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ON PATHOLOGICAL SOLUTIONS TO AN OPTIMAL BOUNDARY CONTROL PROBLEM FOR LINEAR PARABOLIC EQUATION

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Изучена задача оптимального управления для линейного параболического уравнения с неограниченными коэффициентами в главной части эллиптического оператора. Особенностью данного уравнения является то, что матрица потока является кососимметрической, а ее коэффициенты принадлежат к пространству L^2. Показано, что поставленная задача имеет единственное решение, которое нельзя получить, используя L^∞ аппроксимированных задач.

Ключевые слова: параболическое уравнение, оптимальное управление, патологическое решение, неограниченные коэффициенты.

Досліджено задачу оптимального керування для лінійного параболічного рівняння з необмеженими коефіцієнтами в головній частині еліптичного оператора. Особливість даного рівняння полягає в тому, що матриця потоку є кососиметричною, а її коефіцієнти належать до простору L^2. Показано, що поставлена задача керування має єдиний розв’язок, який не можна досягти через границю оптимальних розв’язків для L^∞ апроксимованих задач.

Ключові слова: параболічне рівняння, оптимальне керування, патологічний розв’язок, необмежені коефіцієнти.

INTRODUCTION

In this paper we deal with the following optimal control problem (OCP) for a linear elliptic equation with unbounded coefficients in the main part of elliptic operator

$$I(u,y)=\|y-y_d\|^2_{L^2(0,T;H_0^1(\Omega;\Gamma_D))}+\|u-u_d\|^2_{L^2(0,T;L^2(\Gamma_N))}\to\inf \tag{1}$$

subject to the constraints

$$y_t-div(\nabla y+A(x)\nabla y)=f \ \text{ in } \ (0,T)\times\Omega, \tag{2}$$

$$y(0,\cdot)=y_0 \ \text{ in } \ \Omega, \tag{3}$$

$$y(\cdot,x)=0 \ \text{ on } \ (0,T)\times\Gamma_D, \ \frac{\partial y(\cdot,x)}{\partial\nu_A}=u \ \text{ on } \ (0,T)\times\Gamma_N, \tag{4}$$

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$$u \in L^2(0,T;L^2(\Gamma_N)) \quad (5)$$

where u is a control, $y_d \in L^2(0,T;H_0^1(\Omega))$ and $u_d \in L^2(0,T;L^2(\Gamma_N))$, $f \in L^2(0,T;H^{-1}(\Omega;\Gamma_D))$ are given distributions, A is a skew-symmetric square L^2-matrix.

The characteristic feature of this problem is the fact that the matrix $A(x) = [a_{ij}]_{i,j=1,\dots,N}$ is skew-symmetric, $a_{ij}(x) = -a_{ji}(x)$ and belongs to L^2-space (rather than L^∞). This leads to the existence of elements $y \in L^2(0,T;H_0^1(\Omega;\Gamma_D))$ such that $y \notin L^\infty((0,T)\times\Omega)$ and

$$\lim_{n\to\infty} \int_0^T \int_\Omega (\nabla\varphi_n, A(x)\nabla y)_{R^3}\, dxdt < 0$$

where $\varphi \in C^\infty([0,T];C_0^\infty(\Omega)) \ni \varphi_n \to y$ strongly in $L^2(0,T;H_0^1(\Omega;\Gamma_D))$. As a result, the existence, uniqueness, and variational properties of the weak solution to (2)–(4) usually are drastically different from the corresponding properties of solutions to the parabolic equations with L^∞-matrices in coefficients. In most cases, the situation can change dramatically for the matrices A with unremovable singularity. Typically, in such cases, boundary value problem may admit infinitely many weak solutions which can be divided into two classes: approximable and non-approximable solutions [1–3]. A function $y = y(u)$ is called an approximable solution to the initial-boundary value problem in (2)–(4) if it can be attained by weak solutions to the similar boundary value problems with L^∞—approximated matrix A. However, this type of solutions does not exhaust all weak solutions to the above problem. There is another type of weak solutions, which cannot be approximated by weak solutions of such regularized problems. Usually, such solutions are called non-variational [2–4], singular [5–7], pathological [8, 9], etc.

The purpose of this work is to consider OCP (1)–(5) with a well prescribed skew-symmetric L^2-matrix A and, using the direct method in the Calculus of variations, to show that this problem admits a unique solution possessing a special singular properties. As a result, we prove that this solution cannot be attained through a sequence of optimal solutions to regularized OCP for boundary value problem (23)–(24) with skew-symmetric matrices $A_k \in L^\infty(\Omega, S^3)$ such that $A_k \to A$ strongly in $L^2(\Omega, S^3)$. Thus, this result shows that a numerical analysis of optimal control problems for parabolic equations with unbounded coefficients is a non-trivial matter and it requires the elaboration of special approaches.

