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An Advanced Necessary Test For Convergent Number Series And Some Consequences

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Усиленный необходимый признак сходимости рядов и следствия

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Підсилена необхідна ознака збіжності рядів та наслідки

In the paper an advanced necessary test for convergence of number series with non-negative terms is formulated and proved. The test is written in the form $\lim\limits_{n\to\infty} nu_n = 0$. This formula can be got by using Dzeta–function and has more possibilities than the usual test $\lim\limits_{n\to\infty} u_n = 0$. The new test gives new representations of the necessary test for convergence of non-negative number series.
Keywords: series, convergence, divergence, comparison test, advanced necessary test, limit, Cauchy's integral test.

В работе сформулирован и доказан усиленный необходимый признак сходимости числовых рядов с неотрицательными членами в виде $\lim\limits_{n\to\infty} nu_n = 0$. Признак получен на основе признака сравнения с обобщенно гармоническим рядом. Признак имеет более широкие возможности по сравнению с обычным необходимым признаком сходимости $\lim\limits_{n\to\infty} u_n = 0$. Применение усиленного необходимого признака сходимости к интегральному признаку Коши дает новые формы представлений необходимого признака сходимости рядов с неотрицательными членами.
Ключевые слова: ряд, сходимость, расходимость, признаки сравнения, усиленный необходимый признак, предел, интегральный признак Коши.

У роботі сформульована і доведена посилена необхідна ознака збіжності чисельних рядів з невід'ємними членами у виді $\lim\limits_{n\to\infty} nu_n = 0$. Ця формула отримана на підставі ознаки порівняння з гармонійним рядом і має значно більше можливостей у порівнянні з традиційною необхідною ознакою збіжності $\lim\limits_{n\to\infty} u_n = 0$. Посилена ознака збіжності разом з інтегральною ознакою Коші відкривають можливість отримати цілу низку нових представлень ознаки збіжності рядів з невід'ємними членами.
Ключові слова: ряд, збіжність, ознака порівняння, посилена необхідна ознака, границя, інтегральна ознака Коші.

Introduction

Let us recall the content of the necessary test for convergent number series with non-negative terms. If the series $\sum_{n=1}^{\infty} u_n, (u_n \geq 0)$ converges, then the limit of its common term tends to zero at $n \to \infty$, i.e. $\lim_{n\to\infty} u_n = 0$. Usually this test is used to determine the divergence of series, because it is not sufficient criterion, but is the necessary one [1].

Among sufficient tests, perhaps, the most common is the comparison test, which is usually used in two forms: in the limit form and in the finite form. In both cases, members of the investigated series $\sum_{n=1}^{\infty} u_n$ are compared with members of the known series $\sum_{n=1}^{\infty} v_n$. We are interested in the limit sufficient test, where $\lim_{n\to\infty} \frac{u_n}{v_n}$ is considered. If the series $\sum_{n=1}^{\infty} v_n$, $(v_n > 0)$ converges and the value of $\lim_{n\to\infty} \frac{u_n}{v_n}$ is equal to $C < \infty$ (in a particular case $C = 0$) then the series $\sum_{n=1}^{\infty} u_n$ converges. If the series $\sum_{n=1}^{\infty} v_n$, $(v_n > 0)$ diverges and the value of $\lim_{n\to\infty} \frac{u_n}{v_n}$ is equal to $C > 0$ then the series $\sum_{n=1}^{\infty} u_n$ diverges also [1-2].

There are three standard series, which are used in the comparison test. They are the harmonic one with the common term $v_n = 1/n$, the generalized harmonic one (so called zeta function) with the common term $v_n = \frac{1}{n^{\alpha}}$ and the geometric progression one with the common term $v_n = q^n$.

The limit comparison test with respect to the generalized harmonic series is considered in the paper. This allows intensify the necessary test for convergent number series with non-negative terms and expand significantly the possibilities of the usual necessary test $\lim_{n\to\infty} u_n = 0$.

1 Deduction of the advanced necessary test for convergent number series

Let us represent the generalized harmonic series in the form

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\beta}} = \begin{cases} \beta > 0 - \textit{for convergent series} \\ \beta \leq 0 - \textit{for divergent series}, \end{cases}$$

where instead of the usual parameter α the new parameter β is used: $\alpha = 1 + \beta$, and apply the limit comparison test for an arbitrary number series: $\sum_{n=1}^{\infty} u_n$, $(u_n \geq 0)$: $\lim_{n\to\infty} \frac{u_n}{1/n^{1+\beta}} = \lim_{n\to\infty} n^{1+\beta} \cdot u_n$. If there exists the finite value limit

$$\lim_{n\to\infty} n^{\beta+1} \cdot u_n = C < \infty \qquad (1)$$

and $\beta > 0$, then the series $\sum_{n=1}^{\infty} u_n,\ (u_n \geq 0)$ converges. In other cases at $\beta \leq 0$ the limit is equal to infinity (or does not exist) the series $\sum_{n=1}^{\infty} u_n,\ (u_n \geq 0)$ diverges [1-3].

