PRECISION OF A FDTD METHOD TO SIMULATE COLD MAGNETIZED PLASMAS

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The finite difference time domain (FDTD) method is applied to describe the propagation of the transverse electromagnetic waves through the magnetized plasmas. The numerical dispersion relation is obtained in a cold plasma approximation. The accuracy of the numerical dispersion is calculated as a function of the frequency of the launched wave and time step of the numerical grid. It is shown that the numerical method does not reproduce the analytical results near the plasma resonances for any chosen value of time step if there is not a dissipation mechanism in the system. It means that FDTD method cannot be applied straightforward to simulate the problems where the plasma resonances play a key role (for example, the mode conversion problems). But the accuracy of the numerical scheme can be improved by introducing some artificial damping of the plasma currents. Although part of the wave power is lost in the system in this case but the numerical scheme describes the wave processes in an agreement with analytical predictions.

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INTRODUCTION

The FDTD method [1,2] is widely used to simulate a propagation of the electromagnetic waves through the materials with various dielectric and magnetic properties which may vary in time and space. In classical case it solves the Maxwell's equations in partial differential form. Particularly it is frequently applied in an antenna theory to design new antennas and in a waveguide theory to optimize energy transfer. Sometime the conductive properties of the materials depend on the external electromagnetic impact. In this case the Maxwell's equations are supplemented by the equations which describe an interaction of the material media with the electromagnetic waves. For example, if medium contains the free charges it will be the motion equations for the free charges in the electromagnetic field.

At present the FDTD method is used in plasma physics mainly to solve the Maxwell's equations together with the plasma current equations for all plasma species [3]. It does not allow to reproduce the hot plasma effects which require solving the kinetic equation but gives a space- time distribution of the wave electromagnetic field in cold plasma approach. This approach can be used as a simplified approximation to calculate wave power transmission, reflection and damping (particularly for the problems of antenna-plasma coupling). Also application of the FDTD method in plasma physics can give some advantages to study a wave-particle interaction including the nonlinear processes but it requires a development of this method for the anisotropic and nonuniform media.

It was shown [4, 5] that FDTD method can be used to describe the fast wave propagation through the cold magnetized plasmas including some phenomena of the mode conversion. But description of the plasma resonances which could appear along wave propagation through a nonuniform plasma by the FDTD method is under a question. The problem is that a wavelength tends to zero near the plasma resonances. As a result a fixed space step of the numerical grid does not allow to resolve correctly the small scale wave processes. Just the mode conversion problem operates with plasma resonances and cutoffs. Therefore the additional analysis of the FDTD scheme accuracy in simulations of the magnetized plasmas is required.

1. SET OF EQUATIONS AND DISCRETIZATION SCHEME

The cold magnetized plasma is described by the set of Maxwell's equations coupled by the plasma current equations for plasma species:

$$\begin{cases} \nabla\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t}, \\ \nabla\times\vec{B} = \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t} + \dfrac{4\pi}{c}\left(\vec{J} + \sum\limits_{\alpha}\vec{J}_{\alpha}\right), \\ \dfrac{\partial}{\partial t}\vec{J}_{\alpha} + \nu_{\alpha}\vec{J}_{\alpha} = \dfrac{\omega_{p\alpha}^2}{4\pi}\vec{E} + \left[\vec{J}_{\alpha}\vec{\Omega}_{\alpha}\right] \end{cases} \quad (1)$$

here $\vec{J}$ describes an external current into plasma, ν_{α} is the collisional frequency of the plasma species, $\omega_{p\alpha}$ and $\vec{\Omega}_{\alpha}$ are plasma and cyclotron frequencies respectively.

A standard Yee cell algorithm [1] is applied to solve the set of equations. It means that Faraday equation and Ampere equation (Maxwell-Ampere equation without a displacement current) are discretized in the way that electric $\vec{E}$ and magnetic $\vec{B}$ fields are calculated at different space points shifted from each other by a half of the space numerical grid step. Moreover, the different components of the fields are calculated at different space points. It allows to present the curl numerically in a convenient form (the sense of Yee cell application). For convenient treatment of the boundary conditions a discretization model has been chosen with the components of electric field at the edges of the Yee cell and the components of the magnetic field at the faces of the Yee cell.