NOTATION AND PRELIMINARIES

Let Ω be the unit ball in R^3, $\Omega = \{x \in R^3 : \|x\|_{R^3} < 1\}$. Let $C_0^\infty(\Omega;\Gamma_D)$ be the set of all infinitely differentiable functions $\varphi : \Omega \to R$ with compact supports in Ω. Let $C_0^\infty(\Omega;\Gamma_D) = \{\varphi \in C_0^\infty(R^N) : \varphi = 0 \text{ on } \Gamma_D\}$. We define the Banach space $H_0^1(\Omega;\Gamma_D)$ as the closure of $C_0^\infty(\Omega;\Gamma_D)$ with respect to the norm (see [10])

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$$\|y\|_{H_0^1(\Omega;\Gamma_D)} = \left(\int_\Omega \|\nabla y\|_{R^3}^2 dx \right)^{1/2}.$$

Let $H^{-1}(\Omega;\Gamma_D)$ be the dual space to $H_0^1(\Omega;\Gamma_D)$. Let X be a Banach space and let $T>0$ be a given value. We denote by $L^2(0,T;X)$ the set of measurable functions $y \in (0,T) \to X$ such that $\|u(\cdot)\|_X \in L(0,T)$. Similarly, one can also define the set of distributions $D'(0,T;X)$ on $(0,T)$ with values in X. $L^2(0,T;X)$ is a Banach space with respect to the norm

$$\|y\|_{L^2(0,T;X)} = \left(\int_\Omega \|u(x)\|_X^2 dx \right)^{1/2}.$$

If X is reflexive, the space $L^2(0,T;X)$ is reflexive, too. Moreover, if X is separable, then $L^2(0,T;X)$ is separable. Let $C([0,T];L^2(\Omega))$ be the space of measurable functions on $[0,T]\times\Omega$ such that $y(t,\cdot)\in L^2(\Omega)$ for any $t\in[0,T]$ and such that the map $t\in[0,T]\mapsto y(t,\cdot)\in L^2(\Omega))$ is continuous. Let us define the Banach space

$$W_{\Gamma_D} = \left\{ y : y \in L^2(0,T;H_0^1(\Omega;\Gamma_D)),\ \frac{\partial y}{\partial t} \in L^2(0,T;H^{-1}(\Omega;\Gamma_D)) \right\}$$

equipped with the norm of the graph. Here, the derivative $\partial y/\partial t$ is the distribution in $D'(0,T;H^{-1}(\Omega;\Gamma_D))$. Then the following properties holds true (see [11, 12]).

Theorem 1. 1) The embedding $W_{\Gamma_D} \subset L^2(0,T;L^2(\Omega))$ is compact.

2) One has the embedding $W_{\Gamma_D} \subset C([0,T];L^2(\Omega))$.

3) For any $u,v \in W_{\Gamma_D}$, one has

$$\frac{d}{dt}\int_\Omega u(t,x)v(t,x)\,dx =$$

$$= \left\langle u'(t,\cdot), v(t,\cdot) \right\rangle_{H^{-1}(\Omega;\Gamma_D),H_0^1(\Omega;\Gamma_D)} + \left\langle v'(t,\cdot), u(t,\cdot) \right\rangle_{H^{-1}(\Omega;\Gamma_D),H_0^1(\Omega;\Gamma_D)}.$$

Let $y \in L^2(0,T;H_0^1(\Omega;\Gamma_D)) \cap C([0,T];L^2(\Omega))$. Then the following density result holds: there exists $\Phi \in C^\infty([0,T];C_0^\infty(\Omega;\Gamma_D))$ such that

$$\|y-\Phi\|_{C([0,T];L^2(\Omega))} \le \delta, \quad \|\nabla y - \nabla\Phi\|_{L^2(0,T;L^2(\Omega))} \le \delta, \ \forall \delta > 0.$$

Skew-symmetric matrices. Let S^3 be the set of all skew-symmetric matrices $\mathrm{A(x)} = [a_{ij}]_{i,j=1}^3$, i.e. A is a square matrix with $a_{ij} = -a_{ji}$ and, hence, $a_{ii}=0$. Therefore, the set S^3 can be identified with the Euclidean space R^3.

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Let $L^2(\Omega;S^3)$ be the space of measurable square-integrable functions whose values are skew-symmetric matrices and it is endowed with the norm

$$\|A\|_{L^2(\Omega;S^3)} = \left(\int_\Omega \|A(x)\|^2_{S^3}\, dx \right)^{1/2}.$$

In what follows, we associate with matrix $A \in L^2(\Omega;S^3)$ the bilinear form $\varphi(\cdot,\cdot)_A : L^2(0,T;C_0^1(\Omega)) \times L^2(0,T;C_0^1(\Omega)) \to R$ following the rule

$$\varphi(y,v)_A = \int_0^T \int_\Omega (\nabla v, A(x)\nabla y)_{R^3}\, dxdt, \ \ \forall y,v \in L^2(0,T;C_0^1(\Omega)).$$

It is easy to see that this form is unbounded on $L^2(0,T;H_0^1(\Omega))$, since, in general, the 'integrand' $(\nabla v, A(x)\nabla y)_{R^3}$ is not integrable on $(0,T)\times\Omega$. This motivates an introduction of the following set. We say that a distribution $y \in L^2(0,T;H_0^1(\Omega;\Gamma_D))$ belongs to the set $D(A)$ if

$$\left| \int_0^T \int_\Omega (\nabla\varphi, A(x)\nabla y)_{R^3}\, dxdt \right| \le c(y,A) \left(\int_0^T \int_\Omega \|\nabla\varphi\|^2_{R^3}\, dxdt \right)^{1/2},$$

for all $\varphi \in C^\infty([0,T];C_0^\infty(\Omega;\Gamma_D))$, with some constant c depending on y and A. As a result, having set

