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A Liouville comparison principle for solutions of quasilinear singular parabolic second-order partial differential inequalities

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The purpose of this work is to obtain a Liouville comparison principle for entire weak solutions (u, v) of quasilinear singular parabolic second-order partial differential inequalities of the form $u_t - A(u) - |u|^{q-1}u \geqslant v_t - A(v) - |v|^{q-1}v$ in the half-space $\mathbb{S} = \mathbb{R}_+ \times \mathbb{R}^n$, where $n \geqslant 1$, $q > 0$ and the differential operator A satisfies the α-monotonicity condition. Model examples of the operator A are the well-known p-Laplacian operator, defined by the relation $\Delta_p(w) := \operatorname*{div}_x(|\nabla_x w|^{p-2}\nabla_x w)$, and its well-known modification, defined by $\widetilde{\Delta}_p(w) := = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial w}{\partial x_i}\right|^{p-2}\frac{\partial w}{\partial x_i}\right)$.

1. Introduction and definition. The purpose of this work is to obtain a Liouville comparison principle of elliptic type for entire weak solutions (u, v) of quasilinear singular parabolic second-order partial differential inequalities of the form

$$u_t - A(u) - |u|^{q-1}u \geqslant v_t - A(v) - |v|^{q-1}v \tag{1}$$

in the half-space $\mathbb{S} = (0, +\infty) \times \mathbb{R}^n$, where $n \geqslant 1$ is a natural number and $q > 0$ is a real number. Typical examples of the differential operator A and the main subjects in our study are the p-Laplacian operator defined by

$$\Delta_p(w) := \operatorname*{div}_x(|\nabla_x w|^{p-2}\nabla_x w) \tag{2}$$

and its well-known modification, see, e. g., [1, p. 155], defined by

$$\widetilde{\Delta}_p(w) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial w}{\partial x_i}\right|^{p-2}\frac{\partial w}{\partial x_i}\right). \tag{3}$$

Note that the Laplacian operator is a special case of (2) or (3) with $p = 2$. Also, it is important to note that if $u = u(t, x)$ and $v = v(t, x)$ satisfy the inequalities

$$u_t \geqslant A(u) + |u|^{q-1}u, \tag{4}$$

$$v_t \leqslant A(v) + |v|^{q-1}v, \tag{5}$$

then the pair (u, v) satisfies inequality (1). Thus, all the results obtained in this work for solutions of (1) are valid for the corresponding solutions of system (4)–(5).

As the entire solutions of inequalities (1), (4), and (5), we understand the solutions of these inequalities defined in the whole half-space $\mathbb{S}$. Moreover, as the Liouville results of elliptic type for the solutions of inequalities (1), (4), and (5) in the half-space $\mathbb{S}$, we understand Liouville-type results which, in their formulations, have no restrictions on the behavior of solutions of these inequalities on the hyper-plane $t = 0$. We would like to underline that we impose neither growth conditions on the behavior of solutions of inequalities (1), (4), and (5) or on that of any of their partial derivatives at infinity.

Let A be a differential operator defined by the formula

$$A(w) := \sum_{i=1}^{n} \frac{d}{dx_i} A_i(t, x, \nabla_x w), \tag{6}$$

where $n \geqslant 1$ and $(t, x) \in \mathbb{S}$. Assume that the functions $A_i(t, x, \xi)$, $i = 1, \ldots, n$, satisfy the Carathéodory conditions in $\mathbb{S} \times \mathbb{R}^n$; namely, they are continuous in ξ at almost all $(t, x) \in \mathbb{S}$ and measurable in t, x at all $\xi \in \mathbb{R}^n$.

Definition 1. Let $n \geqslant 1$ and $\alpha > 1$. The operator A given by (6) is said to be α-monotone if $A_i(t, x, 0) = 0$, $i = 1, \ldots, n$, at almost all $(t, x) \in \mathbb{S}$ and the inequalities

$$0 \leqslant \sum_{i=1}^{n} (\xi_i^1 - \xi_i^2)(A_i(t, x, \xi^1) - A_i(t, x, \xi^2)), \tag{7}$$

$$\left(\sum_{i=1}^{n} (A_i(t, x, \xi^1) - A_i(t, x, \xi^2))^2 \right)^{\alpha/2} \leqslant \mathcal{K} \left(\sum_{i=1}^{n} (\xi_i^1 - \xi_i^2)(A_i(t, x, \xi^1) - A_i(t, x, \xi^2)) \right)^{\alpha-1}, \tag{8}$$

where $\mathcal{K}$ is some positive constant, hold for all ξ^1, $\xi^2 \in \mathbb{R}^n$ and almost all $(t, x) \in \mathbb{S}$.

