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Integration of a Rational Fraction of a Special Kind

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Интегрирование рациональной дроби специального вида

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Інтегрування раціонального дробу спеціального вигляду

In the paper new representations for the functions $\cos^n x$ and $\sin^n x$ are obtained, which are effective for evaluation of many integrals, especially $K_n = \int (x^2 + a^2)^{-n} dx$. It is found a primitive of the integral K_n in the explicit form, while in the mathematical analysis this integral is calculated by means of a recurrent equation. Besides, the received representation gives new representation of Wallace's formula.
Keywords: integral, rational fraction, sine, cosine, identity, primitive.

В статье получены представления для функций $\sin^n x$ и $\cos^n x$, которые оказались эффективными для вычисления интегралов вида $K_n = \int (x^2 + a^2)^{-n} dx$. Найдена первообразная интеграла K_n в явном виде, в то время как в математическом анализе этот интеграл вычисляется с помощью рекуррентного соотношения. Кроме того, полученное представление дает новое представление формулы Валлиса.
Ключевые слова: интеграл, рациональная дробь, синус, косинус, тождество, первообразная.

У статті отримано представлення для функцій $\sin^n x$ і $\cos^n x$, які ефективні для обчислення інтегралів вигляду $K_n = \int (x^2 + a^2)^{-n} dx$. Знайдена первісна інтегралу K_n в явному вигляді, тоді як у математичному аналізі інтеграл обчислюють за допомогою рекурентного співвідношення. Крім того, представлення використано для отримання нового представлення формули Валіса.
Ключові слова: інтеграл, раціональний дріб, синус, косинус, тотожність, первісна.

Introduction

Integration of the fraction of the kind $[\int \frac{Mt+N}{(t^2+pt+q)^n} dt,\ n \in N,\ \frac{p^2}{4} - q < 0$ presents the well-known problem in mathematical analysis [1]. After the appropriate replacement of a variable and the transformation of the numerator of the fraction the problem is reduced to the integral K_n:

$$K_n = \int \frac{dx}{(x^2 + a^2)^n},\ n > 1. \tag{1}$$

In the case $n = 1$ integration is not a problem. It is the tabular integral:

$$K_1 = \int \frac{dt}{a^2 + t^2} = \frac{1}{a} a \tan \frac{t}{a} + C.$$

In the general case the integral (1) is calculated by means of a recurrent equality [1-3] in which K_n is expressed by K_{n-1}.

$$K_\lambda = \frac{t}{2a^2(\lambda-1)(t^2+a^2)^{\lambda-1}} + \frac{2\lambda-3}{2a^2(\lambda-1)} K_{\lambda-1}. \quad (2)$$

Knowing K_1 we find from the recursion (2) the next integral $K_2 = \frac{t}{2a^2(t^2+a^2)} + \frac{1}{2a^2}K_1$. And from K_2 we will find K_3 and so on.

In the paper we will consider other way of evaluation of the integral K_λ. For this purpose we introduce some important and useful equalities. It will be done in the next section.

1. A new representation of the functions $\cos^n x$ and $\sin^n x$.

Let's consider the problem of expressing of the functions $\cos^n x$ and $\sin^n x$ by a linear combination of the functions $\cos kx$ and $\sin kx$.

For this Euler's formula $e^{ix} = \cos x + i\sin x$ can be used. Thus, for cosine function $\cos x$ we have the following representation $\cos x = (e^{ix}+e^{-ix})/2$, where i is imaginary unit. Then, $\cos^n x = \frac{1}{2^n}\left(e^{ix}+e^{-ix}\right)^n = \frac{e^{-nix}}{2^n}\left(1+e^{2ix}\right)^n$. Applying Newton's binomial $\left(a+b\right)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$, where – $C_n^k = \frac{n!}{k!(n-k)!}$ are binomial factors [4-5], we will receive

$$\cos^n x = \frac{e^{-nix}}{2^n}\sum_{k=0}^{n} C_n^k e^{2ix(n-k)} = \frac{1}{2^n}\sum_{k=0}^{n} C_n^k e^{ix(n-2k)} = \frac{1}{2^n}\sum_{k=0}^{n} C_n^k (\cos\left(x(n-2k)\right) + i\sin\left(x(n-2k)\right)).$$