In such a way the components of the electromagnetic field will be calculated at following space points:

$$E_x(i+1/2,k,l),\quad E_y(i,k+1/2,l),\quad E_z(i,k,l+1/2),$$

$$B_x(i,k+\frac{1}{2},l+\frac{1}{2}),\quad B_y(i+\frac{1}{2},k,l+\frac{1}{2}),\quad etc.$$

The time discretization is built also in the way that the electric $\vec{E}$ and magnetic $\vec{B}$ fields are calculated at different time moments shifted from each other by a half of the time numerical grid step.

The current equation requires to know all components of the current and the electric field at the same space points. We have chosen the space points to calculate the currents as a center of Yee cell. In such a way all components of the current density will be calculated at the same space points:

$$J_{\alpha x}(i+\frac{1}{2},k+\frac{1}{2},l+\frac{1}{2}),\quad J_{\alpha y}(i+\frac{1}{2},k+\frac{1}{2},l+\frac{1}{2}),\quad etc.$$

As a result the current data from the centers will be averaged to the edges to calculate the electric field from the Maxwell- Ampere equation. And the electric field data from the edges will be averaged to the centers to calculate the currents from third equation of the set (1).

The FDTD scheme has been built with $\vec{E}$ - $\vec{J}_\alpha$ collocation in time. Then the Maxwell- Faraday equation is explicit:

$$-\frac{1}{c}\frac{\partial \vec{B}^{(n+1/2,n-1/2)}}{\partial t} = \nabla\times\vec{E}^{(n)}. \qquad (2)$$

The mixed explicit/implicit scheme has been chosen for the Maxwell- Ampere and current equations:

$$\begin{cases} \frac{1}{c}\frac{\partial \vec{E}^{(n+1,n)}}{\partial t} = \nabla\times\vec{B}^{(n+1/2)} - \frac{4\pi}{c}(\vec{J}+\sum_\alpha \vec{J}_\alpha)^{(n+1,n)}, \\ \frac{\partial}{\partial t}\vec{J}_\alpha{}^{(n+1,n)} + \nu_\alpha \vec{J}_\alpha{}^{(n+1,n)} = \frac{\omega_{p\alpha}^2}{4\pi}\vec{E}^{(n+1,n)} + [\vec{J}_\alpha\vec{\Omega}_\alpha]^{(n+1,n)}. \end{cases} \qquad (3)$$

The set of the Maxwell's equations with the current equations (1) has been discretized for a harmonic signal: all components of the vector values are proportional to $\exp(j(k_x\Delta x\cdot i + k_y\Delta y\cdot l + k_z\Delta z\cdot m - \omega\Delta t\cdot n))$. In our consideration the z axis is directed along the external magnetic field.

Direct calculations issue the discretized dispersion relation:

$$\begin{aligned} &\tilde{k}^2(\varepsilon_1^x d_x^2 + \varepsilon_1^y d_y^2 + \varepsilon_3 d_z^2) \\ &-\frac{b^2}{c^2}(\varepsilon_1^x\varepsilon_3(d_x^2+d_z^2) + \varepsilon_1^y\varepsilon_3(d_y^2+d_z^2) \\ &+\varepsilon_1^x\varepsilon_1^y(d_x^2+d_y^2) - \varepsilon_2^2(d_x^2+d_y^2)) \\ &+\frac{b^4}{c^4}\varepsilon_3(\varepsilon_1^x\varepsilon_1^y - \varepsilon_2^2) = 0, \end{aligned} \qquad (4)$$

where $b \equiv \frac{\sin(\omega\Delta t/2)}{\Delta t/2}$ is the discretized wave frequency, $d_x \equiv \frac{\sin(k_x\Delta x/2)}{\Delta x/2}$, $d_y \equiv \frac{\sin(k_y\Delta y/2)}{\Delta y/2}$, $d_z \equiv \frac{\sin(k_z\Delta z/2)}{\Delta z/2}$ are the components of the discretized wave vector $\tilde{k}$ and $\tilde{k}^2 \equiv d_x^2 + d_y^2 + d_z^2$, c is speed of light.

