ANALYTICAL SOLUTIONS IN THE TWO-CAVITY COUPLING PROBLEM

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Analytical solutions of precise equations that describe the rf-coupling of two cavities through a co-axial cylindrical hole are given for various limited cases. For their derivation we have used the method of solution of an infinite set of linear algebraic equations, based on its transformation into dual integral equations.

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1. INTRODUCTION

Authors of the basic works on RF-coupling [1-3] made a suggestion that in the case, when the hole dimensions are much smaller than the wavelength, the coupling of two resonant volumes can be described with some accuracy by a system of equations of two resonant circuits. They also assumed that coupling coefficients could be calculated within the same accuracy by solving the static problem ($a/\lambda\to 0$). In our papers [4-5] we showed that RF-coupling of two resonators through a center hole of arbitrary dimensions in the frequency domain is strictly described by the system of equations of two resonant circuits with the coupling coefficients that depend on the frequency. On the base of these equations we calculated numerically the relationship of coupling coefficients versus different parameters (frequency, hole radius, etc.). But there is a question on confirming the analytical results of a static approach by obtaining the static coupling coefficients from precise equations in the limit $a/\lambda\to 0$. As we know, this problem was not solved up to this time. For dealing this problem we developed a new mathematical method for solving an infinite set of linear algebraic equations that is based on its transformation into dual integral equations. Besides, expressions are derived for coupling coefficients which are valid up to the second order of the ratio between the hole dimension and the free-space wavelength (a/λ). Such coupling coefficients were also obtained in [6] by using a variation method, but our approach gives the possibility of finding the electrical field distribution on the coupling area.

2. PROBLEM DEFINITION ORIGINAL EQUATIONS.

Let us consider the coupling of two cavities through a circular hole with the radius *a* in a separating wall of a thickness *t*. For simplicity sake, we will consider the case of two identical cavities, with *b* being the cavity radii and *d* their length. In [4,5] it was demonstrated that if the field is expanded with the short-circuit resonant cavity modes and E_{010}-modes are selected as fundamental, the precise set of equations will consist of two equations for the amplitudes of E_{010} -modes, where coupling coefficients are defined by soluting the infinite set of linear algebraic equations. Let us generalize the case considered in [4,5], choosing as fundamental the E_{0pq} -modes of closed cavities (q is the number of field variations across the radius, p is the number of field variations along the longitudinal coordinates). Using the method similar to that one in [4,5], one can show that the set of equations describing the system under consideration has the form:

$$\varepsilon_p Z_{q,p} a_{q,p}^{(1;2)} = = -\omega_{q,0}^2 \frac{4}{3\pi J_1^2(\lambda_q)} \frac{a^3}{b^2 d}\left[a_{q,p}^{(1;2)}\Lambda_1 - (-1)^p a_{q,p}^{(2;1)}\Lambda_2 \right] , \quad (1)$$

where

$$Z_{q,p} = \omega_{q,p}^2 - \omega^2,\ \omega_{q,p}^2 = c^2\left[\lambda_q^2 / b^2 + (p\pi/d)^2\right],$$

$$\varepsilon_p = \begin{cases} 2, & p = 0 \\ 1, & p \neq 0 \end{cases},\ p = 0,1\ldots\infty,\ J_0(\lambda_s) = 0,\ s = 1,2\ldots\infty,$$

$a_{q,p}^{(i)}$ is the amplitude of E_{0qp} -mode in the *i*-th cavity (*i*=1,2). The normalized coupling coefficients Λ_i are determined by the expression:

$$\Lambda_i = \Lambda_i(\omega) = J_0^2(\theta_q)\sum_{s=1}^{\infty} w_s^{(i)} / \left(\lambda_s^2 - \theta_q^2\right), \quad (2)$$

where $w_s^{(i)}$'s are the solution of the following set of linear equations:

$$w_m^{(1;2)} + \sum_s G_{m,s}\left(w_s^{(1;2)} f_m^{(1)} + w_s^{(2;1)} f_m^{(2)} \right) = = 3\pi f_m^{(1)} / \left(\lambda_m^2 - \Omega_*^2 \right), \quad (3)$$

$$f_m^{(j)} = \frac{\mu_m}{sh(q_m)} \begin{cases} ch(q_m) - ch(q_m t/L), & j = 1 \\ ch(2 q_m d/L) - 1, & j = 2, \end{cases}$$

