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S. D. Borysenko, M. Ciarletta, G. Iovane, A. M. Piccirillo, V. Zampoli

Control of hyperbolic equations

(Presented by Corresponding Member of the NAS of Ukraine S. I. Lyashko)

The new analogs of Wendroff's type inequalities for discontinuous functions are considered. The impulse influence on the behaviour of the solutions of hyperbolic equations with nonlinearities of the Lipschitz and Hölder types is investigated.

Introduction. In the present article we found new analogs of the Wendroff inequality for discontinuous functions with finite jumps on some curves $\Gamma_j \subset \mathbb{R}^2_+$ and discontinuities of the Lipschitz and non-Lipschitz types. New conditions of boundedness for solutions of impulsive nonlinear hyperbolic equations are obtained.

Our paper is devoted to a generalization of results [1–15], and it is based on new analogs of a Wendroff type inequality.

We consider some set $D^* \subset \mathbb{R}^2$, where $D^* = D \setminus \Gamma$, $D = \bigcup\limits_j D_j$, $j = 1, 2 \ldots$; $\Gamma = \bigcup\limits_j \Gamma_j$, $\Gamma_j = \{(x, y)\colon \varphi_j(x, y) = 0, j = 1, 2 \ldots\}$, $\Gamma_k \cap \Gamma_{k+1} = \varnothing$, $k = 1, 2, \ldots$;

$\varphi_j(x, y)$ are real-valued continuously differentiable functions such that grad $\varphi_j(x, y) > 0$, for all $j = 1, 2 \ldots$;

$D_1 = \{(x, y)\colon x \geqslant 0, y \geqslant 0, \varphi_1(x, y) < 0\}$;

$D_k = \{(x, y)\colon x \geqslant 0, y \geqslant 0, \varphi_{k-1}(x, y) > 0, \varphi_k(x, y) < 0, \forall\, k > 2, k \in \mathbb{N}\}$;

$G_p = \{(u, v)\colon (x, y) \in D_p, 0 \leqslant u \leqslant x, 0 \leqslant v \leqslant y, p \in \mathbb{N}\}$; μ_{φ_n} is the Lebesgue–Stieltjes measure concentrated on the curves $\{\Gamma_n\}$.

Let us consider a real-valued nonnegative, discontinuous, nondecreasing function $u(x, y)$ in D^*, which has finite jumps on the curves $\{\Gamma_j\}$.

Previous results. Lipschitz type discontinuities.

Proposition 1. *Let us suppose that a function $u(x_1, x_2)$ satisfies the following integro-sum inequality in D^*:*

$$u(x_1, x_2) \leqslant q(x_1, x_2) + g(x_1, x_2) \iint_{G_n} \overline{\psi}(\tau, s) W[u(\tau, s)]\, d\tau ds + \sum_{j=1}^{n-1} \int\limits_{\Gamma_j \cap G_n} \beta_j(x, y) u(x_1, x_2) d\mu_{\varphi_j}, \tag{1}$$

where $q(x_1, x_2)$ is positive and nondecreasing, $g(x_1, x_2) \geqslant 1$, $\beta_j(x_1, x_2) \geqslant 0$, $\overline{\psi}(\tau, s) \geqslant 0$; the function W belongs to the class Φ_1 of functions such that:

1. $W(\sigma_1\sigma_2) \leqslant W(\sigma_1)W(\sigma_2) \qquad \forall\, \sigma_1, \sigma_2 > 0$;
2. $W\colon [0, \infty[\to [0, \infty[, \qquad W(0) = 0$;
3. *W is nondecreasing.*

Moreover, $u(x_1, x_2)$ is a nonnegative discontinuous function, which has finite jumps on the curves $\{\Gamma_j\},\ j = 1, 2, \ldots$.

