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DIFFERENTIAL EQUATIONS FOR INTEGRAL SCALES OF SEMIEMPIRICAL DEVELOPED-TURBULENCE THEORY

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The calculation of turbulent shear flows and of turbulent diffusion in these flows calls for the theory that enables the anisotropic properties of the flows to be adequately taken into account. The information provided by classical semiempirical theories of Boussinesque, Prandtle, Taylor, Carman is limited to mere averaged velocities, and appears insufficient, as it gives no way of calculating the spatial and temporal distributions of turbulence intensity and turbulent diffusion coefficients. In this regard, of interest are the semiempirical theories, where in parallel with the Reynolds equations use is made of the equations for turbulent characteristics, in particular, the turbulent energy (b)-balance equation [1].

Similarly to classical semiempirical theories, these theories also comprise the notion of the coefficient of turbulent diffusion for different substances (momentum, energy, etc.), however, the dependence of this coefficient on statistical characteristics of turbulence is quite different and is based on the following obvious considerations. It is known that the kinematic coefficient of molecular viscosity is related to the average molecular velocity U_m and the average free path length of molecules l_m by $v=U_m l_m$. It can be assumed that the coefficients of turbulent viscosity would be described by similar relations, where the corresponding characteristics of disordered turbulent flows will play the role of molecular flow characteristics U_m and l_m.

Thus the mean-square value of pulsation velocity, i.e., $\sqrt{b}$, is obviously analogous to U_m. Instead of l_m, one should use the turbulence scale l, i.e., the length dimensionality describing the mean distance that the turbulent formations can be displaced without disturbing their individuality (i.e., the Prandtle "way of mixing", which is coincident in the order of magnitude with the "correlation length" calculated from the spatial correlation function). In this case, the anisotropic turbulence can be characterized by different scales in different directions.

At the same time, it is the semiempirical theories that are now gaining acceptance [2]. In these theories, the anisotropy of the flow can be considered in terms of pulsation velocity components, differing in direction, but at one (isotropic) scale. The very construction of the semiempirical theory in this way shows its limited usefulness, because in reality the anisotropy in shear flows must exert an effect not only on pulsation velocity values, but on the scales as well.

Strictly speaking, at each point of the flow, along with the pulsation velocity components, a scale ellipsoid must be determined, i.e., a symmetric second-rank tensor l_{ij} (scale tensor) must be assigned, the components of which have the length dimensionality. Therefore, to hold the rigidity in the determination of anisotropic properties of the flow, one needs not only the equations for the pulsation velocity components but also the equations for turbulence scales; however, this will result in a bulky set of equations.

The set of equations for turbulence energy and scales may appear more compact to consider the anisotropic properties of flows. Then the tensor of turbulent viscosity coefficients can be expressed as

$$v_{ij} = \sqrt{b} l_{ij} \tag{1}$$

This approach appears more natural, because the turbulence energy is the parameter quite real for shear flows.

With the tensor v_{ij}, we can express the Reynolds stresses in terms of the turbulence energy, scales and derivatives of averaged velocities, assuming (as is generally the case) that the dependence of these stresses on the derivatives $\partial U_i/\partial x_j$ is linear, namely,

$$\langle u_i u_j \rangle = \frac{2}{3} b\delta_{ij} - \frac{1}{2}\sqrt{b}\left(l_{i\alpha}\Phi_{\alpha j} + l_{j\alpha}\Phi_{\alpha i}\right) \tag{2}$$

where $\Phi_{ij} = \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}$ is the strain tensor, U_i is the ith pulsation velocity component value, U_j is the jth averaged velocity component value, δ_{ij} is the Kronecker symbol, <...> is the probability-theoretic averaging.

This formula was proposed by Monin [3]. It explicitly takes into account the symmetry of the Reynolds stress tensor, yet it often neglects the significant fact that generally (except for the isotropic turbulence) normal stresses differ between themselves in magnitude. In a number of cases, one can simply take as a first approximation that $l_{ij} = l\delta_{ij}$ is the isotropic tensor, i.e., take the hypothesis expressed by the formula

$$\langle u_i u_j \rangle = \frac{2}{3} b\delta_{ij} - l\sqrt{b}\Phi_{ij} \tag{3}$$

as the base (the Reynolds stress tensor just the same remaining anisotropic).