Now we will show the last term is null:

$$\sum_{k=0}^{n} C_n^k \sin\left(x(n-2k)\right) = 0. \quad (3)$$

At $k = n/2$ it is obvious. For $k \neq n/2$ because to the property $C_n^k = C_n^{n-k}$ of the factors C_n^k to everyone k in the sum (3) there is existed the same term of opposite on a sign.

As a result we will receive a simple formula:

$$\cos^n x = \frac{1}{2^n}\sum_{k=0}^{n} C_n^k \cos\left(\left(n-2k\right)x\right), \quad (4)$$

This representation can be considered as a generalization of the formula $\cos^2 x = (1+\cos 2x)/2$ for any $n = 2k$. Really, in the case $n = 2$ from (4) the well-known formula is followed $\cos^2 x = \frac{1}{2^2}\sum_{k=0}^{2} C_2^k \cos\left((2-2k)x\right) = \frac{1+\cos 2x}{2}$.

Let's move out from the sum (4) the term with $k = n/2$:

$$\cos^n x = \frac{C_n^{n/2}}{2^n} + \frac{1}{2^n}\sum_{\substack{k=0 \\ k\neq n/2}}^{n} C_n^k \cos\left((n-2k)x\right). \quad (5)$$

The term with $k = n/2$ corresponds to a case when a value of the cosine function is unit.

The next step is to get the expression for $\sin^n x$ like to (4) - (5). We will transform the function $\sin x = (e^{ix} - e^{-ix})/2i$ which follows from Euler's formula like the cosine function:

$$\sin^n x = \frac{e^{-ixn}}{2^n i^n}\left(e^{i2x}-1\right)^n = \frac{e^{-ixn}}{2^n i^n}\sum_{k=0}^{n} C_n^k(-1)^k e^{i2x(n-k)} = \frac{1}{2^n i^n}\sum_{k=0}^{n} C_n^k(-1)^k e^{ix(n-2k)}.$$

Let's represent imaginary unit i in the exponential form $i = e^{i\pi/2}$ and after this we will apply Euler's formula

$$\sin^n t = \frac{1}{2^n} e^{-in\frac{\pi}{2}} \sum_{k=0}^{n} C_n^k(-1)^k e^{it(n-2k)} = \frac{1}{2^n}\sum_{k=0}^{n} C_n^k(-1)^k e^{it(n-2k)-in\frac{\pi}{2}},$$

$$e^{it(n-2k)-in\frac{\pi}{2}} = \cos\left(t(n-2k)-n\frac{\pi}{2}\right) + i\sin\left(t(n-2k)-n\frac{\pi}{2}\right).$$

It is not difficult to show $\sum_{k=0}^{n} C_n^k(-1)^k \sin\left(t(n-2k)-n\frac{\pi}{2}\right)=0$. Therefore, there is a similar formula as (4) only for $\sin^n x$:

$$\sin^n x = \frac{1}{2^n}\sum_{k=0}^{n}(-1)^k C_n^k \cos\left(x(n-2k)-n\frac{\pi}{2}\right). \tag{6}$$

This formula can be considered as a generalization of the formula for $\sin^2 x = (1-\cos 2x)/2$ for any $n = 2k$.

$$\sin^2 x = \frac{1}{2^2}\sum_{k=0}^{2}(-1)^k C_2^k \cos(x(2-2k)-\pi) = \frac{1}{2^2}\sum_{k=0}^{2}(-1)^{k+1} C_2^k \cos(x(2-2k)) = \frac{1-\cos 2x}{2}.$$