Then for arbitrary $\{0 < x_1 < \infty, 0 < x_2 < \infty\}$, the following estimate is fulfilled:

$$u(x_1, x_2) \leqslant q(x_1, x_2) g(x_1, x_2) \Psi_i^{-1}\left\{\iint\limits_{D_i} \frac{\overline{\psi}(\tau, s)}{q(\tau, s)} W[q(\tau, s) g(\tau, s)]\, d\tau ds\right\}, \tag{2}$$

$$\forall\, x \in D_i\colon \quad \iint\limits_{D_i} \frac{\overline{\psi}(\tau, s)}{q(\tau, s)} W[q(\tau, s) g(\tau, s)]\, d\tau ds \in \mathrm{Dom}(\Psi_i^{-1}),$$

$$\Psi_0(V) \stackrel{def}{=} \int\limits_1^V \frac{d\sigma}{W(\sigma)}, \qquad \Psi_i(V) \stackrel{def}{=} \int\limits_{C_i}^V \frac{d\sigma}{W(\sigma)}, \qquad i = 1, 2,$$

where

$$V = (V_1, V_2), \qquad \sigma = (\sigma_1, \sigma_2)$$

and

$$C_i = \left(1 + \int\limits_{\Gamma_i \cap G_n} \beta_j(x_1, x_2) g(x_1, x_2) du_{\varphi_i}\right) \Psi_i^{-1}\left\{\iint\limits_{G_{i+1} \setminus G_i} \frac{\overline{\psi}(\tau, s)}{q(\tau, s)} W[q(\tau, s) g(\tau, s)]\, d\tau ds\right\}.$$

Proposition 2. *Let us suppose that a nonnegative discontinuous function $u(x_1, x_2)$, which has finite jumps on the curves $\{\Gamma_j\}$, satisfies inequality* (1), *where the functions W belongs to the class $\overline{\Phi}_1$ of functions such that:*

1. $W\colon [0, \infty[\to [0, \infty[$ *is continuous and nondecreasing*;
2. $\forall\, t > 0, \quad u \geqslant 0, \quad t^{-1}(W(u)) \leqslant W(t^{-1}u)$;
3. $W(0) = 0$.

If all functions q, g, $\overline{\psi}$, β_j satisfy the conditions of Proposition 1, then, for arbitrary $\{0 \leqslant x_1 \leqslant x_1^*,\ 0 \leqslant x_2 \leqslant x_2^*\}$ the following inequality is justified:

$$u(x_1, x_2) \leqslant q(x_1, x_2) g(x_1, x_2) \Psi_i^{-1}\left\{\iint\limits_{D_i} \overline{\psi}(\tau, s) g(\tau, s)\, d\tau ds\right\}, \qquad i = 0, 1, \ldots,$$

where

$$\overline{\Psi_0}(V) = \int\limits_1^V \frac{d\sigma}{W(\sigma)}, \qquad \overline{\Psi}_i(V) = \int\limits_{C_i}^V \frac{d\sigma}{W(\sigma)}, \qquad i = 1, 2, \ldots,$$

$$C_i = \left(1 + \int\limits_{\Gamma_i \cap G_n} \beta_j(x_1, x_2) g(x_1, x_2)\, d\mu_{\varphi_i}\right) \overline{\Psi}_{i-1}^{-1}\left\{\iint\limits_{D_i} \overline{\psi}(\tau, s) g(\tau, s)\, d\tau ds\right\},$$

$$(x_1^*, x_2^*) = x^* = \sup_x \Biggl\{ x\colon \iint\limits_{G_{i+1}\setminus G_i} \overline{\psi}(\tau,s) g(\tau,s) d\tau ds \in \mathrm{Dom}(\overline{\Psi}_i^{-1}(V)),\ i = 1, 2, \ldots \Biggr\}.$$

Non-Lipschitz type discontinuities. Let us consider the following inequality:

$$u(x_1,x_2) \leqslant \varphi(x_1,x_2) + \iint\limits_{G_n} f(\sigma_1,\sigma_2) u^\alpha(\sigma_1,\sigma_2)\, d\sigma_1 d\sigma_2 +$$

$$+ \sum_{j=1}^{n-1} \int\limits_{\Gamma_j \bigcap G_n} \beta_j(x_1,x_2) u^m(x_1,x_2)\, d\mu_{\varphi_j}. \tag{3}$$