It should be noted that in the literature, because of the lack of data on the scale tensor l_{ij}, it is formula (2) that is used to calculate turbulent flows by the set of equations including the turbulent energy equation. If the use is made of the common hypothesis that the turbulent diffusion coefficients of different substances are equal to the turbulent viscosity coefficient to an accuracy of numerical factors, then the introduction of the tensor l_j allows one to write in terms of principal unknowns (e.g.,

[4]) both the third moments of the velocity field

$$\frac{1}{2}\langle u_\beta u_\beta u_i \rangle = -\alpha_b \sqrt{b} l_{i\alpha} \frac{\partial b}{\partial x_a} \qquad (4)$$

(where α_b is a certain numerical coefficient) and the density components of the turbulent impurity flow

$$\langle \vartheta u_i \rangle = -\alpha_\vartheta \sqrt{b} l_{i\alpha} \frac{\partial \Theta}{\partial x_a} \qquad (5)$$

where ϑ is the concentration pulsation, Θ is the concentration average.

The construction of closed semiempirical theory of inhomogeneous turbulence around the set of Reynolds equations, the continuity equation and the turbulent energy-balance equation calls for the presence of data on space-time variation of scales, that cannot be obtained from the principal equations of the set. These data can be obtained either by processing the experimental data or by solving differential equations for scales. Currently some studies have been advanced, which propose the equations for scales. Let us consider a homogeneous isotropic turbulence, for which all points and directions are of equal importance, therefore, all single-point characteristics are constant in space. In this case, two integral scales are of importance: longitudinal Λ_f and transverse Λ_g. They are expressed in terms of longitudinal and transverse correlation coefficients *f(x)* and *g(x)* by the following formulas

$$\Lambda_g = \int_0^\infty g(x)\,dx, \qquad \Lambda_f = \int_0^\infty f(x)\,dx, \qquad (6)$$

However, in the consideration of inhomogeneous nonisotropic turbulence these definitions become meaningless because of nonequivalence of different points and directions in the flow. Generalizing the isotropic case, Rotta [5] has defined the "averaged scale" of turbulence *L* by the following relationship $L(\vec{x},t) = \frac{3\pi}{8b(\vec{x},t)} \int_0^\infty F(k,\vec{x},t)\frac{dk}{k}, \qquad (7)$

where *k* is the wave vector modulus, $\vec{x}$ is the radius vector of the point under consideration, $F(k,\vec{x},t)$ is the function of three-dimensional spectrum (sum of diagonal elements of the spectral function tensor, integrated over all possible directions of the wave vector $\vec{k}$). The differential equation derived for the thus introduced "average scale" inadequately describes its behavior in the vicinity of a hard wall. In [6], Glushko has eliminated this drawback by modifying formula (7) relationship

$$L(x_1,x_2,x_3,t) = \frac{3}{16\pi b(\vec{x},t)} \int_{-\infty}^{\infty}\int_{-2x_2}^{2x_2}\int_{-\infty}^{\infty} \frac{Q(\vec{x},\vec{\xi},t)}{\xi^2} d\xi \ ,(8)$$

where *Q* is the half-sum of diagonal moments of the correlation function tensor for the points $\vec{x}-\frac{\vec{\xi}}{2}$ and $\vec{x}+\frac{\vec{\xi}}{2}$.

The physical meaning of the modification consists in the explicit consideration of the condition of attachment on the wall (mathematically, this means the restriction of the region of variation $\vec{\xi}$). Expressions (7) and (8) are equivalent at the centre of the flow. In fact, as $Q(\vec{x},\vec{\xi},t)$ is essentially different from zero for $|\vec{\xi}|<L$, then at $x_2 \gg L$ (this holds at the centre of the flow) the upper and lower limits can be considered infinite, and thus the restriction on possible $\vec{k}$ values is removed.

Note that formulae (7) and (8) take into account only the inhomogeneity of the flow, leaving quite apart its anisotropy. This situation is not wholly satisfactory, because turbulence can be characterized by different scales in different directions.

The flow anisotropy can be taken into account by introducing the scale tensor. Here, of importance is the issue which interpretation should be given to these scales. If the coordinate frame is chosen so that only diagonal components of the scale tensor are nonzero, then each principal direction in space has the sole scale. Apparently, in the determination of these tensor components one can proceed from the following qualitative considerations:

- the scale in the given direction must characterize the same-direction correlation of both longitudinal and transverse components of pulsation velocity;
- the scale must make allowance for the presence of flow-limiting surfaces, on which the averaged velocity and the pulsation velocity become zero;
- in the case of isotropic homogeneous flow, the scale must be coincident with either transverse or longitudinal integral scale.

