation scheme for solving vector optimization problems. The objective vector function and the feasible set of the problem are substituted by their approximations. Accurate calculating of the objective functions or constraints of the problem is often impossible for finite (or reasonable) time and, therefore, the problem needs to be approximated. This situation is typical for stochastic multiobjective optimization. Approximate problems themselves are solved approximately with some accuracy, i.e. their approximately nondominated solutions are found. It is shown that under natural conditions, uniform convergence of approximation functions and set convergence of feasible domains, the found solutions approximate from above and from below approximately nondominated solutions of the original problem.

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Vector optimization has a great variety of applications. Such problems naturally appear in stochastic optimization, where the optimization problem contains random parameters. In the latter case the vector objective function may include mean value, median, variance, quantiles and other characteristics of the random objective function. The difficulty is that these quantities usually cannot be calculated exactly and are non-convex as functions of variable parameters. These circumstances set additional difficulties for solving corresponding vector optimization problems.

We consider an approximation approach to solving vector optimization problems. The standard approach to such problems is to optimize one criterion under constraints on the others or to scalarize the problem, i.e. to combine all criteria into one scalar criterion. This paper describes a completely different approach, where the feasible set is approximated by a discrete grid (deterministic or random) and the vector function is approximately calculated on this grid. The obtained discrete problem is exactly solved by Pareto type optimization.

The paper studies conditions for convergence of the approximation method when the objective functions and the feasible set are replaced by their more and more fine approximations.

Sufficient conditions are established for Pareto-optimal solutions of the approximate problems to converge in set convergence sense to the Pareto optimal solution of the original problem (with some accuracy). Namely, it is required for the approximate functions to converge uniformly to the original function and for the feasible set approximations (possibly discrete) to converge to elements of the

original feasible set, at least, in the vicinity of the solution. The result confirms a natural hypothesis that the approximation accuracy should increase when approaching to the solution.

**Keywords:** vector optimization, stochastic multicriteria optimization, Pareto optimality, discrete approximation, epsilon-dominance.

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