Proposition 3. *If the function $u(x_1,x_2)$ satisfies inequality* (3) *with $f \geqslant 0$, $\beta_j \geqslant 0$, $\alpha = 1$, $m > 0$, then the following estimates are valid:*

$$u(x_1,x_2) \leqslant \varphi(x_1,x_2) \prod_{j=1}^{\infty} \int\limits_{\Gamma_j \cap G_{j+1}} \varphi^{m-1}(x_1,x_2) \beta_j(x_1,x_2)\, d\mu_{\varphi_j} \times$$

$$\times \exp\left[\int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1,\sigma_2)\, d\sigma_1 d\sigma_2 \right], \qquad \textit{if} \qquad 0 < m \leqslant 1, \tag{4}$$

$$u(x_1,x_2) \leqslant \varphi(x_1,x_2) \prod_{j=1}^{\infty} \left(1 + \int\limits_{\Gamma_j \cap G_{j+1}} \varphi^{m-1}(x_1,x_2) \beta_j(x_1,x_2)\, d\mu_{\varphi_j} \right) \times$$

$$\times \exp\left[m \int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1,\sigma_2)\, d\sigma_1, d\sigma_2 \right], \qquad \textit{if} \qquad m \geqslant 1. \tag{5}$$

Proposition 4. *If the function $u(x_1,x_2)$ satisfies inequality* (3) *with $\alpha = m > 0$, $m \neq 1$ and the conditions of the above theorem are valid, then the following estimates hold:*

$$u(x_1,x_2) \leqslant \varphi(x_1,x_2) \prod_{j=1}^{\infty} \left(1 + \int\limits_{\Gamma_j \cap G_{j+1}} \beta_j(x_1,x_2) \varphi^{m-1}(x_1,x_2)\, d\mu_{\varphi_j} \right) \times$$

$$\times \left[1 + (1-m) \int\limits_0^{x_1} \int\limits_0^{x_2} \varphi^{m-1}(\sigma_1,\sigma_2) f(\sigma_1,\sigma_2)\, d\sigma_1 d\sigma_2 \right]^{1/(1-m)} \qquad \textit{for} \qquad 0 < m < 1; \tag{6}$$

$$u(x_1,x_2) \leqslant \varphi(x_1,x_2) \prod_{j=1}^{\infty} \left(1 + m \int\limits_{\Gamma_j \cap G_{j+1}} \beta_j(x_1,x_2) \varphi^{m-1}(x_1,x_2)\, d\mu_{\varphi_j} \right) \times$$

$$\times \left[1 - (m-1) \left[\prod_{j=1}^{\infty} \left(1 + m \int\limits_{\Gamma_j \cap G_{j+1}} \beta_j(x_1,x_2) \varphi^{m-1}(x_1,x_2)\, d\mu_{\varphi_j} \right) \right]^{m-1} \times \right.$$

$$\left. \times \int\limits_0^{x_1} \int\limits_0^{x_2} \varphi^{m-1}(\sigma_1,\sigma_2) f(\sigma_1,\sigma_2)\, d\sigma_1 d\sigma_2 \right]^{1/(1-m)} \qquad \textit{for} \qquad m > 1 \tag{7}$$

such that

$$\int_0^{x_1}\int_0^{x_2} \varphi^{m-1}(\sigma_1,\sigma_2) f(\sigma_1,\sigma_2)\, d\sigma_1 d\sigma_2 \leqslant \frac{1}{m},$$
$$\prod_{j=1}^{\infty}\left(1 + m \int\limits_{\Gamma_j\cap G_{j+1}} \beta_j(x_1,x_2)\varphi^{m-1}(x_1,x_2)\, d\mu_{\varphi_j}\right) < \left(1+\frac{1}{m-1}\right)^{1/(1-m)}. \tag{8}$$

Proposition 5. *Let us suppose that a function $u(x_1,x_2)$ satisfies the inequality*

$$u(x_1,x_2) \leqslant \overline{\varphi}(x_1,x_2) + g(x_1,x_2)\iint_{G_n} \overline{\psi}(\tau,s) u^m(\tau,s)\, d\tau ds +$$
$$+\sum_{j=1}^{n-1}\int\limits_{\Gamma_j\cap G_n} \beta_j(x_1,x_2) u^n(x_1,x_2)\, d\mu_{\varphi_j}, \tag{9}$$