In view of the mentioned requirements, let us define the scale in the direction x_i as a mean integral scale of correlation between like pulsation-velocity components in the given direction, i.e.,

$$l_{ii}(\vec{x},t) = \frac{3}{8b(\vec{x},t)} \iiint_{\Omega(\vec{x})} \frac{1}{2}\sum_{k=1}^{3} R_{kk}(\vec{x},\vec{\xi},t) \frac{\delta(\vec{\xi})}{\delta(\xi_i)} d\vec{\xi} \ ,(9)$$

where *b* is the specific turbulent energy, *Ω (x)* is the domain of ξ variation

$$*R_{ij} = \langle u_i' u_j' \rangle = \left\langle u_j\left(\vec{x}-\frac{\vec{\xi}}{2}\right) u_i\left(\vec{x}+\frac{\vec{\xi}}{2}\right)\right\rangle,$$

$\vec{\xi} = (\xi_1,\xi_2,\xi_3)$ is the distance between the points under consideration, $\delta(\xi_1)\delta(\xi_2)\delta(\xi_3) = \delta(\vec{\xi})$ is the Dirac delta function having the following property

$$\int_{-\varepsilon}^{\varepsilon} \delta(\xi)\,d\xi = 1, \text{ where } \varepsilon > 0 .$$

We now show that this definition of l_{ij} in the case of isotropic homogeneous turbulence is coincident with the definition of the transverse integral scale Λ_g. It is known that for the turbulence of this kind

$$R_{ij}\left(\left|\vec{\xi}\right|\right)=\frac{2}{3}b\left(\frac{f-g}{\xi^2}\xi_i\xi_j+g\delta_{ij}\right),$$

where f and g are, respectively, the longitudinal and transverse correlation coefficients, δ_{ij} is the Kronecker symbol.

Taking the half-sum of diagonal elements of the tensor R_{ij} and making use of the known relationship [7] between f and g, we obtain

$$\frac{1}{2}\sum_{k=1}^{3}R_{kk}\left(\left|\vec{\xi}\right|\right)=\frac{b}{3}\left(3f+\left|\vec{\xi}\right|\frac{\partial f}{\partial\left|\vec{\xi}\right|}\right),$$

and since

$$\frac{\partial f}{\partial\left|\vec{\xi}\right|}=\frac{\left|\vec{\xi}\right|}{\xi_i}\frac{\partial f}{\partial\xi_i},$$

then

$$l_{ii}=\frac{1}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(3f+\frac{\left|\vec{\xi}\right|^2}{\xi_i}\frac{\partial f}{\partial\xi_i}\right)\frac{\delta\left(\vec{\xi}\right)}{\delta\left(\xi_i\right)}d\vec{\xi}=$$

$$=\frac{1}{8}\int_{-\infty}^{\infty}\left(3f+\xi_i\frac{\partial f}{\partial\xi_i}\right)d\xi_i$$

Taking into consideration that f is the even function of ξ_i, we find

$$l_{ii}=\frac{1}{4}\int_{0}^{\infty}\left(3f+\xi_i\frac{\partial f}{\partial\xi_i}\right)d\xi_i=$$

$$=\frac{1}{4}\left\{\begin{matrix}3\Lambda_f-\\-\int_0^{\infty}f\left(\left|\xi_i\right|\right)d\xi_i\end{matrix}\right\}=\frac{\Lambda_f}{2}=\Lambda_g,$$

where $\Lambda_{f(g)}$ is, respectively, the longitudinal (transverse) integral scale. By definition (9), the functional dependences of l_{ij} on the coordinates can be obtained in three ways, namely:

- from the experimental data on correlation functions;
- with the use of theoretical dependences for correlation functions;
- by equations formed for l_{ij}.

Let us illustrate the first way of setting up functional dependences l_{ij} on the coordinates using the experimental data of Conte-Bellaux [8] who measured, among other important turbulence characteristics, the spatial distributions of turbulent energy b in the plane flow, the average squared components of pulsation velocities $\langle u_i^2\rangle$, the correlation coefficients

$$r_{ij}\left(\vec{x},\gamma,0,0\right)=\frac{\left\langle u_i\left(x_1,x_2,x_3\right)u_j\left(x_1+\gamma,x_2,x_3\right)\right\rangle}{\sqrt{u_i^2\left(x_1,x_2,x_3\right)}\sqrt{u_j^2\left(x_1+\gamma,x_2,x_3\right)}}$$

As is seen from expression (9), the calculation of l_{ij} requires the correlation functions of the form

$$R_{ij}\left(\vec{x},\vec{\xi}\right)==\left\langle u_j\left(\vec{x}-\frac{\vec{\xi}}{2}\right)u_i\left(\vec{x}+\frac{\vec{\xi}}{2}\right)\right\rangle.$$