$$q_m = \mu_m L/a,\quad L = 2d + t,\quad \mu_m = \sqrt{\lambda_m^2 - \Omega^2},$$

$$\Omega = \omega a/c,\quad \Omega_*^2 = \Omega^2 - (\pi a p/d)^2 .$$

$$G_{m,s} = B_{m,s} - \frac{\delta_{m,s}}{2\mu_m} cth(\mu_m d/a) + $$

$$+ \frac{2\pi a^2 \theta_q^3 J_0^2(\theta_q)}{d\, b\, \varepsilon_p \chi_q (\lambda_m^2 - \theta_q^2)(\lambda_s^2 - \theta_q^2)[\mu_m^2 + (\pi a p/d)^2]},$$

$$B_{m,s} = \pi \frac{a}{b} \sum_{\ell=1}^{\infty} \frac{\theta_\ell^2 J_0^2(\theta_\ell) R_\ell}{\chi_\ell (\lambda_m^2 - \theta_\ell^2)(\lambda_s^2 - \theta_\ell^2)},$$

$$\theta_\ell = \lambda_\ell \, a/b\,, \quad \chi_\ell = \pi \, \lambda_\ell J_1^2(\lambda_\ell)/2\,, \quad \nu_\ell = \sqrt{\theta_\ell^2 - \Omega^2}\,,$$

$$R_\ell = \begin{cases} \theta_q \, cth \, (\nu_q d/a)/\nu_q - \\ -2a \, \theta_q /\{\varepsilon_p \, d \, (\nu_q^2 + (\pi ap/d)^2\}\,, & \ell = q\,, \\ \theta_q \, cth \, (\nu_q d/a)/\nu_q\,, & \ell \neq q\,. \end{cases}$$

The coefficients $w_s^{(i)}$ have a simple physical sense. Really, it is easy to show that the tangential electric field component in the left cross-section of the coupling hole $E_r^{(-)}(r)$ has the form:

$$E_r^{(-)}(r) = E_{ind}^{(1)} - E_{ind}^{(2)} = \tilde{E}_{0.q,p}^{(1)} Q^{(1)}(r) - \tilde{E}_{0.q,p}^{(2)} Q^{(2)}(r), \quad (4)$$

where $Q^{(i)}(r) = \dfrac{1}{3\pi} \sum\limits_s \dfrac{J_1(\lambda_s r/a)}{J_1(\lambda_s)} w_s^{(i)}$, $\tilde{E}_{0,q,p}^{(1)}$ is the value of the longitudinal (perpendicular to the hole) electric field of $(0, q, p)$-mode in the first cavity on the left coupling hole cross-section for $r = a$, while $\tilde{E}_{0,q,p}^{(2)}$ is the same value for the right-hand cavity on the right coupling hole cross-section for the same radius. From the expression (4) it follows that the tangential electric field component on the left coupling hole cross-section[1] is equal to the difference of two induced fields, each of which is proportional to the perpendicular electric field components of $E_{0,q,p}$-modes taken to be fundamental. There, the coefficients $w_s^{(i)}$ are the ones in the expansion of appropriate functions with the complete set of functions $\left\{ J_1(\lambda_s r/a) \right\}$. Thus, the two-cavity coupling problem, rigorously formulated on the base of the electric field expansion with the short-circuit resonant cavity mode, is reduced to the induced field definition on the right and left cylindrical hole cross-sections.

3. INFINITELY THIN WALL CASE

An important role in the problem of cavity coupling plays the case of infinitely thin wall dividing the cavities ($t = 0$). In this case, from Eq (3) it follows that $w_m^{(1)} = w_m^{(2)} = w_m$. Here the set of equations for w_m will take the form[2]:

$$\sum_{s=1}^{\infty} w_s B_{m,s} = 3\pi \Big/ \left\{ 2\left(\lambda_m^2 - \Omega_*^2 \right) \right\}. \quad (5)$$

For the case t=0 $\Lambda_1 = \Lambda_1 = \Lambda$, where

$$\Lambda = J_0^2(\theta_q) \sum_s w_s \Big/ \left(\lambda_s^2 - \theta_q^2 \right). \quad (6)$$