where $\overline{\varphi}$ is a positive and nondecreasing function, $g(x_1,x_2) \geqslant 1$, $\beta_j(x_1,x_2) \geqslant 0$, $\overline{\psi}(\tau,s) \geqslant 0$; the function $u(x_1,x_2)$ is nonnegative and has finite jumps on the curves Γ_j, $j = 1,2,\ldots$; $m,n > 0$. With these conditions, the following estimates take place:

$$u(x_1,x_2) \leqslant \overline{\varphi}(x_1,x_2) g(x_1,x_2) (\Psi_i^{-1} \iint_{D_i} \overline{\psi}(\tau,s)\overline{\varphi}^{m-1}(\tau,s) g^m(\tau,s) d\tau ds) \tag{10}$$

$$\forall\,(x_1,x_2)\in D_i\colon \iint_{D_i} \overline{\psi}(\tau,s) g(\tau,s)\, d\tau ds \in \mathrm{Dom}(\Psi_i^{-1}),$$

$$\Psi_i(V) = \int_{C_i}^{V} \sigma^{-m} d\sigma, \qquad C_0 = 1,$$

$$C_i = \left(1 + \int\limits_{\Gamma_i\cap G_{i+1}} \beta_i(x_1,x_2) g^n(x_1,x_2)\overline{\varphi}^{n-1}(x_1,x_2) d\mu_{\varphi_i}\right) \times$$
$$\times\, \Psi_{i-1}^{-1}\left(\iint_{G_{i+1}\setminus G_i} \overline{\psi}(\tau,s)\overline{\varphi}^{m-1}(\tau,s) g^m(\tau,s)\, d\tau ds\right), \qquad i = 1,2,\ldots,$$

where $m = 1$;

$$\Psi_i^{-1}(V) = C_i \exp V, \qquad i = 1,2,\ldots;$$

if $0 < m < 1$ $\forall\,(x_1,x_2) \in D_i$:

$$\Psi_i^{-1}(V) = (C_i + (1-m)V)^{\frac{1}{1-m}}, \qquad i = 1,2,\ldots;$$

if $m > 1$,

$$\Psi_i^{-1}(V) = [C_i - (m-1)V]^{-\frac{1}{m-1}}, \qquad i = 1,2,\ldots,$$

$$\forall\,(x_1,x_2)\in D_i\colon \iint_{D_i}\overline{\psi}(\tau,s)\overline{\varphi}^{m-1}(\tau,s)g^m(\tau,s)\,d\tau ds<\frac{C_i}{m-1}.$$

Applications. Lipschitz type discontinuities. Let us suppose that the evolution of some real processes may be described by hyperbolic partial differential equations with impulse perturbations concentrated on the surfaces

$$\begin{aligned}
&\frac{\partial^2 u(x_1,x_2)}{\partial x_1\partial x_2}=H(x,u(x)), \qquad (x_1,x_2)\in\Gamma_i,\\
&u(x_1,0)=\phi_1(x_1),\\
&u(0,x_2)=\phi_2(x_2),\\
&\phi_1(0)=\phi_2(0),\\
&\Delta u\big|_{(x_1,x_2)\in\Gamma_i}=\int\limits_{\Gamma_i\cap G_n}\beta_i(x_1,x_2)u(x_1,x_2)d\mu_{\phi_i}.
\end{aligned}\tag{11}$$

Here $\Delta u\big|_{(x_1,x_2)\in\Gamma_i}$ are the characteristic values of finite jumps $u(x)(x=(x_1,x_2))$, when the solution of (11) meets the hypersurfaces $\Gamma_i\colon u(x)\bigcap\Gamma_i$.

We investigate equation (11) in the domain $D^*\subset\mathbb{R}^2_+$, which was described in Introduction.

Denote, by $\phi(x_1,x_2)$ the boundary conditions in (11). Every solution of (11), satisfying the boundary conditions, is also a solution of the Volterra integro-sum equation:

$$u(x_1,x_2)=\phi(x_1,x_2)+\iint\limits_{G_n}H(\tau,s,u(\tau,s))\,d\tau ds+\sum_{j=1}^{n-1}\int\limits_{\Gamma_j\cap G_n}\beta_j(x_1,x_2)u(x_1,x_2)\,d\mu_{\varphi_j}.\tag{12}$$

Let us suppose that

$$|H(\tau,s,u(\tau,s))|\leqslant\psi(\tau,s)W[|u(\tau,s)|],\tag{13}$$

where $\psi(\tau,s)\geqslant 0$, $W(\sigma)\in\Phi_1$.