$$[y,\varphi] = \int_0^T \int_\Omega (\nabla\varphi, A(x)\nabla y)_{R^N}\, dxdt, \ \forall y \in D, \forall \varphi \in C^\infty([0,T];C_0^\infty(\Omega)), \tag{6}$$

we observe that the bilinear form $[y,\varphi]$ can be defined for all $\varphi \in L^2(0,T;H_0^1(\Omega;\Gamma_D))$ using the standard rule

$$[y,\varphi] = \lim_{\varepsilon\to\infty} [y,\varphi_\varepsilon] \tag{7}$$

where $\{\varphi_\varepsilon\}_{\varepsilon>0} \in C^\infty([0,T];C_0^\infty(\Omega;\Gamma_D))$ and $\varphi_\varepsilon \to \varphi$ converges strongly in $L^2(0,T;H_0^1(\Omega;\Gamma_D))$. In this case the value $[y,\varphi]$ is finite for every $y \in D(A)$, although the 'integrand' $(\nabla\varphi, A(x)\nabla y)_{R^3}$ need not be integrable on $(0,T)\times\Omega$, in general. This fact leads us to the conclusion

$$[y,y] < +\infty, \quad \forall y \in D(A).$$

At the same time, if we temporary assume that $A \in L^\infty(\Omega;S^3)$, then the bilinear form $[y,\varphi]$ is obviously bounded on $L^2(0,T;H_0^1(\Omega;\Gamma_D))$, i.e. in this case $D(A) \equiv L^2(0,T;H_0^1(\Omega;\Gamma_D))$. Indeed, in view of the Bunjakowski inequality, we get

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$$\|[y,v]\| \le \|A\|_{L^\infty(\Omega,S^3)} \int_0^T \int_\Omega \|\nabla y\|_{R^3} \|\nabla v\|_{R^3} dxdt \le$$
$$\le \|A\|_{L^\infty(\Omega,S^3)} \|y\|_{L^2(0,T;H_0^1(\Omega;\Gamma_D))} \|v\|_{L^2(0,T;H_0^1(\Omega;\Gamma_D))}.$$

Moreover, if $y = v$ then $[y,y] = -[y,y]$, and, therefore, $[y,y]=0$ for all $y \in L^2(0,T;H_0^1(\Omega;\Gamma_D))$. However, as it is shown in the next section, there exist skew-symmetric L^2-matrices A such that the equality $\|[y,y]\| = [y,y]$ does not hold true for some $y \in D(A)$.

We define the divergence $div\ A$ of a skew-symmetric matrix $A \in L^2(\Omega;S^3)$ as a vector-valued distribution $d \in H^{-1}(\Omega; R^3)$ by the following rule

$$\langle d_i, \varphi \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = -\int_\Omega (a_i, \nabla\varphi)_{R^3} dx, \quad \forall \varphi \in C_0^\infty(\Omega)$$

where a_i stands for the i-th row of the matrix A. We say that a matrix $A \in L^2(\Omega;S^3)$ belongs to the space $H(\Omega;div;S^3)$ if $d := divA \in L^1(\Omega;R^3)$, that is

$$H(\Omega;div;S^3) = \left\{A \,\middle|\, A \in L^2(\Omega;S^3), div\,A \in L^1(\Omega;R^3)\right\}.$$

MOTIVATING EXAMPLE

Our main intention in this section is to show that for a given positive scalar value $\alpha \in R$ there exist a skew-symmetric matrix $A \in L^2(\Omega;S^3)$ and a function $y_d \in L^2(0,T;H_0^1(\Omega))$ such that

$$y_d \in D(A) \text{ and } [y_d, y_d] = -\alpha < 0$$

where the bilinear form $[y,v]$ is defined by (6). We divide our analysis into several steps.

Step 1. We define a skew-symmetric matrix A as follows

$$A(x) = \begin{pmatrix} 0 & a(x) & 0 \\ -a(x) & 0 & -b(x) \\ 0 & b(x) & 0 \end{pmatrix} \tag{8}$$

where $a(x) = \dfrac{x_1}{2\|x\|_{R^3}^2}$ and $b(x) = \dfrac{x_3}{2\|x\|_{R^3}^2}$. Since

$$\|a(x)\|_{L^2(\Omega)}^2 = \int_\Omega \left(\frac{x_1}{2\|x\|_{R^3}^2} \right)^2 dx = \int_0^1 \int_0^{2\pi} \int_0^\pi \frac{\rho^2 \cos^2\varphi \sin^2\psi}{4\rho^4} \rho^2 \sin\psi \; d\psi\, d\varphi\, d\rho < +\infty,$$

it follows that $a \in L^2(\Omega)$. By analogy, it can be shown that $b \in L^2(\Omega)$. Moreover, it is easy to see that the skew-symmetric matrix A, we define by (8), satisfies the property $A \in H(\Omega;div;S^3)$, i.e. $A \in L^2(\Omega;S^3)$ and $div\,A \in L^1(\Omega;R^3)$. Indeed, in

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view of the definition of the divergence $div\,A$ of a skew-symmetric matrix, we have $div\,A = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$, where $d_i = div\,a_i = \dfrac{x_i x_2}{\|x\|^4_{R^3}}$ and a_i is i-th column of A. As a result, we get

$$\|div\,a_i\|_{L^1(\Omega)} = \int_0^1\int_0^{2\pi}\int_0^{\pi} \left| \frac{\rho^2 f_i(\varphi,\psi)\sin\varphi\sin\psi}{\rho^4} \right| \rho^2 \sin\psi \; d\psi\, d\varphi\, d\rho < +\infty,$$

for the corresponding $f_i = f_i(\varphi,\psi)$ $i = 1,2,3$. Therefore, $div\,A \in L^1(\Omega; R^3)$.