The test (1) may be represented in the other form if it is applied by L'Hospital's rule. Indeed according to the necessary test for convergent number series $u_n \to 0$ at $n \to \infty$, Then $u_n^{-1} = \frac{1}{u_n} \to \infty$ at $n \to \infty$ and

$$\lim_{n\to\infty} n^{\beta} \lim_{n\to\infty} n \cdot u_n = \lim_{n\to\infty} n^{\beta} \lim_{n\to\infty} \frac{n}{u_n^{-1}} = \left\{\frac{\infty}{\infty}\right\} = \lim_{n\to\infty} n^{\beta} \lim_{n\to\infty} \frac{n'}{\left(u_n^{-1}\right)'} = \lim_{n\to\infty} \frac{n^{\beta}}{\left(u_n^{-1}\right)'}.$$

As the result the test (1) will get the form

$$\lim_{n\to\infty} \frac{n^{\beta}}{\left(u_n^{-1}\right)'} = C < \infty \tag{2}$$

If the series $\sum_{n=1}^{\infty} u_n,\ (u_n > 0)$ converges then the equality (1) takes place at $\beta > 0$. The test (1) can be written in the form

$$\lim_{n\to\infty} n^{\beta+1} u_n = \lim_{n\to\infty} n^{\beta} \cdot n u_n = \lim_{n\to\infty} n^{\beta} \cdot \lim_{n\to\infty} n u_n = C < \infty. \tag{3}$$

Suppose that $\lim_{n\to\infty} n u_n \neq 0$. In this case the equality (3) will be correct, when $\lim_{n\to\infty} n^{\beta}$ is equal to a number at $\beta > 0$. But there is no such $\beta > 0$, because $\lim_{n\to\infty} n^{\beta} = \infty$ for any $\beta > 0$. Thus, when $\lim_{n\to\infty} n u_n \neq 0$, then the series $\sum_{n=1}^{\infty} u_n$ diverges. So we have the new advanced necessary test for convergent number series with non-negative terms $\sum_{n=1}^{\infty} u_n,\ (u_n \geq 0)$:

$$\lim_{n\to\infty} n u_n = 0. \tag{4}$$

By the same way the formula (2) is followed the another form of the new advanced necessary test for convergent number series with non-negative terms $\sum_{n=1}^{\infty} u_n,\ (u_n \geq 0)$:

$$\lim_{n\to\infty} \frac{1}{\left(u_n^{-1}\right)'} = 0. \tag{5}$$

These two forms of the new advanced necessary test for convergent number series have more opportunities with respect to the casual necessary test for convergent number series $\lim_{n\to\infty} u_n = 0$. For example, for the number series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ the casual necessary test $(\lim_{n\to\infty} u_n = 0)$ gives $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. This means the series must converge, but it diverges. At the same time according to the formulae (4) and (5) the new advanced necessary test gives: $\lim_{n\to\infty} n u_n = \lim_{n\to\infty} \frac{n}{\sqrt{n}} = \infty$ and $\lim_{n\to\infty} \frac{1}{\left(u_n^{-1}\right)'} = \lim_{n\to\infty} 2\sqrt{n} = \infty$. It means the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

2 The new advanced necessary test for convergent number series is applied to Cauchy's integral test

According to Cauchy's integral test if the function $u(x)$ is nonnegative and decreasing at $x \geq 1$ then the series $\sum_{n=1}^{\infty} u(n)$ converges if and only if the improper integral $\int_{1}^{+\infty} u(x)dx$ converges.

Let the series $\sum_{n=1}^{\infty} u(n)$ converges therefore the improper integral $\int_{1}^{+\infty} u(x)dx$ converges also. The last is integrated by parts:

$$\int_{1}^{+\infty} u(x)dx = \lim_{x\to+\infty} xu(x) - u(1) - \int_{1}^{+\infty} xu'(x)dx.$$

According to the new advanced necessary test for convergent number series (4) $\lim_{x\to+\infty} xu(x) = 0$. Since the left-hand side integral converges then the right-hand side integral $\int_{1}^{+\infty} xu'(x)dx$ must converge also and according to Cauchy's integral test the series $\sum_{n=1}^{\infty} nu'_n$ converges.