By using the result of Proposition 1, we obtain the following statement:

Proposition 6. *If $H(x,u(x))$ in (11) satisfies condition (13), then, for all solutions of equation (13), the following inequality is valid for all $x_1>0$, $x_2>0$*

$$|u(x_1,x_2)|\leqslant|\phi(x_1,x_2)|\Psi_i^{-1}\left\{\iint\limits_{D_i}\frac{\psi(\tau,s)}{|\phi(\tau,s)|}W[|\phi(\tau,s)|]\,d\tau ds\right\}\tag{14}$$

$$\forall\,x\in D_i\colon\quad \iint\limits_{D_i}\frac{\psi(\tau,s)}{|\phi(\tau,s)|}W[||\phi(\tau,s)|]\,d\tau ds\in\operatorname{Dom}(\Psi_i^{-1}),$$

where

$$\Psi_0(V_1)=\int\limits_1^{V_1}\frac{d\sigma}{W(\sigma)},\qquad \Psi_i(V_1)=\int\limits_{C_i}^{V_1}\frac{d\sigma_1}{W(\sigma_1)}\qquad i=1,2,\ldots$$

$$C_i = \left(1 + \int\limits_{\Gamma_i \cap G_{i+1}} \|\beta_i(x_1, x_2)|\, d\mu_{\phi_i}\right) \Psi_{i-1}^{-1}\left(\iint\limits_{G_{i+1}\setminus G_i} \frac{\psi(\tau, s)}{\|\phi(\tau, s)|} W[\|\phi(\tau, s)|]\, d\tau ds\right).$$

By using the result of Proposition 2, we obtain:

Proposition 7. *If the function H satisfies* (13)*, where W belongs to the class of functions $\overline{\Phi}_1 \colon W \in \overline{\Phi}_1$, then all solutions of equation* (11) *satisfy such estimate:*

$$\|u(x_1, x_2)| \leqslant \|\phi(x_1, x_2)|\overline{\Psi}_j^{-1}\left(\iint\limits_{D_j} \psi(\tau, s) d\tau ds\right), \qquad \forall\, j = 0, 1, \ldots,$$

where

$$\overline{\Psi}_0(V) = \int\limits_1^V \frac{d\sigma}{W(\sigma)}, \qquad \overline{\Psi}_i(V) = \int\limits_{C_1}^{V_1} \frac{d\sigma}{W(\sigma)}, \qquad i = 1, 2, \ldots,$$

$$C_i = \left(1 + \int\limits_{\Gamma_i \cap G_{i+1}} \|\beta_i(x_1, x_2)|\, d\mu_{\phi_i}\right) \overline{\Psi}_{i-1}^{-1}\left(\iint\limits_{G_{i+1}\setminus G_i} \psi(\tau, s)\, d\tau ds\right),$$

$$\forall\, x\colon \quad 0 < x < x^*\colon \quad x^* = \sup_x\left\{x\colon \iint\limits_{G_{i+1}\setminus G_i} \overline{\psi}(\tau, s)\, d\tau ds \in \mathrm{Dom}(\overline{\Psi}_i^{-1}),\ i = 1, 2, \ldots\right\}.$$

From Proposition 3, the next result follows:

Proposition 8. *Let us suppose that the following conditions take place:*

A) $\|H(x_1, x_2, u(x_1, x_2))| \leqslant f(x_1, x_2)\|u(x_1, x_2)|^\alpha$, $\alpha = \mathrm{const} > 0$, *where f is a continuous nonnegative function in $\mathbb{R}_+^2$.*

B) $\exists\, M = \mathrm{const} > 0\colon \|\phi(x_1, x_2)| \leqslant M$. *Then, for solutions of equations* (11)*, the following estimates take place:*

