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Mironenko's Limit Test for Numerical and Power Series

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Предельный признак Мироненко для числовых и степенных рядов

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Гранична ознака Мироненко для числових та степеневих рядів

The report describes the derivation and application of Mironenko's test for series with positive terms. Mironenko's test has the same limitations that have d'Alembert's ratio test and Cauchy's radical test. But in practical sense Mironenko's test has some advantages, moreover, it can be used for power series research.
Keywords: series, limit test, d'Alembert's ratio test, Cauchy's radical test, limit.

В статье описывается вывод и применение признака Мироненко для рядов с положительными членами. Признак Мироненко имеет те же ограничения, что и признак Даламбера и радикальный признак Коши. Однако в практическом смысле признак Мироненко имеет ряд преимуществ и, кроме того, он может быть использован при исследовании степенных рядов.
Ключевые слова: ряд, предельный признак, признак Даламбера, радикальный признак Коши, предел.

У статті описується походження і застосування ознаки Мироненко для рядів з позитивними членами. Ознака Мироненко має ті ж обмеження, які мають ознака Даламбера та радикальна ознака Коші. Але в практичному сенсі тест Мироненко має деякі переваги. Крім того, він може бути використаний для дослідження степеневих рядів.
Ключові слова: ряд, гранична ознака, ознака Даламбера, радикальна ознака Коші, межа.

Introduction

Tests of the convergence of series with positive terms such as Cauchy's and D'alembert's tests are usually applied to quickly converging series. In the limit form for any series $\sum_{n=1}^{\infty} u_n$, $u_n > 0$ these tests can be written down in the forms of inequalities:

$$\lim_{n\to\infty} \sqrt[n]{u_n} \le 1,\ u_n \ge 0\ ,$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} \le 1\ ,\ u_n > 0\,.$$

The first inequality is called Cauchy's radical test, the second one is D'alembert's test. For both tests the sign of equality means uncertainty in a question of convergence of the series.

1 Mironenko's limit comparison test for series with positive terms

Mironenko's limit comparison test for series with positive terms is the union of d'Alembert's ratio test and Cauchy's radical test [1].

It can be obtained by using arguments similar to d'Alembert's ratio test or Cauchy's radical test. We will proceed accordingly and show that d'Alembert's ratio test and Cauchy's radical test are followed from Mironenko's test.

We consider the series with positive terms $\sum_{n=1}^{\infty} u_n$ and compare it with the geometric progression series $\sum_{n=1}^{\infty} q^n$, which converges when $q < 1$ and diverges when $q \geq 1$. Assume that inequality $u_n \leq q^n$ is satisfied for all n or starting with some arbitrary number N_o. In consequence of continuous and monotonic increase of the logarithmic function $\ln x$ when $x > 0$ we have the inequality $\ln u_n \leq \ln q^n = n \ln q = np, \;\; p = \ln q$. From the last inequality we have got

$$\frac{\ln u_n}{n} \leq p . \qquad (1)$$

Remark. According to the comparison test of convergence for series with positive terms, if $\frac{\ln u_n}{n} > p = \ln q, \; q > 1,$ then series $\sum_{n=1}^{\infty} u_n$ diverges.

When $n \to \infty$, then according to the necessary comparison test for series with positive terms we have $u_n \to 0$ or $\ln u_n \to -\infty$. This means that under the limit sign the uncertainty $\left\{\frac{\infty}{\infty}\right\}$ takes place and all conditions of L'Hospital's rule are fulfilled:

$$\lim_{n\to\infty} \frac{(\ln u_n)'}{n'} = \lim_{n\to\infty} (\ln u_n)' = p . \qquad (2)$$

So we have came to the equality

$$\lim_{n\to\infty} (\ln u_n)' = p , \qquad (3)$$

which expresses Mironenko's limit test: if $p < 0$, then the series $\sum_{n=1}^{\infty} u_n$ converges, if $p > 0$, the series diverges, and finally if $p = 0$, the test can not answer the question of the series convergence.

Cauchy's radical test follows from the inequalities (1) taking into account the remark

$$\frac{\ln u_n}{n} \leq \ln q \Rightarrow \ln \sqrt[n]{u_n} \leq \ln q \Rightarrow \sqrt[n]{u_n} \leq q$$

and applying subsequent transition to the limit at $n \to \infty$.

If there be given the limit [2]

$$\lim_{n\to\infty} \sqrt[n]{u_n} = q, \quad u_n \geq 0$$

and $q < 1$, then the series $\sum_{n=1}^{\infty} u_n$ converges, if $q > 1$, the series diverges, and finally if $q = 1$, the test can not answer the question of the series convergence.

Let us use Mironenko's limit test in the form $\lim_{n\to\infty} \frac{u_n'}{u_n} = p$ to derive d'Alembert's ratio test.

At large value of n the derivative can be represented as $u_{n+1} - u_n$ or $u_n - u_{n-1}$. In any of these embodiments we can get d'Alembert's ratio test.

Substitute this expression in the previous inequality and we obtain

$$\lim_{n\to\infty}\frac{u_{n+1}-u_n}{u_n}=p\Rightarrow\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=1+p \text{ или } \lim_{n\to\infty}\frac{u_n-u_{n-1}}{u_n}=p\Rightarrow\lim_{n\to\infty}\frac{u_{n-1}}{u_n}=1-p.$$

We can demonstrate on simple examples the advantages of Mironenko's test in comparison with d'Alembert's and Cauchy's tests.