We can move out the term with $k = n/2$ from the sum (6) like we did it for the cosine function:

$$\sin^n x = \frac{(-1)^{n/2} C_n^{n/2}}{2^n}\cos\left(n\frac{\pi}{2}\right) + \frac{1}{2^n}\sum_{\substack{k=0\\ k\neq n/2}}^{n}(-1)^k C_n^k \cos\left(x(n-2k)-n\frac{\pi}{2}\right). \tag{7}$$

Let's consider some properties of the formulas (4) - (7). First of all, we will write down the equalities for even and odd values of n. When $n = 2m$ is an even number the formula (5) looks as follows:

$$\cos^{2m} x = \frac{C_{2m}^m}{2^{2m}} + \frac{1}{2^{2m}}\sum_{\substack{k=0\\ k\neq m}}^{2m} C_{2m}^k \cos(2(m-k)x),$$

Using the property $C_{2m}^k = C_{2m}^{2m-k}$ of binomial factors we will get:

$$\cos^{2m} x = \frac{C_{2m}^m}{2^{2m}} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1} C_{2m}^k \cos(2(m-k)x). \tag{8}$$

If the number $n = 2m+1$ is odd the property of symmetry of binomial factors gives the equality:

$$\cos^{2m+1} x = \frac{1}{2^{2m}}\sum_{k=0}^{m} C_{2m+1}^k \cos((2m+1-2k)x). \tag{9}$$

Similar formulas we will receive for a sine function:

$$\sin^{2m} x = \frac{C_{2m}^m}{2^{2m}} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^{k+m} C_{2m}^k \cos(2(m-k)x), \tag{10}$$

$$\sin^{2m+1} x = \frac{1}{2^{2m}}\sum_{k=0}^{m}(-1)^{k+m} C_{2m+1}^k \sin(x(2m+1-2k)). \tag{11}$$

Here the equality $\cos\left(x-(2m+1)\frac{\pi}{2}\right)=(-1)^m\sin x$ was used.

2 The integral K_n

Now we return to the In integral K_n (1) in which we will make the substitution $t=a\cdot\tan x$, $dt=a\cos^{-2}xdx, (t^2+a^2)^n=a^{2n}\cos^{-2n}x$. The result we will write down in terms of the variable t again:

$$K_n=\frac{1}{a^{2n-1}}\int\cos^{2(n-1)}tdt. \tag{12}$$

In the integral (12) we will substitute the expression (8):

$$K_n=\frac{1}{a^{2n-1}}\int\cos^{2(n-1)}tdt=\frac{C_{2(n-1)}^{n-1}}{2^{2(n-1)}a^{2n-1}}\int dt+\frac{1}{2^{2n-3}a^{2n-1}}\sum_{k=0}^{n-2}C_{2(n-1)}^{k}\int\cos\left(2(n-1-k)t\right)dt.$$

Integration becomes a very simple:

$$K_n=\frac{1}{2^{2(n-1)}a^{2n-1}}\left(C_{2(n-1)}^{n-1}t+\sum_{k=0}^{n-2}C_{2(n-1)}^{k}\frac{\sin\left(2t((n-1-k)\right)}{n-1-k}\right)+C. \tag{13}$$

This is one of the main result of the paper. Comparison of the formula (13) with the recursion (2) allows to make a conclusion on some advantage of our theory. For example, if there will be a necessity to calculate a definite integral (especially at $n>>1$) it easier to make it using the formula (13), instead of (2).

Besides, our approach allows to continue logically researches and to receive, for example, Wallice's formula, meanwhile it is impossible to do by of the classical method.

Some sequence values of K_n are given bellow according toour theory.

$$K_2=\frac{1}{2^2a^3}\left(\sum_{k=0,k\neq1}^{2}C_2^k\frac{\sin\left(2t(1-k\right)}{2(1-k)}+C_2^1t\right)=\frac{1}{4a^3}\left(2t+\sin 2t\right)+C,$$

$$K_3=\frac{1}{2^4a^5}\left(\sum_{k=0,k\neq2}^{4}C_4^k\frac{\sin\left(2t(2-k\right)}{2(2-k)}+C_4^2t\right)=\frac{1}{32a^5}\left(12t+\sin 4t+8\sin 2t\right)+C \text{ etc.}$$