1. $$\|u(x_1, x_2)| \leqslant M\prod_{j=1}^{\infty}\left(1 + \int\limits_{\Gamma_i \cap G_{j+1}} \|\beta_j(x_1, x_2)| d\mu_{\phi_i}\right) \exp\left[\int\limits_0^{x_1}\int\limits_0^{x_2} f(\sigma_1, \sigma_2)\, d\sigma_1 d\sigma_2\right],$$

   if $\alpha = 1$;

2. $$\|u(x_1, x_2)| \leqslant M\prod_{j=1}^{\infty}\left(1 + \int\limits_{\Gamma_i \cap G_{j+1}} \|\beta_j(x_1, x_2)| d\mu_{\phi_i}\right) \times$$

   $$\times\left[1 + (1 - \alpha)M^{\alpha - 1}\int\limits_0^{x_1}\int\limits_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2\right]^{1/(1-\alpha)}, \qquad \text{if} \qquad 0 < \alpha < 1;$$

3. $$\|u(x_1, x_2)| \leqslant M\prod_{j=1}^{\infty}\left(1 + \int\limits_{\Gamma_i \cap G_{j+1}} \|\beta_j(x_1, x_2)| d\mu_{\phi_i}\right)\Bigg\{1 + (\alpha - 1)M^{\alpha - 1} \times$$

$$\times \left[\prod_{j=1}^{\infty} \left(1 + \int\limits_{\Gamma_i \cap G_{j+1}} \|\beta_j(x_1, x_2)|\, d\mu_{\phi_i} \right) \right]^{\alpha-1} \int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 \Bigg\}^{-1/(\alpha-1)}$$

for $\alpha > 1$ *and arbitrary* $(x_1, x_2) \in D^*$ *such that*

$$\int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2 < \left\{ (\alpha - 1) M^{\alpha-1} \left[\prod_{j=1}^{\infty} \left(1 + \int\limits_{\Gamma_i \cap G_{j+1}} \|\beta_j(x_1, x_2)|\, d\mu_{\phi_i} \right) \right]^{\alpha-1} \right\}^{-1}.$$

From Proposition 8, we obtain the following statement:

Proposition 9. *Let us consider the following conditions for equation* (11)*:*

1. $\|H(x_1, x_2, u(x_1, x_2))| \leqslant \psi(x_1, x_2)\|u(x_1, x_2)|^{\alpha}$*;*

2. $\exists\, M = \mathrm{const} > 0 : \|\varphi(x_1, x_2)| \leqslant M$*;*

3. $\exists\, \xi,\ \eta$*:*

$$\prod_{j=1}^{\infty} \left(1 + \int\limits_{\Gamma_i \cap G_{j+1}} \|\beta_j(x_1, x_2)| d\mu_{\varphi_i} \right) \leqslant \xi < \infty;$$

$$\int\limits_0^{x_1} \int\limits_0^{x_2} \psi(\sigma_1, \sigma_2)\, d\sigma_1 d\sigma_2 \leqslant \eta < \infty.$$

Then all solutions $u(x_1, x_2)$ *of equation* (11) *are bounded for* $0 < \alpha \leqslant 1$*. If additionally*

$$\prod_{j=1}^{\infty} \left(1 + \int\limits_{\Gamma_i \cap G_{j+1}} \|\beta_j(x_1, x_2)| d\mu_{\varphi_i} \right) < \frac{M^{1-\alpha}}{(\alpha - 1)\eta},$$

all solutions of equation (11) *are bounded also for* $\alpha > 1$*.*

Hölder type discontinuities. Let us consider such problem:

$$\begin{aligned}
&\frac{\partial^2 u(x_1, x_2)}{\partial x_1 \partial x_2} = F(x, u(x)), \qquad x = (x_1, x_2) \in \Gamma_i, \\
&u(x_1, 0) = \psi_1(x_1), \\
&u(0, x_2) = \psi_2(x_2), \\
&\psi_1(0) = \psi_2(0), \\
&\Delta u|_{x \in \Gamma_i} = \int\limits_{\Gamma_i \cap G_n} \beta_i(x) u^m(x) d\mu_{\phi_i}, \qquad m > 0.
\end{aligned} \tag{15}$$

In (15) we suppose that the boundary conditions $\psi(x_1, x_2)$ are bounded, i. e.

$$|\psi(x_1, x_2)| \leqslant M = \mathrm{const} < \infty,$$

and $F(x, u)$ satisfies the estimate:

$$|F(x, u)| \leqslant f(x_1, x_2)|u(x_1, x_2)|^{\alpha}, \tag{16}$$

with $f \geqslant 0$, $\alpha = \mathrm{const} > 0$.