Step 2 deals with the choice of the function $y_d \in L^2(0,T; H^1_0(\Omega))$. We define it by the rule

$$y_d(t,x) = t\sqrt{\frac{52\alpha}{\pi T^3(1-\exp(-2\pi))}}\left(1 - \|x\|^5_{R^3}\right)\frac{x_2^2}{x_1^2 + x_2^2}$$
$$\exp\left(-\frac{\pi}{2} - \arctan\frac{\sqrt{x_1^2 + x_2^2} - x_1}{x_2}\right), \tag{9}$$

for all $(t,x) \in (0,T)\times\Omega$. It is easy to see that

$$v_0\left(\frac{x}{\|x\|_{R^3}}\right) = \sqrt{\frac{52\alpha}{\pi T^3(1-\exp(-2\pi))}}\,\frac{x_2^2}{x_1^2 + x_2^2}\exp\left(-\frac{\pi}{2} - \arctan\frac{\sqrt{x_1^2 + x_2^2} - x_1}{x_2}\right)$$
$$= \sqrt{\frac{52\alpha}{\pi T^3(1-\exp(-2\pi))}}\sin^2\varphi\exp(-\varphi/2), \quad \forall\varphi \in [0,2\pi]$$

with respect to the spherical coordinates. Hence, $v_0 \in C^1(\partial\Omega)$, and, as immediately follows from (9), it provides that

$$y_d \in L^2(0,T; L^2(\Omega)) \text{ and } y_d(t,\cdot) = 0 \text{ on } \partial\Omega \quad \forall t \in [0,T]. \tag{10}$$

By direct computations, we get

$$v_0\left(\frac{x}{\|x\|_{R^3}}\right) = \frac{1}{\|x\|^3_{R^3}}\begin{bmatrix} \dfrac{\partial v_0}{\partial z_1}\left(\|x\|^2_{R^3} - x_1^2\right) - \dfrac{\partial v_0}{\partial z_2}x_1 x_2 \\ \dfrac{\partial v_0}{\partial z_2}\left(\|x\|^2_{R^3} - x_2^2\right) - \dfrac{\partial v_0}{\partial z1}x_1 x_2 \\ -\dfrac{\partial v_0}{\partial z_1}x_1 x_3 - \dfrac{\partial v_0}{\partial z_2}x_2 x_3 \end{bmatrix}, \quad \forall x \neq 0. \tag{11}$$

Hence, there exists a constant $C^* > 0$ such that $\left\|\nabla v_0\left(\dfrac{x}{\|x\|_{R^3}}\right)\right\|_{R^3} \leq \dfrac{C^*}{\|x\|_{R^3}}$.

Thus,

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$$\|\nabla y_d\|_{R^3} \le t\left|v_0\left(\frac{x}{\|x\|_{R^3}}\right)\right|\left\|\nabla\left(1-\|x\|_{R^3}^5\right)\right\|_{R^3} + t\left(1-\|x\|_{R^3}^5\right)\left\|\nabla v_0\left(\frac{x}{\|x\|_{R^3}}\right)\right\| \le C_1 + \frac{C_2}{\|x\|_{R^3}}.$$

As a result, we infer that $\nabla y_d \in L^2(0,T;L^2(\Omega;\ R^3))$, i.e. we finally have $y_d \in L^2(0,T;H_0^1(\Omega))$.

Step 3. We show that the function y_d, which was introduced before, belongs to the set $D(A)$. To do so, we have to prove the estimate

$$\left|\int_0^T\int_\Omega (\nabla\varphi, A(x)\nabla y_d)_{R^3}\,dxd\right| \le \widetilde{C}\left(\int_0^T\int_\Omega |\nabla\varphi|_{R^3}^2\,dxdt\right)^{1/2},$$

for all $\varphi \in C^\infty([0,T];C_0^\infty(\Omega))$.

To this end, we make use of the following transformations

$$\int_0^T\int_\Omega (\nabla\varphi, A\nabla\psi)_{R^3}\,dx\,dt = -\int_0^T \langle div(A\nabla\psi),\varphi\rangle_{H^{-1}(\Omega);H_0^1(\Omega)}dt =$$

$$= \int_0^T \left\langle div\begin{bmatrix}(a_1)^t\nabla\psi\\ (a_2)^t\nabla\psi\\ (a_3)^t\nabla\psi\end{bmatrix},\varphi\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} dt =$$

$$= \int_0^T\sum_{i=1}^3\left\langle div\,a_i,\varphi\frac{\partial\psi}{\partial x_i}\right\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} dt + \underbrace{\int_0^T\int_\Omega\sum_{i=1}^3\sum_{j=1}^3\left(a_{ij}\frac{\partial^2\psi}{\partial x_i\partial x_j}\right)dx\,dt}_{\substack{=0\\ \text{since } A\in L^2(\Omega;S^3)}} =$$

$$= \int_0^T\int_\Omega (div\ A,\nabla\psi)_{R^3}\,\varphi\,dx\,dt$$

due to the fact that $div\,A \in L^1(\Omega;R^3)$, which are obviously true for all $\psi,\varphi \in C^\infty([0,T];C_0^\infty(\Omega))$. Since