3.1. SMALL COUPLING HOLE CASE ($a \to 0$)

If in Eqs. (5,6) the hole radius tends to zero[3], then Eq. (5) will become:

$$\sum_{s=1}^{\infty} w_s \int_0^{\infty} d\theta \frac{\theta^2 J_0^2(\theta)}{\left(\lambda_s^2 - \theta^2 \right)\left(\lambda_m^2 - \theta^2 \right)} = \frac{3\pi}{2\lambda_m^2}. \quad (7)$$

In order to get the solution for Eq. (7) we will introduce an integer odd function $f_1(z)$ the values of which in the points $z = \lambda_s$ are equal to $f_1\left(\lambda_s\right) = w_s J_1\left(\lambda_s\right)$. Let us assume that at $|z| \to \infty$ $f_1(z)$ grows not faster than exp(z), then, in accordance with the Cauchy theorem, the function $\left(f(z)/J_0(z) \right)$ can be expanded into the series over mere fractions

$$f_1(z)/J_0(z) = 2z \sum_{n=1}^{\infty} w_n \Big/ \left(\lambda_n^2 - z^2 \right). \quad (8)$$

Using (8), and, also, multiplying Eq. (7) by $J_1\left(\lambda_m x\right)/J_1\left(\lambda_m\right)$, where $0 < x < 1$, and doing summation over sub-index m, we will get

$$\int_0^{\infty} f_1(z) J_1(xz) dz = 3\pi\, x/2, \; 0 < x < 1. \quad (9)$$

By multiplying (8) by $z J_1(xz)$ and integrating over z from 0 to ∞, we will obtain at $x > 1$:

$$\int_0^{\infty} z f_1(z) J_1(xz) dz = 0, \quad x > 1. \quad (10)$$

In this way, the set of linear algebraic equations (7) with a complicated matrix coefficients that cannot be expressed via elementary functions and can be calculated only numerically, has been reduced to two integral equations (9,10). Having determined the kind of the function $f_1(z)$, there is no need in calculating the sum (6), since

$$\Lambda = \sum_s w_s / \lambda_s^2 = \lim_{z \to 0} \left[f_1(z) / \{2zJ_0(z)\} \right]. \quad (11)$$

The method of solving the dual integral equations of the type (9,10) on the base of the Mellin transformation, as well as the property of Cauchy-type integrals, can be found in [7]. The brief summary of their solutions is given in [8]. We shall dwell briefly on a simpler method of resolving this system.

Since $f_1(z)$ is the odd function it can be represented in the form $f_1(z) = \int_0^\infty \sin(zt)\eta(t)dt$. Substituting this expression in Eq. (10) we obtain the following integral equation for $\eta(t)$: $\int_x^\infty dt\, \eta(t)/\sqrt{t^2 - x^2} = 0, \; x > 1$. The solution of this equation is $\eta(t) = 0$ for $t > 1$. Consequently, any function of the type

$$f_1(z) = \int_0^1 \sin(zt)\eta(t) dt \quad (12)$$

satisfies Eq. (10). Substituting (12) into (9), we obtain the first kind Volterra equation of Abelian type

$$\int_0^x dt\eta(t)/\sqrt{x^2 - t^2} = 3\pi\, x^2/2, \; 0 < x < 1, \quad (13)$$

the solution for which can be found in the analytical form. Omitting the intermediate formulae, we shall give the final expression for the function $f_1(z)$:

[1]The same is true for the right cross section

[2]We have neglected terms of order a^5 in the expression for $G_{m,s}$

[3]In this case, as follows from Eqs.(1), the coupling coefficients will be proportional to a^3

$f_1(z) = 6\,j_1(z)$, where $j_n(z)$ is the spherical Bessel function of order n.

The normalized coupling coefficients, as follows from (11), is equal to $\Lambda = 1$. Since $w_s = f_1(\lambda_s)/J_1(\lambda_s)$, then from (4) we will obtain

$$E_r^{(-)}(r) = \frac{E_{0,q,p}^{(1)}(r=0) - E_{0,q,p}^{(2)}(r=0)}{\pi}\frac{r}{\sqrt{a^2 - r^2}}.$$

Thus, based on rigorous electrodynamics description of the two-cavity coupling system we are the first to prove, by the way of the limit transition $a \to 0$, the correctness of the equations formulated in [1-3] using the quasi-static approximation, and to obtain the expression for the tangential electric field on the hole.