Note that condition (7) is the well-known monotonicity condition in PDE theory, while condition (8) is the proper α-monotonicity condition for evolution differential operators considered first in [2], see also [3]. Note also that the α-monotonicity condition (8) in the case where $\xi^2 = 0$ is, in turn, a special case of the very general growth condition for quasilinear differential operators considered first in [4].

We now present the algebraic inequalities, from which it follows immediately that the p-Laplacian operator Δ_p and its modification $\widetilde{\Delta}_p$ satisfy the α-monotonicity condition for $\alpha = p$ and $1 < p \leqslant 2$.

Lemma 1. *Let $n \geqslant 1$ and $1 < \alpha \leqslant 2$, and let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be arbitrary vectors in $\mathbb{R}^n$ of length $|a| = \sqrt{a_1^2 + \cdots + a_n^2}$ and $|b| = \sqrt{b_1^2 + \cdots + b_n^2}$. Then there exists a positive constant $\mathcal{K}$ such that the inequalities*

$$\left(\sum_{i=1}^{n} (a_i |a|^{\alpha-2} - b_i |b|^{\alpha-2})^2 \right)^{\alpha/2} \leqslant \mathcal{K} \left(\sum_{i=1}^{n} (a_i - b_i)(a_i |a|^{\alpha-2} - b_i |b|^{\alpha-2}) \right)^{\alpha-1} \tag{9}$$

and

$$\left(\sum_{i=1}^{n} (a_i |a_i|^{\alpha-2} - b_i |b_i|^{\alpha-2})^2 \right)^{\alpha/2} \leqslant \mathcal{K} \left(\sum_{i=1}^{n} (a_i - b_i)(a_i |a_i|^{\alpha-2} - b_i |b_i|^{\alpha-2}) \right)^{\alpha-1} \tag{10}$$

hold.

Remark 1. The statements of Lemma 1 were proved in [2], see also [5].

It is important to note that there exist α-monotone differential operators with arbitrary degeneracy. For example, the weighted p-Laplacian operator defined by

$$\Delta_p^*(w) := \operatorname*{div}_x(d(t,x)|\nabla_x w|^{p-2}\nabla_x w), \tag{11}$$

see, e. g., [6, p. 55], where $d(t,x)$ is an arbitrary function measurable, non-negative, and uniformly bounded in $\mathbb{S}$, is α-monotone with $\alpha = p$ for any fixed $1 < p \leqslant 2$.

Below, we consider inequality (1) with the differential operator A, which is α-monotone.

Definition 2. Let $n \geqslant 1$, $q > 0$ and $\alpha > 1$, and let the operator A be α-monotone. A pair (u,v) of functions $u = u(t,x)$ and $v = v(t,x)$ is called an entire weak solution of inequality (1) in $\mathbb{S}$, if these functions are defined and measurable in $\mathbb{S}$, belong to the function space $L_{q,\mathrm{loc}}(\mathbb{S})$, with u_t, $v_t \in L_{1,\mathrm{loc}}(\mathbb{S})$ and $|\nabla_x u|^\alpha$, $|\nabla_x v|^\alpha \in L_{1,\mathrm{loc}}(\mathbb{S})$, and satisfy the integral inequality

$$\int\limits_{\mathbb{S}} \left[u_t\varphi + \sum_{i=1}^{n} \varphi_{x_i} A_i(t,x,\nabla_x u) - \varphi|u|^{q-1}u \right] dtdx \geqslant$$

$$\geqslant \int\limits_{\mathbb{S}} \left[v_t\varphi + \sum_{i=1}^{n} \varphi_{x_i} A_i(t,x,\nabla_x v) - \varphi|v|^{q-1}v \right] dtdx \tag{12}$$

for every non-negative function $\varphi \in C^\infty(\mathbb{S})$ with compact support in $\mathbb{S}$, where $C^\infty(\mathbb{S})$ is the space of all functions defined and infinitely differentiable in $\mathbb{S}$.