$$\left|\int_0^T\int_\Omega (div\ A,\nabla\psi)_{R^3}\,\varphi\,dx\,dt\right| = \left|\int_0^T\int_\Omega (\nabla\varphi, A\nabla\psi)_{R^3}\,dx\,dt\right| \le$$
$$\le C\|A\|_{L^2(\Omega;S_{skew}^3)}\|\psi\|_{L^2(0,T;H_0^1(\Omega))},$$

it follows that, using the continuation principle, we can extend the previous equality with respect to ψ to the following one

$$\int_0^T\int_\Omega (\nabla\varphi, A\nabla y_d)_{R^3}\ dx\,dt = \int_0^T\int_\Omega \varphi(div\,A,\nabla y_d)_{R^3}\ dx\,dt,\ \forall\varphi\in C^\infty([0,T];C_0^\infty(\Omega)). \quad (12)$$

Let us show that $(div\,A,\nabla y_d)_{R^3} \in L^\infty((0,T)\times\Omega)$. In this case, relation (12) implies the estimate

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$$\left|\int_0^T\int_\Omega (\nabla\varphi, A\nabla y_d)_{R^3}\,dx dt\right| \le \left\|(div\,A,\nabla y_d)_{R^3}\right\|_{L^\infty((0,T)\times\Omega)}\int_0^T\int_\Omega|\varphi|\,dx dt \le$$
$$\le \widetilde{C}\left(\int_0^T\int_\Omega|\nabla\varphi|^2_{R^N}\,dx dt\right)^{1/2}, \qquad (13)$$

for all $\varphi\in C^\infty([0,T];C_0^\infty(\Omega))$, which means that the element y_d belongs to the set $D(A))$.

Indeed, as follows from (11), we have the equality

$$\left(\nabla v_0\left(\frac{x}{\|x\|_{R^3}}\right),\frac{x}{\|x\|^3_{R^3}}\right)_{R^3}=0. \qquad (14)$$

Thus, the gradient of the function $\nabla v_0\left(\frac{x}{\|x\|_{R^3}}\right)$ is orthogonal to the vector field $Q=\frac{x}{\|x\|^3_{R^3}}$ outside the origin. Therefore,

$$(\nabla y_d, div\,A)_{R^3} := t\left(\nabla\left[\left(1-\|x\|^5_{R^3}\right)v_0\left(\frac{x}{\|x\|_{R^3}}\right)\right],\frac{x}{\|x\|^3_{R^3}}\frac{x_2}{\|x\|_{R^3}}\right)_{R^3}=$$
$$=t\left(\nabla\left(1-\|x\|^5_{R^3}\right),\frac{x}{\|x\|^3_{R^3}}\right)_{R^3}\times v_0\left(\frac{x}{\|x\|_{R^3}}\right)\frac{x_2}{\|x\|_{R^3}}+$$
$$+t\left(1-\|x\|^5_{R^3}\right)\left(\nabla v_0\left(\frac{x}{\|x\|_{R^3}}\right),\frac{x}{\|x\|^3_{R^3}}\right)_{R^3}\frac{x_2}{\|x\|_{R^3}}=I_1+I_2$$

where $I_2=0$ by (14). Since $\nabla\left(1-\|x\|^5_{R^3}\right)=-5\|x\|^3_{R^3}x$, $\frac{x_2}{\|x\|_{R^3}}=\sin\varphi\sin\psi$ with respect to the spherical coordinates, and function v_0 is smooth, it follows that there exists a constant $C_0>0$ such that $\left|(\nabla y_d, div\,A)_{R^3}\right|\le C_0$ almost everywhere in $(0,T)\times\Omega$. Thus,

$$(div\,A,\nabla y_d)_{R^3}\in L^\infty((0,T)\times\Omega)$$

and we have obtained the required property.

Step 4. Using results of the previous steps, we show that the function y_d satisfies the condition $[y_d,y_d]=-\alpha<0$. Indeed, $\{\varphi_\varepsilon\}_{\varepsilon\to 0}\in C^\infty([0,T];C_0^\infty(\Omega))$ be a sequence such that

$$\varphi_\varepsilon\to y_d \text{ strongly in } L^2(0,T;H_0^1(\Omega)). \qquad (15)$$

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Then by continuity, we have

$$[y_d, y_d] = \lim_{\varepsilon\to 0} \int_0^T \int_\Omega (\nabla\varphi_\varepsilon, A\nabla y_d)_{R^3}\, dx\,dt \overset{by (12)}{=} \lim_{\varepsilon\to 0} \int_0^T \int_\Omega \varphi_\varepsilon (div\, A, \nabla y_d)_{R^3}\, dx\,dt\,.$$

Since $(div\, A, \nabla y_d)_{R^3} \in L^\infty((0,T)\times\Omega)$, in view of the property (15), we can pass to the limit in the right-hand side of this relation. As a result, we get

$$[y_d, y_d] = \int_0^T \int_\Omega y_d (div\, A, \nabla y_d)_{R^3}\, dxdt = \frac{1}{2}\int_0^T \int_\Omega (div\, A, \nabla y_d^2)_{R^3} dxdt\,. \tag{16}$$