By using Propositions 3 and estimates A)–D), we obtain the following statement:

Proposition 10. *Let us suppose that, for problem* (15), *the assumptions in Introduction about curves* Γ_i, *domains* B_k, G_k *and functions* φ_k *are valid. Moreover, let* F *satisfy inequality* (16).

I. Then the following estimates take place:

A') $\alpha = 1, \quad m \leqslant 1 \quad \Rightarrow$

$$\Rightarrow |u(x_1, x_2)| \leqslant M \prod_{j=1}^{\infty} \left(1 + M^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)| d\mu_{\varphi_j}\right) \exp\left[\int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2)\, d\sigma_1 d\sigma_2\right],$$

B') $\alpha = 1, \quad m \geqslant 1 \quad \Rightarrow$

$$\Rightarrow |u(x_1, x_2)| \leqslant M \prod_{j=1}^{\infty} \left(1 + M^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)|\, d\mu_{\varphi_j}\right) \exp\left[m \int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2)\, d\sigma_1 d\sigma_2\right],$$

C') $0 < \alpha = m < 1 \quad \Rightarrow |u(x_1, x_2)| \leqslant M \prod_{j=1}^{\infty} \left(1 + M^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)| d\mu_{\varphi_j}\right) \times$

$$\times \left[1 + (1-m) M^{m-1} \int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2) d\sigma_1 d\sigma_2\right]^{1/(1-m)},$$

D') $\alpha = m > 1 \quad \Rightarrow \quad |u(x_1, x_2)| \leqslant M \prod_{j=1}^{\infty} \left(1 + mM^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^(*)(x_1, x_2)|\, d\mu_{\varphi_j}\right) \times$

$$\times \left[1 - (m-1) M^{m-1} \left[\prod_{j=1}^{\infty} \left(1 + mM^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)|\, d\mu_{\varphi_j}\right)\right]^{m-1} \times \right.$$

$$\left. \times \int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2)\, d\sigma_1 d\sigma_2 \right]^{-1/(m-1)},$$

for all $x_1,\ x_2 > 0$:

$$\int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2)\, d\sigma_1 d\sigma_2 \leqslant \frac{1}{mM^{m-1}}, \tag{17}$$

$$\prod_{j=1}^{\infty} \left(1 + mM^{m-1} \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^*(x_1, x_2)|\, d\mu_{\varphi_j}\right) < \left(1 + \frac{1}{m-1}\right)^{1/(1-m)}. \tag{18}$$

II. All solutions $u(x_1, x_2)$ *of (15) are bounded in cases* A')–C') *only if the values*

$$\prod_{j=1}^{\infty} \left(1 + \int\limits_{\Gamma_j \cap G_{j+1}} |\beta_j^*|\, d\mu_{\varphi_j}\right), \qquad \int\limits_0^{x_1} \int\limits_0^{x_2} f(\sigma_1, \sigma_2)\, d\sigma_1 d\sigma_2$$

are bounded. Referring to case D'), (17), (18) guarantee conditions of boundedness for all solutions of (15).

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NTU of Ukraine "Kiev Polytechnical Institute"
Second University of Naples, Italy
University of Salerno, Italy

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С. Д. Борисенко, М. Чарлетта, Ж. Іоване, А. М. Пічірілло, В. Замполі

Імпульсні управління гіперболічними рівняннями

Розглянуто нові аналоги нерівностей Вендрофа для розривних функцій. Досліджено вплив імпульсного збурення на поведінку розв'язків гіперболічних рівнянь з нелінійностями як ліпшицевого так і гельдерового характеру.

С. Д. Борисенко, М. Чарлетта, Ж. Иоване, А. М. Пичирилло, В. Замполи

Импульсное управление гиперболическими уравнениями

Рассмотрены новые аналоги неравенств Вендрофа для разрывных функций. Исследовано воздействие импульсных возмущений на поведение решений гиперболических уравнений с нелинейностями как липшицевого так и гельдерового характера.