Analogous definitions of solutions of inequalities (4) and (5), as special cases of inequality (1) for $v \equiv 0$ or $u \equiv 0$, follow immediately from Definition 2.

2. Results.

Theorem 1. *Let $n \geqslant 1$, $2 \geqslant \alpha > 1$ and $1 < q \leqslant \alpha - 1 + \alpha/n$, let the operator A be α-monotone, and let (u,v) be an entire weak solution of inequality* (1) *in $\mathbb{S}$ such that $u \geqslant v$. Then $u = v$ in $\mathbb{S}$.*

As we have observed above, since any solutions $u = u(t,x)$, $v = v(t,x)$ of inequalities (4), (5) are a solution (u,v) of inequality (1), then the following statement is a direct corollary of Theorem 1.

Theorem 2. *Let $n \geqslant 1$, $2 \geqslant \alpha > 1$ and $1 < q \leqslant \alpha - 1 + \alpha/n$, let the operator A be α-monotone, and let $u = u(t,x)$ be an entire weak solution of inequality* (4) *and $v = v(t,x)$ be an entire weak solution of inequality* (5) *in $\mathbb{S}$ such that $u \geqslant v$. Then $u = v$ in $\mathbb{S}$.*

We call the results in Theorems 1 and 2, which evidently have a comparison principle character, a Liouville-type comparison principle, since, in particular cases where either $u \equiv 0$ or $v \equiv 0$, it becomes a Liouville-type theorem of elliptic type for the solutions of inequality (5) or (4), respectively. In addition, since we impose no conditions in Theorems 1 and 2 on the behavior of the entire solutions of inequality (1) and system (4), (5) on the hyper-plane $t = 0$, we can formulate, as direct corollaries of the results in Theorems 1 and 2, the following comparison principle, which can be called, in turn, a Fujita comparison principle, for the weak solutions of the Cauchy problem for inequality (1) and system (4), (5).

Theorem 3. *Let $n \geqslant 1$, $2 \geqslant \alpha > 1$ and $1 < q \leqslant \alpha - 1 + \alpha/n$, let the operator A be α-monotone, and let (u,v) be an entire weak solution of the Cauchy problem, with arbitrary initial data for $u = u(t,x)$ and $v = v(t,x)$, for inequality* (1) *in $\mathbb{S}$ such that $u \geqslant v$. Then $u = v$ in $\mathbb{S}$.*

Remark 2. The initial data for $u = u(t,x)$ and $v = v(t,x)$ in Theorem 3 may be different.

Theorem 4. *Let $n \geqslant 1$, $2 \geqslant \alpha > 1$ and $1 < q \leqslant \alpha - 1 + \alpha/n$, let the operator A be α-monotone, and let $u = u(t,x)$ be a weak solution of the Cauchy problem, with arbitrary initial* *data, for inequality* (4) *and* $v = v(t,x)$ *be a weak solution of the Cauchy problem, with arbitrary initial data, for inequality* (5) *in* $\mathbb{S}$ *such that* $u \geqslant v$*. Then* $u = v$ *in* $\mathbb{S}$.

Note that the results in Theorems 1–4 are sharp, and that the hypotheses on the parameter α in these theorems in fact force α to be greater than $2n/(n+1)$. The sharpness of the results for $n \geqslant 1$, $2 \geqslant \alpha > 1$ and $q > \alpha - 1 + \alpha/n \geqslant 1$ follows, for example, from the existence of non-trivial non-negative self-similar entire solutions of the equation

$$w_t - \Delta_p(w) = |w|^{q-1}w \tag{13}$$

for $p = \alpha$ in $\mathbb{S}$, see, e. g., [7]. One can find a Fujita-type theorem on the non-existence of non-trivial non-negative entire solutions of the Cauchy problem for Eq. (13), which was obtained as a very interesting generalization of the famous blow-up result in [8] to quasilinear parabolic equations. The sharpness of the results for $n \geqslant 1$, $2 \geqslant \alpha > 1$, and $0 < q \leqslant 1$ follows, for example, from the fact that the function $u(t,x) = e^t$ is a positive entire classical super-solution of (13) in $\mathbb{S}$.