Let $\Omega_\varepsilon = \left\{x \in R^3 \middle| \varepsilon < \|x\|_{R^3} < 1\right\}$ and let $\Gamma_\varepsilon = \left\{\|x\|_{R^3} = \varepsilon\right\}$ be the sphere of radius ε centered at the origin. Then

$$\begin{aligned}
&\int_0^T \int_{\Omega_\varepsilon} (div\, A, \nabla y_d^2)_{R^3} dx\,dt = \int_0^T \int_{\Gamma_\varepsilon} (div\, A, \nu)_{R^3} y_d^2\, dH^2\, dt = \\
&= \int_0^T \left[\int_{\Gamma_\varepsilon} (div\, A, \nu)_{R^3} \left(1 - \|x\|_{R^3}^5\right) v_0^2\left(\frac{x}{\|x\|_{R^3}}\right) dH^2\, t^2 dt\right] = \\
&= \frac{T^3}{3} \int_{\Gamma_\varepsilon} (div\, A, \nu)_{R^3} v_0^2 \left(\frac{x}{\|x\|_{R^3}}\right) dH^2 + o(1) = \\
&= \frac{T^3}{3} \int_{\Gamma_\varepsilon} \left(\frac{x}{\|x\|_{R^3}^3}, \left(-\frac{x}{\|x\|_{R^3}}\right)\right)_{R^3} \frac{x_2}{\|x\|_{R^3}} v_0^2\left(\frac{x}{\|x\|_{R^3}}\right) dH^2 + o(1) \\
&= -\frac{T^3}{3\varepsilon^2} \int_{\Gamma_\varepsilon} \frac{x_2}{\|x\|_{R^3}} v_0^2\left(\frac{x}{\|x\|_{R^3}}\right) dH^2 + o(1) = -\frac{T^3}{3}\int_\Gamma b_0(x) v_0^2(x) dH^2 + o(1)
\end{aligned} \tag{17}$$

where $b_0 = \sin\varphi \sin\psi$ and $v_0^2 = \dfrac{52\alpha}{\pi T^3 (1 - \exp(-2\pi))} \sin^4\varphi \exp(-\varphi)$. Since

$$\int_{\partial\Omega} b_0\, v_0^2\, dH^2 = \frac{52\alpha}{\pi T^3 (1-\exp(-2\pi))}\left(\int_0^{2\pi} \sin^5\varphi\, e^{-\varphi} d\varphi \int_0^{2\pi} \sin^2\psi\, d\psi\right) = 6\alpha T^{-3} > 0\,,$$

it remains to combine this result with (16), (17), and relation

$$\int_0^T \int_\Omega (div\, A, \nabla y_d^2)_{R^3}\, dxdt = \lim_{\varepsilon\to 0} \int_0^T \int_{\Omega_\varepsilon} (div\, A, \nabla y_d^2)_{R^3}\, dxdt\,.$$

As a result, we finally infer $[y_d, y_d] = -\alpha < 0$.

SETTING OF THE OPTIMAL CONTROL PROBLEM AND ITS PRELIMINARY ANALYSIS

Let Ω be the unit ball in R^3. We assume that its boundary $\Gamma = \left\{\|x\|_{R^3} = 1\right\}$ is divided onto two disjoint parts $\partial\Omega = \Gamma_D \cup \Gamma_N$. Let the sets Γ_D and Γ_N have positive 2-dimensional measures.

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Let $f \in L^2(0,T;H^{-1}(\Omega;\Gamma_D))$ and $y_0 \in L^2(\Omega)$ be given distributions, let $A \in L^2(\Omega;S^3)$ and $y_d \in L^2(0,T;H_0^1(\Omega))$ be defined by (8) and (9), respectively. The optimal control problem we consider in this paper is to minimize the discrepancy (tracking error) between a given distribution $y_d \in L^2(0,\, T;\, H_0^1(\Omega))$ and a solution y of the Neumann-Dirichlet boundary value problem for parabolic equation (2)–(4) by choosing an appropriate boundary control $u \in L^2(0,T;L^2(\Gamma_N))$ where

$$\frac{\partial y}{\partial \nu_A} = \sum_{i,j=1}^{3} \left(\delta_{ij} + a_{ij}(x)\right)\frac{\partial y}{\partial x_j}\cos(\nu, x_i)\,,$$

δ_{ij} is the Kronecker's delta, $\cos(\nu, x_i)$ is the i-th directing cosine of ν, and ν is the outward unit normal vector at Γ_N to the ball Ω.

More precisely, we are concerned with OCP (1)–(5). The distinguishing feature of this problem is the special choice of matrix A and distribution f. As we will see later on, this entails a number of pathologies with respect to the standard properties of optimal control problems for parabolic equation. In particular, this leads to the non-uniqueness of weak solutions to the corresponding initial boundary value problem and a singular properties of an optimal pair. As a result, numerical approximation of the solution to OCP (1)–(5) is getting non-trivial.

To begin with, we introduce the following notion.

Definition 1. We say that (u, y) is an admissible pair to OCP (1)–(5) if $u \in L^2(0,T;L^2(\Gamma_N))$, $y \in W$,

$$y(0,\cdot) = y_0 \in L^2(\Omega) \text{ almost everywhere in } \Omega\,, \tag{18}$$

and the integral identity

$$\begin{aligned}&\int_0^T\int_\Omega y_t\varphi\,dxdt + \int_0^T\int_\Omega (\nabla\varphi, \nabla y + A(x)\nabla y)_{R^N}\,dxdt = \\ &= \int_0^T \langle f, \varphi\rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)}dt + \int_0^T\int_{\Gamma_N} u\varphi\,dH^2dt\end{aligned} \tag{19}$$

holds true for each $\varphi \in C^\infty([0,T];C_0^\infty(\Omega;\Gamma_N))$.