They can easily be related to the correlation coefficients $r_{ij}(x,\xi)$ measured in experiment, namely:

$$R_{ee}\left(\vec{x},\vec{\xi}\right)=\sqrt{R_{ee}\left(\vec{x}-\frac{\vec{\xi}}{2},0\right)}\sqrt{R_{ee}\left(\vec{x}+\frac{\vec{\xi}}{2},0\right)}r_{ee}\left(\vec{x}-\frac{\vec{\xi}}{2},\vec{\xi}\right).$$

Since the experiment was conducted in the plane flow, all single-point characteristics, including l_{ij}, are merely the functions of one coordinate x_2 (the axis x_2 is directed along the averaged velocity variation), and for two-point characteristics the dependence on the first argument is related only to the function of x_2.

The experimental data were used to plot

$$\frac{1}{2}\sum_{k=1}^{3}R_{kk}\left(\vec{x},\xi_i\right)b\left(\vec{x}\right)$$

versus $|\xi_i|$ for different x_2/D values (where D is the half-width of the channel). Finding the area under the curve (e.g., with a planimeter) that corresponds to the direction i and to a certain x_2/D value and multiplying by a constant factor, we obtain the $l_{ii}\left(\frac{x_2}{D}\right)$ values.

Based on the definition (1) and the data obtained, one can ascertain the known fact about the excess of the longitudinal coefficient v_{11} over the transverse coefficient v_{22} by factors of about 3 to 4.

For comparison, the data measured by Conte-Bellaux are used for the correlation lengths defined as
It should be noted that the definition (10) provides a consideration of correlation in one direction only, this being insufficient for finding the diagonal elements of scale tensor, as two of the scales must characterize certain “averaged correlation lengths” in transverse directions, and this is not provided by the definition. Besides, it masks the anisotropy of the flow, because even in the case of homogeneous isotropic turbulence the correlation length L_1 will be twice as large as the lengths L_2 and L_3.

Using the introduced integral turbulence scale (9) as the base, differential equations can be set up to take into account the anisotropic properties of the flow.

In the derivation of equations for diagonal components of integral scale tensor (DCIST), according to (9), we use the equation for $Q(\vec{x},\vec{\xi},t)$, that is found by folding the equations for two-point moments R_{ij}), multiply it by $3/8\,b(\vec{x},t)$ and integrate over $\Omega(\vec{x})$.In the integration one should take into account that the average velocity and the pulsation velocity are equal to zero at the boundary of the region. It is assumed that all two-point characteristics are dependent on the ratio $\xi_i/l_i\left(\vec{x},t\right)$. Then, similarly to [2,5], we write down each characteristic as follows:

$$\Gamma(\vec{x},\vec{\xi},t)=\Gamma(\vec{x},0,t)\,f^{(\Gamma)}\left(\frac{\xi_1}{l_{11}},\frac{\xi_2}{l_{22}},\frac{\xi_3}{l_{33}}\right),$$

where $f^{(\Gamma)}$ satisfies the condition

$$\frac{3}{8b(\vec{x},t)}\iiint\limits_{\Omega(\vec{x})} f^{(\Gamma)}\frac{\delta(\vec{\xi})}{\delta(\xi_i)}d\vec{\xi}=\varsigma^{(\Gamma)}=Const\ .$$

This means that the dependence on the direction in space is determined by the dependence on $l_{ii}(\vec{x},t)$. The mentioned manipulations lead to the set of equations for $b\,l_{ii}(\vec{x},t)$.

$$\frac{\partial(bl_{ii})}{\partial t}+U_k\frac{\partial(bl_{ii})}{\partial x_k}+\varsigma_{ki}R_{ki}(\vec{x},0,t)\,l_{ii}\frac{\partial U_i}{\partial x_k}=$$
$$=\frac{\partial}{\partial x_k}\left\{\nu\frac{\partial(bl_{ii})}{\partial x_k}+\varsigma_{ki}^{(b)}I_{ki}^{(b)}(\vec{x},0,t)\,l_{ii}\right\}+ \quad , \quad (11)$$
$$+T_s-\varsigma_s l_{ii}\varepsilon_t$$