3. Integrals I_n and J_n. Wallace's formula

Let's apply the theory to the integrals $I_n=\int_0^{\pi/2}\cos^n tdt$ $J_n=\int_0^{\pi/2}\sin^n tdt$. We use the representations of the functions $\cos^n x$ in the forms (8) – (9) and the functions $\sin^n x$ in the forms (10) – (11). For the function (8) we take into account $\int_0^{\pi/2}dt\cos\left(2(m-k)t\right)=0$.

$$I_{2m}=\frac{C_{2m}^m}{2^{2m}}\int_0^{\pi/2}dt+\frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}C_{2m}^k\int_0^{\pi/2}dt\cos\left(2(m-k)t\right)=\frac{\pi}{2^{2m+1}}C_{2m}^m,$$

$$I_{2m+1}=\frac{1}{2^{2m}}\sum_{k=0}^{m}C_{2m+1}^k\int_0^{\pi/2}dt\cos\left((2m+1-2k)t\right)=\frac{1}{2^{2m}}\sum_{k=0}^{m}C_{2m+1}^k\frac{\sin\left((2m+1-2k)\frac{\pi}{2}\right)}{2m+1-2k}=\frac{1}{2^{2m}}\sum_{k=0}^{m}\frac{C_{2m+1}^k}{2m+1-2k}.$$

For function (10) we also to take into account equality $\int_0^{\pi/2} dt\cos(2(m-k)t) = 0$.

$$J_{2m} = \frac{C_{2m}^m}{2^{2m}}\int_0^{\pi/2} dt + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}(-1)^{k+m}C_{2m}^k\int_0^{\pi/2} dt\cos(2(m-k)t) = \frac{\pi}{2^{2m+1}}C_{2m}^m.$$

It is not wonder, that, $I_{2m} = J_{2m}$. The equality follows from the obvious geometrical equality $\int_0^{\pi/2} \cos^n tdt = \int_0^{\pi/2} \sin^n tdt$. At last,

$$J_{2m+1} = \frac{1}{2^{2m}}\sum_{k=0}^{m} (-1)^{k+m}C_{2m+1}^k\int_0^{\pi/2} dt\sin((2m+1-2k))t = -\frac{1}{2^{2m}}\sum_{k=0}^{m} (-1)^{k+m}C_{2m+1}^k\frac{\cos((2m+1-2k)\pi/2)-1}{2m+1-2k} =$$

$$= -\frac{1}{2^{2m}}\sum_{k=0}^{m} (-1)^{k+m}C_{2m+1}^k\frac{\cos((m-k)\pi+\pi/2)-1}{2m+1-2k} = \frac{1}{2^{2m}}\sum_{k=0}^{m}\frac{(-1)^{k+m}C_{2m+1}^k}{2m+1-2k}.$$

Integration of the obvious inequality $\sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x$ over the interval, $x \in [0, \frac{\pi}{2}]$ yields the integral inequalities:

$$I_{2m+1} \le I_{2m} \le I_{2m-1}. \tag{14}$$

We use the formulas for the integral I_n in the cases $I_{2m+1}, I_{2m}, I_{2m-1}$:

$$\sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x\ , \tag{15}$$

that is equavivalent to the expanded form:

$$\frac{1}{C_{2m}^m}\sum_{k=0}^{m}\frac{(-1)^{m+k}C_{2m+1}^k}{2m+1-2k} \le \frac{\pi}{2} \le -\frac{4}{C_{2m}^m}\sum_{k=0}^{m-1}\frac{(-1)^{m+k+1}C_{2m-1}^k}{2m-1-2k}.$$

For example, $m=1$ $\frac{8}{3} \le \pi \le 4 \Rightarrow 2{,}667 \le \pi \le 4$, $m=2$ $\frac{128}{45} \le \pi \le \frac{32}{9} \Rightarrow 2{,}844 \le \pi \le 3{,}556$ etc.