We denote by Ξ the set of all admissible pairs for the OCP (1)–(5).

It is worth to note that in view of definition of the space W and Theorem 1, the condition (18) has a sense. Moreover, as was shown in [13], if (u, y) is an admissible pair, then $y \in D(A)$.

Definition 2. *We say that OCP (1)–(5) is regular if it admits at least one admissible pair, i.e.* $\Xi \neq \varnothing$.

We also say that a pair $(u^0, y^0) \in L^2(0,T;L^2(\Gamma_N)) \times D(A)$ is optimal for problem (1)–(5) if

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$$(u^0,y^0)\in\Xi \text{ and } I(u^0,y^0)=\inf_{(u,y)\in\Xi} I(u,y).$$

As immediately follows from (19) and the definition of bilinear form $[y,\varphi]$ (see also the extension rule (7)), every admissible pair $(u,y)\in\Xi$ is related by the following energy equality

$$\begin{aligned}&\frac{1}{2}\int_0^T\int_\Omega \left(y^2\right)_t \varphi\,dxdt+\|y\|^2_{L^2(0,T;H_0^1(\Omega;\Gamma_D))}+[y,y]=\\&=\int_0^T\langle f,\varphi\rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)}dt+\int_0^T\int_{\Gamma_N} u\varphi dH^2dt.\end{aligned}\tag{20}$$

However, as was shown in previous section, the value $[y,y]$ is not of constant sign on $D(A)$. Hence, energy equality (20) does not allow us to derive any priory estimate for the admissible solutions. In spite of this, the following result proves that OCP (1)–(5) is well-posed under the special choice of distributions $y_d\in L^2(0,T;H_0^1(\Omega))$, $u_d\in L^2(0,T;L^2(\Gamma_N))$, $y_0\in L^2(\Omega)$, and $f\in L^2(0,T;H^{-1}(\Omega))$.

Theorem 2. Let $A\in L^2(\Omega;S^3)$ and $y_d\in L^2(0,T;H_0^1(\Omega))$ be defined by (8) and (9), respectively. Assume that $y_0\equiv 0$ in Ω and distributions $f\in L^2(0,T;H^{-1}(\Omega))$ and $u_d\in L^2(0,T;L^2(\Gamma_N))$ are given by the rule

$$f:=(y_d)_t-div(\nabla y_d+A\nabla y_d),\tag{21}$$

$$u_d:=\gamma^1_{\Gamma_N}(y_d)\tag{22}$$

where

$$\gamma^1_{\Gamma_N}:L^2(0,T;H_0^1(\Omega;\Gamma_D))\to L^2(0,T;H^{-1/2}(\Gamma_N))$$

is the trace operator such that

$$\gamma^1_{\Gamma_N}(y)=\left.\frac{\partial y}{\partial\nu_A}\right|_{\Gamma_N}:=\sum_{i,l=1}^{3}\left(\delta_{ij}+a_{ij}(x)\right)\frac{\partial y}{\partial x_j}\cos(\nu,x_i)$$

provided $y\in L^2(0,T;H_0^1(\Omega;\Gamma_D))\cap L^2(0,T;C^1(\overline{\Omega}))$.

Then the pair

$$(u^0,y^0):=(u_d,y_d)\in L^2(0,T;L^2(\Gamma_N))\times D(A)$$

is a unique solution to OCP (1)–(5).

Proof. As follows from (9), the function y_d is smooth near the boundary $\partial\Omega$ and $\left(y_d\right)_t\in L^2(0,T;H_0^1(\Omega))$. Hence, $u_d:=\gamma^1_{\Gamma_N}(y_d)\in L^2(\Gamma_N)$ and $y_d\in W$ (see (10)). Moreover, the inclusion $y\in D(A)$ (see estimate (13)) implies:

$$div(A\nabla y_d)\in L^2(0,T;H^{-1}(\Omega)).$$

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Therefore, in view of the inclusion $(y_d)_t \in L^2(0,T;H_0^1(\Omega))$, we have $f \in L^2(0,T;H^{-1}(\Omega))$. Since $y_d(0,\cdot)=0$ in Ω and

$$\int_0^T \langle f,\varphi\rangle_{H^{-1}(\Omega;\Gamma_D);H_0^1(\Omega;\Gamma_D)} dt = \int_0^T\int_\Omega (y_d)_t \varphi dxdt +$$
$$\int_0^T\int_\Omega (\nabla\varphi, \nabla y_d + A(x)\nabla y_d)_{R^N} dxdt - \int_0^T \int_{\Gamma_N} \gamma_{\Gamma_N}^1 (y_d)\varphi dH^2 dt$$

for all $\varphi \in C^\infty([0,T];C_0^\infty(\Omega;\Gamma_D))$, it follows that the pair (u_d,y_d) satisfies relations (18)–(19). Thus, (u_d,y_d) is an admissible solution to OCP (1)–(5) in the sense of Definition 2. To conclude the proof, it is enough to note that

$$I(u,y)\ge 0 \quad \forall (u,y)\in\Xi, \quad I(u_d,y_d)=0,$$

and the cost functional $I:\Xi\to R$ is strictly convex.