Take into account, that in this case the advanced necessary test (4) for convergent number series $\sum_{n=1}^{\infty} nu'_n$ takes the form: $\lim_{n\to\infty} n^2 u'_n = 0$. We see that convergence of the series $\sum_{n=1}^{\infty} nu'_n$ is necessary and sufficient condition for convergence of the series $\sum_{n=1}^{\infty} u_n$.

Indeed, let us assume that series $\sum_{n=1}^{\infty} nu'_n$ converges therefore the improper integral $\int_{1}^{+\infty} xu'(x)dx$ converges also. The last is integrated by parts:

$$\int_{1}^{+\infty} xu'(x)dx = \lim_{x\to+\infty} xu(x) - u(1) - \int_{1}^{+\infty} u(x)dx.$$

Since $\lim_{x\to+\infty} xu(x) = 0$ is necessary and sufficient condition for convergence of the series $\sum_{n=1}^{\infty} u_n$, then the series $\sum_{n=1}^{\infty} u_n$ converges. If the condition $\lim_{x\to+\infty} xu(x) = 0$ is not satisfied then the series $\sum_{n=1}^{\infty} u_n$ diverges and the series $\sum_{n=1}^{\infty} nu'_n$ diverges also.

Let us apply this reasoning to the series $\sum_{n=1}^{\infty} nu'(n)$, given that $\lim_{n\to\infty} n^2 u'_n = 0$ is necessary and sufficient condition for convergence of the series $\sum_{n=1}^{\infty} nu'(n)$. Let the series $\sum_{n=1}^{\infty} nu'(n)$ converges therefore the improper integral $\int_{1}^{+\infty} xu'(x)dx$ converges also. The last is integrated by parts:

$$\int_1^{+\infty} xu'(x)dx = \frac{1}{2}\left(\lim_{x\to+\infty} x^2u'(x) - u'(1)\right) - \frac{1}{2}\int_1^{+\infty} x^2u''(x)dx.$$

Let $\lim_{n\to\infty} n^2u'_n = 0$. Since the left-hand side integral converges then the right-hand side integral $\int_1^{+\infty} x^2u''(x)dx$ converges also and according to Cauchy's integral test the series $\sum_{n=1}^{\infty} n^2u''(n)$ converges. These arguments can be repeated k times. As result two important statements are obtained.

1. In order to the series with non-negative terms $\sum_{n=1}^{\infty} u(n)$ converges it is necessary and sufficient, that the series $\sum_{n=1}^{\infty} n^k u^{(k)}(n)$, $(k = 1,2,...)$ converges too.

2. The series with non-negative terms $\sum_{n=1}^{\infty} u(n)$ converges if and only if the advanced necessary test in the form $\lim_{n\to+\infty} n^{k+1}u^{(k)}(n) = 0$, $(k = 1,2,...)$ takes place.

These statements can be demonstrated by the example of the convergent series $\sum_{n=1}^{\infty} \frac{1}{n\ln^2 x}$ with the common term $u_n = \frac{1}{n\ln^2 x}$. If the common term is differentiated, then

$$u'_n = \frac{-\ln n - 2}{n^2\ln^3 n} \underset{n\to\infty}{\to} -\frac{1}{n^3\ln^2 n}, \quad u''_n = \frac{2\ln^2 n + 6\ln n + 6}{n^3\ln^4 n} \underset{n\to\infty}{\to} \frac{2}{n^3\ln^2 n}.$$

The series $\sum_{n=1}^{\infty} n^k u^{(k)}(n)$ ($k = 1,2$) have the next forms:

$$\sum_{n=1}^{\infty} nu'(n) = \sum_{n=1}^{\infty} \frac{-\ln n - 2}{n\ln^3 n}, \quad \sum_{n=1}^{\infty} n^2u''(n) = \sum_{n=1}^{\infty} \frac{2\ln^2 n + 6\ln n + 6}{n\ln^4 n}.$$

For sufficiently large values of n the common term of each of the series behaves like the common term of the initial series, namely $u_n \approx nu'_n \approx n^2u''_n \underset{n\to\infty}{\to} \frac{1}{n\ln^2 n}$.

Findings

1. An advanced necessary test for convergence of number series with non-negative terms is formulated and proved. The test is much more powerful than the casual necessary test.
2. The advanced necessary test gives rich possibilities to get equivalent converged series with respect to the given series. These series cannot be obtained by applying the usual way in the series theory.

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