The sequence of integrals monotonously $\{I_m\}$ decreases, and $\lim_{m\to\infty} I_m = 0$. From here follows, that there is a limit:

$$\frac{\pi}{2} = \lim_{m\to\infty}\frac{1}{C_{2m}^m}\sum_{k=0}^{m}\frac{(-1)^{m+k}C_{2m+1}^k}{2m+1-2k}. \tag{16}$$

Let's compare this formula with Wallace's formula [1]

$$\frac{\pi}{2} = \lim_{n\to\infty}\frac{1}{2n+1}\left[\frac{(2n)!}{(2n-1)!}\right]^2 \tag{17}$$

It is easy for receiving from inequalities (14) which can be written down in an explicit form, using a recurrent parity between integrals $I_n = \frac{n-1}{n}I_{n-2}$

$$\frac{(2n)!}{(2n+1)!} \le \frac{(2n-1)!}{(2n)!}\frac{\pi}{2} \le \frac{(2n-2)!}{(2n-1)!} \Rightarrow \frac{1}{2n+1}\left[\frac{(2n)!}{(2n-1)!}\right]^2 \le \frac{\pi}{2} \le \frac{1}{2n}\left[\frac{(2n)!}{(2n-1)!}\right]^2. \tag{18}$$

The sign means $m!!$ product of the first natural m numbers taking into account parity, for example $(2m)!! = 2m\cdot(2m-2)\cdot\ldots\cdot 2$.

Comparing formulas (14) and (15) and inequalities (13) and (16), we will receive:

$$\frac{1}{C_{2m}^m}\sum_{k=0}^{m}\frac{(-1)^{m+k}C_{2m+1}^k}{2m+1-2k} = \frac{1}{2m+1}\left[\frac{(2m)!}{(2m-1)!}\right]^2,\ \frac{C_{2m}^m}{2^{2m}} = \frac{(2m-1)!!}{(2m)!!}.$$

These equalities are easily checked for the various m .

The APPENDIX. Identity $(\cos^2 x+\sin^2 x)^m = 1$.

The approach allows to write down often used identity in $(\cos^2 x+\sin^2 x)^m = 1$ a kind:

$$\cos^{2m} x+\sin^{2m} x = 2\cdot\frac{(2m-1)!!}{(2m)!!}+\frac{1}{2^{2m-1}}\sum_{k=0}^{m-1} C_{2m}^{k}\left(1+(-1)^{k+m}\right)\cos\left(2(m-k)x\right).$$

In particular, at we have $m=1$ identity, $\cos^2 x+\sin^2 x = 1$ at - $m = 2,3$ often used equalities:

$$\cos^4 x+\sin^4 x = \frac{1}{2^3}\frac{(4)!}{(2!)^2}+\frac{1}{2^3}\sum_{k=0}^{1} C_4^k\left(1+(-1)^{k+2}\right)\cos\left(2(2-k)x\right) = \frac{3}{4}+\frac{1}{4}\cos 4x = 1-2\sin^2 x\cos^2 x,$$

$$\cos^6 x+\sin^6 x = \frac{1}{2^5}\frac{(6)!}{(3!)^2}+\frac{1}{2^5}\sum_{k=0}^{2} C_6^k\left(1+(-1)^{k+3}\right)\cos\left(2(3-k)x\right) = \frac{5}{8}+\frac{3}{8}\cos(4x) = 1-3\sin^2 x\cos^2 x.$$

Conclusions

1. Representations of the functions $\cos^n x$ and $\sin^n x$ in the form of liniar combinations of the functions $\cos kx$ and $\sin kx$ are received.

2. New representation of the left part of identity from $(\cos^2 x+\sin^2 x)^n = 1$ which a number of known parities follows is received $(\cos^2 x+\sin^2 x)^n = 1$.

3. Cjomparison of the formula (13) with a recurrent equation (2) allows to make a conclusion about some advantage of our theory. For example, if you need to calculate a definite integral (especially when $n >> 1$).

4. Our approach gives a new representation for well-known Wallace’s formula.

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