$$T_s=-\nu bl_{ii}\sum_{k=1}^{3}\left\{\frac{\alpha_k}{l_{kk}^2}\begin{Bmatrix}(1-\delta_{ik})\beta_i+\\+\delta_{ik}\left[\frac{\xi_i}{l_{ii}}f\left(\frac{\xi_i}{l_{ii}}\right)\right]\end{Bmatrix}\right\},\ \xi_i\in S \quad (12)$$

where ν is the kinematic viscosity coefficient; $\varsigma_{ki}\varsigma^{(b)}_{ki}\varsigma_s$, α_k, β_i are the constants, S is the surface of the region Ω, $I_k^{(b)}=\left\langle u_k(u_iu_i)+\frac{2}{\rho}pu_k\right\rangle/2$, p - p is the pressure pulsation , $\varepsilon_t=\nu\left\langle\left(\frac{\partial u_i}{\partial x_k}\right)^2\right\rangle$.

The turbulent energy balance equation has the form

$$\frac{\partial b}{\partial t}+U_k\frac{\partial b}{\partial x_k}+R_{ki}(\vec{x},0,t)\frac{\partial U_i}{\partial x_k}=$$
$$=\frac{\partial}{\partial x_k}\left\{\nu\frac{\partial b}{\partial x_k}+I_{ki}^{(b)}(\vec{x},0,t)\right\}-\varepsilon_t \quad . \quad (13)$$

To derive equations for DCIST, we need only take the difference between (11) and (13) multiplied by l_{ii} and transform it in accordance with the above-given closure hypotheses. In the general case, to calculate the turbulent shear flow, the derived equation should be solved along with the equation for averaged velocities and turbulent energy. Besides, it is essential to determine the unknown constants entering into the closure equation. Further, we consider experimental studies of a steady-state plane-parallel turbulent flow between two walls spaced at $2D$. For convenience, we transform the problem into a formally nonstationary task with the boundary conditions

$$\frac{\partial l_i}{\partial y}=\frac{\partial E}{\partial y}=0 \text{ at } y=1 \text{ (middle of the flow)}$$

$$l_i=E=0 \text{ at } y=0 \text{ (wall)}$$

where $y=\frac{x_2}{D}$, $l_i=\frac{l_{ii}}{D}$, $E=\frac{b}{U_f^2}$, $t'=\frac{U_f}{D}t$, $\mathrm{Re}=\frac{U_fD}{\nu}$ is the analog of the Reynolds number, U_f is the dynamic velocity.

Along with the boundary conditions, we assign the initial conditions for l_i, E, approaching the experimental values obtained from ref. [7] by means of corresponding transformations. Using the physical properties of the plane-parallel shear flow with respect to the balance between the terms of turbulent energy equation, and based on the initial values for l_i and E, we obtain the constants σ_{kl}, σ_3, c, æ, $\alpha^{(b)}$, $\varsigma_k^{(b)}$, δ, β_i, α_k.

The set of equations was solved by the finite-difference method using the implicit scheme. At each time step, four equations for l_i and E are solved. The equations are solved by the method of matrix run [7-8] with the use of the Samarsky difference scheme for solving parabolic-type quasilinear equations [9].

The DCIST and E values, found at each time step, are used to calculate the related coefficients of difference equations; the calculations are repeated in the same sequence until the steady-state profiles for l_i and E are obtained.

In calculations, the following constant values were used: $\delta=0.15$; $c=3.93$; æ=0.06; $\frac{1}{2}\sum_{k=1}^{3}R_{kk}(\vec{x},\xi_i)b(\vec{x})$; $\sigma_{12}=0.21$; $\sigma_3=0.2$; $\beta_1=0.1$; $\beta_3=0.42$; $\alpha_2=0.16$; $\alpha^{(b)}=0.6-0.000098(z_2-120)^2$,

$$z_2=\begin{cases}50, & y\le 0.1\\ 500y, & 0.1<y<0.25\\ 120, & y\ge 0.25\end{cases}\qquad f\left(\frac{2y}{l_2}\right)=\left(\frac{l_2}{2y}\right)^3.$$

With these values, the functional dependences for the DCIST were found. They were compared to the dependences plotted using the experimental data from ref. [7]. Comparison was also made between the bution of the turbulent viscosity coefficient $\nu_T=\frac{\delta}{2}\sqrt{b}\,(l_{11}+l_{22})$, the experimental distribution plotted in [10] by the data of Lauder and Nunner for the flow in the tube, and also the function for the velocity deficit $\frac{U_1-U_0}{U_f}$ based on the Conte-Bellaux data [10] (where U_0 is the velocity in the middle of the flow). The comparison of calculated results with the measured data shows that the proposed model provides a fair description of the turbulent shear flow.

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