1. The test convenient to use when the general term of the series has a lot of factors. Due to the properties of the logarithmic function multiplication and division are converted to the sum and difference, which is convenient to compute the derivative.

Example 1. Need to establish convergence or divergence of the series

$$\sum_{n=1}^{\infty}\frac{(n+2)\ln(n+1)}{2^{n+3}\sqrt{n^2+3n+5}}.$$

Here $\ln u_n=\ln(n+2)+\ln\ln(n+1)-(n+3)\ln 2-\frac{1}{2}\ln\left(n^2+3n+5\right)$,

$(\ln u_n)'=\frac{1}{n+2}+\frac{1}{\ln(n+1)}-\ln 2-\frac{2n+3}{2\left(n^2+3n+5\right)}$, $\lim_{n\to\infty}(\ln u_n)'=-\ln 2<0.$ The series convergence.

2. Test can be used when the general term has the factorial function $n!$. At large n possible to use asymptotic formula for the derivative of the function $n!$: $(n!)'=n!\ln n$. This formula allows to apply test to Cauchy's series as well as d'Alembert's series which contain as multipliers factorial function.

Example 2. Need to establish convergence or divergence of the series $\sum_{n=1}^{\infty}\frac{e^n}{n!}$.

Here $\ln u_n=\ln e^n-\ln n!=n-\ln n!$,

$(\ln u_n)'=1-\frac{n!\ln n}{n!}=1-\ln n$, $\lim_{n\to\infty}(\ln u_n)'=-\infty>0.$ The series convergence.

3. Test can be used to power series without distinguishing if series refers to d'Alembert's ratio test or Cauchy's radical test.

Example 3. Need to establish the radius of convergence of the power series.

$$\sum_{n=1}^{\infty}\frac{x^{2n}}{3^{n+3}\sqrt{n^2+3n+5}}.$$

Here $\ln u_n=\ln x^{2n}-(n+3)\ln 3-\frac{1}{2}\ln\left(n^2+3n+5\right)$,

$(\ln u_n)'=2\ln|x|-\ln 3-\frac{2n+3}{2\left(n^2+3n+5\right)}$, $\lim_{n\to\infty}(\ln u_n)'=2\ln|x|-\ln 3=\ln\frac{|x|^2}{3}<0\Rightarrow\frac{|x|^2}{3}<1$ The radius of convergence is $-\sqrt{3}<x<\sqrt{3}$.

Example 4. Need to establish the radius of convergence of the power series. $\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n)!}$.

Here $\ln u_n=\ln x^{2n-1}-\ln(2n)!=(2n-1)\ln|x|-\ln(2n)!$,

$(\ln u_n)'=2\ln|x|-\frac{((2n)!)'}{(2n)!}=2\ln|x|-\ln 2n$, $\lim_{n\to\infty}(\ln u_n)'=-\infty<0$ The radius of convergence is $-\infty<x<+\infty$.

4. Mironenko's test has the same limitations that have d'Alembert's ratio test and Cauchy's radical test.

Example 5. Need to establish convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!e^n}$.

Here $\ln u_n = n\ln n - n\ln e - \ln n!$, $(\ln u_n)' = \ln n + 1 - 1 - \frac{(n!)'}{n!} = 0$, $\lim_{n\to\infty}(\ln u_n)' = 0$. Test can not answer the question of convergence of the series.

Conclusions

1. Mironenko's limit comparison test for series with positive terms is obtained. The test has the same opportunities as d'Alembert's ratio test and Cauchy's radical test, but it has some advantages in comparison with them.
2. d'Alembert's ratio test and Cauchy's radical test are follow from Mironenko's test.

Appendix. Calculating the derivative of $n!$ we will proceed from the Stirling's formula

$$n! = \sqrt{2\pi n}n^n e^{-n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \frac{139}{51840n^3} + o(n^{-4})\right),$$

which confine only a first approximation. Need to logarithm and differentiate $\ln u! = \ln\sqrt{2\pi n} + \ln n^n + \ln e^{-n} + \ln\left(1 + \frac{1}{12n}\right)$.

We assume approximatione $\ln(1 + 1/12n) \approx 1/12n$, when

$$\ln u! = \ln\sqrt{2\pi} + \frac{1}{2}\ln n + n\ln n - n + \frac{1}{12n},\quad (\ln n!)' = \ln n + \frac{1}{2n} - \frac{1}{12n^2},$$

Finally, $(n!)' = n!\left(\ln n + \frac{1}{2n} - \frac{1}{12n^2}\right) \underset{n>>1}{\approx} n!\ln n$

References

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RESUME

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Mironenko's Limit Test For Numerical And Power Series

Background: Mironenko's limit comparison test for series with positive terms is the union of d'Alembert's ratio test and Cauchy's radical test.

Materials and methods: Methods of differential calculus, L'Hospital's rule, arguments similar to d'Alembert's ratio test and Cauchy's radical test were used in the paper.

Results: One more limit comparison test for series with positive terms is obtained.

Conclusion: Mironenko's limit comparison test for series with positive terms has the same opportunities as d'Alembert's ratio test and Cauchy's radical test, but it has also some advantages in comparison with them.

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