ON GAP IN ATTAINABILITY OF AN OPTIMAL PAIR

The question we are going to discuss in this section is about some pathological properties that can be inherited by optimal pair to the problem, (1)–(5) provided the skew-symmetric matrix is given by the rule (8). Since $A\in L^2(\Omega;S^3)$, it follows that there exists a sequence of skew-symmetric matrices $\{A_k\}_{k\in N} \in L^\infty(\Omega;S^3)$ such that $A_k \to A$ strongly in $L^2(\Omega;S^3)$. Hence, it is reasonably, from numerical point of view, to consider the following sequence of constrained minimization problems associated with matrices A_k.

$$\left\{\left\langle \inf_{(u,y)\in\Xi_k} I_k(u,y)\right\rangle, k\to\infty\right\}. \tag{23}$$

Here,

$$I_k(u,y) := I(u,y) \quad \forall (u,y)\in L^2(0,T;L^2(\Gamma_N))\times L^2(0,T;H_0^1(\Omega;\Gamma_D)), \quad \forall k\in N, \tag{24}$$

and $(u,y)\in\Xi_k$ if and only if

$$\left\{\begin{array}{c} y_t - \mathrm{div}(\nabla y + A_k\nabla y) = f \text{ in } (0,T)\times\Omega, \\ y(0,\cdot) = y_0 \text{ in } \Omega, \\ y(\cdot,x)=0 \text{ on } (0,T)\times\Gamma_D, \dfrac{\partial y(\cdot,x)}{\partial \nu_{A_k}} = u \text{ on } (0,T)\times\Gamma_N, \\ u\in L^2(0,T;L^2(\Gamma_N)),\ y\in L^2(0,T;H_0^1(\Omega;\Gamma_D)), \\ y_t \in L^2(0,T;H^{-1}(\Omega;\Gamma_D)). \end{array}\right\}. \tag{25}$$

Theorem 3. Let $u_d\in L^2(0,T;L^2(\Gamma_N))$, $f\in L^2(0,T;H^{-1}(\Omega;\Gamma_D))$, $y_0\in L^2(\Omega)$, and $y_d\in L^2(0,T;H_0^1(\Omega;\Gamma_D))$ be given distributions. Then for every

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$k \in N$ there exists a unique minimizer $\left(u_k^0, y_k^0\right) \in \Xi_k$ to the corresponding constrained minimization problem (23) such that the sequence of optimal pairs $\left\{\left(u_k^0, y_k^0\right) \in \Xi_k\right\}_{k \in N}$ is relatively compact with to the product of the weak topologies on

$$L^2(0,T;L^2(\Gamma_N)) \times L^2(0,T;H_0^1(\Omega;\Gamma_D))$$

and each of its cluster pairs (u^*, y^*) possesses the properties:

$$(u^*, y^*) \in \Xi, \ \ [y^*, y^*] \geq 0 . \tag{26}$$

The proof of this theorem is similar to proof of Proposition 4.1 and Proposition 4.2 in [13].

Final remarks. As immediately follows from theorem 3, some admissible pairs $(u^*, y^*) \in \Xi$ can be attained by optimal solutions to the approximate OCPs (23). Hence, we can conclude that the original optimal control problem (1)–(5) is regular for every $u_d \in L^2(0,T;L^2(\Gamma_N))$, $f \in L^2(0,T;H^{-1}(\Omega;\Gamma_D))$, $y_0 \in L^2(\Omega)$, and $y_d \in L^2(0,T;H_0^1(\Omega;\Gamma_D))$.

The next observation is crucial in our paper and it deals with the inequality $(26)_2$. As Theorem 3 proves, for any approximation $\{A_k\}_{k \in N}$ of the matrix $A \in L^2(\Omega;S^3)$ with properties $\{A_k\}_{k \in N} \subset L^\infty(\Omega;S^3)$ and $A_k \to A$ strongly in $L^2(\Omega;S^3)$, the optimal solutions to the regularized OCPs (23)–(25) always leads us in the limit to some admissible solution (u^*, y^*) of the original OCP (1)–(5). Moreover, in general, this limit pair can depend on the choice of the approximative sequence $\{A_k\}_{k \in N}$. That's why it is reasonably to call such pairs attainable admissible solutions to OCP (1)–(5). However, as follows from Theorem 2, the pair (u^*, y^*) is not optimal, in general. Indeed, if $y_0 \equiv 0$ in Ω and distributions $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_d \in L^2(0,T;L^2(\Gamma_N))$ are given by the rule (21)–(22), then (u_d, y_d) is a unique optimal pair to OCP (1)–(5). As was shown in previous section, in this case we have $[y_d, y_d] = -\alpha < 0$, where α is a given strictly positive value, whereas $[y^*, y^*] \geq 0$ for any attainable pair (u^*, y^*). Thus, for given f, y_d, y_0, u_d the optimal pair (u^0, y^0) to OCP (1)–(5) cannot be attained through any L^∞-approximation of the matrix $A \in L^2(\Omega;S^3)$.

CONCLUSIONS

The given example of the optimal control problem for linear parabolic equation has an unbounded coefficient such that its unique optimal solution has a non-variational character. Namely, the shown solutions, which can be attained through any L^∞-approximation of the stream matrix, is not exhaustive for all set of solutions to the above problem.

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