3.2. THE CASE OF SMALL FINITE VALUES OF COUPLING HOLE RADIUS

The above method presents the opportunity to obtain analytical expressions for the normalized coupling coefficients with accuracy of the order of $(a/\lambda)^2$. If a/λ is small, though finite, then, the coefficients w_s in (5) will be dependent on the hole radius value a: $w_s = w_s(a)$. Let us introduce the function of two variables: $\psi(a,z) = 2zJ_0(z)\sum_{n=1}^{\infty} w_n(a)/\left(\lambda_n^2 - z^2\right)$. We assume that relative to the variable z the function $\psi(a,z)$ obeys the conditions formulated in Subsec.3.1. Using the technique similar to that described in Subsec.3.1, the set (5) can be reduced to:

$$\sum_{l=1}^{\infty} \theta_l J_1(x\theta_l)\psi(a,\theta_l)/\chi_l = 0, \quad 1 < x < b/a, \qquad (14)$$

$$\pi\frac{a}{b}\sum_{l=1}^{\infty}\frac{J_1(x\theta_l)\psi(a,\theta_l)R_l}{\chi_l} = \frac{3\pi J_1(x\Omega_*)}{\Omega_* J_0(\Omega_*)}, \quad 0 < x < 1. \qquad (15)$$

Letting $a \to 0$ in Eqs. (14,15), we derive a set of equations (9,10), and, consequently, $\psi(0,z) = f_1(z)$. Then we represent $\psi(a,z)$ in the form $\psi(a,z) = \psi(0,z) + a^2\varphi(a,z)$. From (14,15) it follows that $\varphi(0,z)$ satisfies the following equations:

$$\int_0^{\infty}\theta J_1(x\theta)\varphi(0,\theta)d\theta = 0, \; x > 1, \qquad (16)$$

$$\int_0^{\infty} J_1(x\theta)\varphi(0,\theta)d\theta = F(x), \; 0 < x < 1, \qquad (17)$$

where $F(x) = \frac{3\pi x}{8a^2}\left[\Omega_*^2 - \Omega^2 - \frac{x^2}{4}\left(2\Omega_*^2 - \Omega^2\right)\right]$..

The solution of Eqs.(16,17) has the form

$$\varphi(0,z) = \frac{\Omega_*^2 - \Omega^2}{4a^2}f_1(z) - \frac{2\Omega_*^2 - \Omega^2}{2a^2}f_2(z), \qquad (18)$$

where $f_2(z) = j_3(z)/2 - 3j_1(z)/(2z^2)$.

The normalized coupling coefficients Λ, accurate to the order of $(a/\lambda)^2$, are determined by the relationship:

$$\Lambda \approx 1 - \frac{1}{5}\left(\frac{a\lambda_q}{b}\right)^2 - \frac{3}{20}\left(\frac{a\omega_{q,p}}{c}\right)^2 - \frac{1}{20}\left(\frac{a\omega}{c}\right)^2. \qquad (19)$$

For the case $\omega \approx \omega_{q,p}$ the expression (19) agrees with that for the generalized polarizability, obtained in [6] at $b \to \infty$ via the variation technique. Note that the expression (19) is true for the frequency ω that is not close to the resonant frequencies of the non-fundamental modes of closed cavities: $\omega \neq \omega_{n,m}$ if $(m,n) \neq (q,p)$. Knowing $\psi(a,z)$, and, consequently, $w_s(a)$, the form of the tangential electric field around the hole can be reconstructed:

$$E_r^{(-)} = \frac{E_{0,q,p}^{(1)}(r=0) - E_{0,q,p}^{(2)}(r=0)}{\pi}$$
$$\left\{\left[1 - \frac{1}{4}\left(\frac{a}{b}\lambda_q\right)^2 + \frac{\Omega_*^2 - 2\Omega^2}{12}\right]\frac{r}{\sqrt{a^2 - r^2}} + \frac{2\Omega_*^2 - \Omega^2}{6}\frac{r}{a}\sqrt{1 - \left(\frac{r}{a}\right)^2}\right\}. \qquad (20)$$

4. CONCLUSION

Based on our method of reducing the infinite linear algebraic equation set to dual integral equations, we obtained, in different limited cases, the rigorous analytical solutions regarding the two-cavity coupling problem. Along with the general theory significance, the obtained solutions are of applied interest, since they can be used for a better convergence of the original equation solution (3), which are true for arbitrary dimensions of the coupling hole.

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