In addition to Theorem 1, we obtain an *a priori* estimate for solutions of (1). So, in what follows, for $q > 1$ and $2 \geqslant \alpha > 1$, let

$$\omega = \frac{\alpha(q-1)}{q-\alpha+1} \tag{14}$$

and

$$P(R) = \{(t,x) \in \mathbb{S}\colon t^{2/\omega} + |x|^2 < R^{2/\omega}\}$$

for all $R > 0$. It is clear that $0 < \omega \leqslant 2$.

Theorem 5. *Let* $n \geqslant 1$*,* $2 \geqslant \alpha > 1$*,* $q > \max\{1, \alpha - 1 + \alpha/n\}$*, and* $\alpha - 1 > \nu > 0$*, and let the operator* A *be* α*-monotone. Then there exists a constant* C *such that the inequality*

$$\int\limits_{P(R)} (u-v)^{q-\nu} dtdx \leqslant CR^{\frac{n+\omega}{\omega} - \frac{q-\nu}{q-1}} \tag{15}$$

holds for every entire weak solution (u,v) *of* (1) *in* $\mathbb{S}$ *such that* $u \geqslant v$*, for all* $R > 0$.

The following statement is a simple corollary of Theorem 5.

Corollary 1. *Let* $n \geqslant 1$*,* $2 \geqslant \alpha > 1$*, and* $q > \max\{1, \alpha - 1 + \alpha/n\}$*, and let the operator* A *be* α*-monotone. Then there exists no entire weak solution* (u,v) *of* (1) *in* $\mathbb{S}$ *such that* $u - v$ *is bounded below by a positive constant.*

As we have noted above, since any solutions $u = u(t,x)$, $v = v(t,x)$ of inequalities (4), (5) are a solution (u,v) of inequality (1), similar results for solutions $u = u(t,x)$, $v = v(t,x)$ of inequalities (4), (5) follow directly from Theorem 5 and Corollary 1. We note that all the results obtained are new. To prove them, we essentially use the concept of α-monotonicity for differential operators and continue to develop an approach in [9], [10], the elliptic analog of which was proposed in [2].

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Принцип порівнювання Ліувілля для розв'язків квазілінійних, сингулярних, параболічних нерівностей другого порядку в частинних похідних

Встановлюється принцип порівнювання Ліувілля для цілих, слабких розв'язків (u, v) *квазілінійних, сингулярних, параболічних нерівностей другого порядку в частинних похідних виду* $u_t - A(u) - |u|^{q-1}u \geqslant v_t - A(v) - |v|^{q-1}v$ *в напівпросторі* $\mathbb{S} = \mathbb{R}_+ \times \mathbb{R}^n$, *де* $n \geqslant 1$, $q > 0$ *і диференціальний оператор* A *задовольняє умову* α*-монотонності. Модельними прикладами оператора* A *є добре відомий* p*-лапласіан, визначений співвідношенням* $\Delta_p(w) := \operatorname*{div}_x(|\nabla_x w|^{p-2}\nabla_x w)$, *та його добре відома модифікація, визначена співвідношенням* $\widetilde{\Delta}_p(w) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial w}{\partial x_i}\right|^{p-2}\frac{\partial w}{\partial x_i}\right)$.

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Принцип сравнения Лиувилля для решений квазилинейных, сингулярных, параболических неравенств второго порядка в частных производных

Устанавливается принцип сравнения Лиувилля для целых, слабых решений (u, v) *квазилинейных, сингулярных, параболических неравенств второго порядка в частных производных вида* $u_t - A(u) - |u|^{q-1}u \geqslant v_t - A(v) - |v|^{q-1}v$ *в полупространстве* $\mathbb{S} = \mathbb{R}_+ \times \mathbb{R}^n$, *где* $n \geqslant 1$, $q > 0$ *и дифференциальный оператор* A *удовлетворяет условию* α*-монотонности. Модельными примерами оператора* A *являются хорошо известный* p*-лапласиан, определенный соотношением* $\Delta_p(w) := \operatorname*{div}_x(|\nabla_x w|^{p-2}\nabla_x w)$, *и его хорошо известная модификация, определенная соотношением* $\widetilde{\Delta}_p(w) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial w}{\partial x_i}\right|^{p-2}\frac{\partial w}{\partial x_i}\right)$.