The components of the discretized electromagnetic tensor ε can be presented in convenient form:

$$\varepsilon_1^x = 1 - \sum_\alpha \frac{a_y^2 a_z^2 \tilde{b}\,\tilde{\omega}_{p\alpha}^2}{b(\tilde{b}^2 - \tilde{\Omega}_{\alpha z}^2)}, \qquad (5)$$

$$\varepsilon_1^y = 1 - \sum_\alpha \frac{a_x^2 a_z^2 \tilde{b}\,\tilde{\omega}_{p\alpha}^2}{b(\tilde{b}^2 - \tilde{\Omega}_{\alpha z}^2)}, \qquad (6)$$

$$\varepsilon_2 = -\sum_\alpha \frac{a_x a_y a_z^2 \tilde{\omega}_{p\alpha}^2 \tilde{\Omega}_{\alpha z}}{b(\tilde{b}^2 - \tilde{\Omega}_{\alpha z}^2)}, \qquad (7)$$

$$\varepsilon_3 = 1 - \sum_\alpha \frac{a_x^2 a_y^2 \tilde{\omega}_{p\alpha}^2}{b\tilde{b}}, \qquad (8)$$

where the coefficients $a_x \equiv \cos(k_x\Delta x/2)$, $a_y \equiv \cos(k_y\Delta y/2)$, $a_z \equiv \cos(k_z\Delta z/2)$ appear due to averaging the electric field values from the edges to the centers in the current equations and averaging the current density values from the centers to the edges in the Maxwell- Ampere equation, $\tilde{\omega}_{p\alpha} \equiv \omega_{p\alpha}\cdot A$ is "numerical" plasma frequency, $\tilde{\Omega}_{\alpha z} \equiv \Omega_{\alpha z}\cdot A$ is "numerical" cyclotron frequency, $\tilde{b} \equiv b + j\nu_\alpha \cdot A$ and $A \equiv \cos(\omega\Delta t/2)$ is a multiplier coming from a level of the explicit/implicit presentation.

Let's consider the transverse propagation of the electromagnetic waves through the magnetized plasmas. The transverse propagation means that $k_z = 0$. Moreover, to resolve the shortest wavelengths of the problem the steps of the space grid should be enough small to provide $d_x \approx k_x$, $d_y \approx k_y$, $d_z \approx k_z$, $a_x \approx a_y \approx a_z \approx 1$ which mean $\varepsilon_1^x \approx \varepsilon_1^y \equiv \varepsilon_1$. Finally the general dispersion relation (4) is reduced to well known analytical form [6] but, please, pay an attention to the "numerical" definitions of the usual notations:

$$\varepsilon_1 k^4 - \frac{b^2}{c^2}k^2(\varepsilon_1\varepsilon_3 + (\varepsilon_1^2 - \varepsilon_2^2)) + \frac{b^4}{c^4}\varepsilon_3(\varepsilon_1^2 - \varepsilon_2^2) = 0, \qquad (9)$$

which issues the ordinary $k^2 = \varepsilon_3\frac{b^2}{c^2}$ and the extraordinary $k^2 = \frac{b^2}{c^2}\frac{(\varepsilon_1^2 - \varepsilon_2^2)}{\varepsilon_1}$ waves.

2. ACCURACY OF THE NUMERICAL DISPERSION FOR THE EXTRAORDINARY WAVE

The numerical dispersion of the studied FDTD scheme is compared with the analytical dispersion according to the motivation in [7]. The accuracy of the numerical scheme is defined as:

$$\eta = \frac{\left|\mathrm{Re}(n_{an}^2) - \mathrm{Re}(n_{num}^2)\right|}{\mathrm{Re}(n_{an}^2)}, \qquad (10)$$

where n_{an}^2 is square of the analytical refraction index and n_{num}^2 is square of the numerical refraction index (9) for the extraordinary wave from.

The hydrogen homogeneous plasma with typical fusion parameters has been chosen for the accuracy tests. The densities of the electrons and ions are equal to $3\cdot10^{13}$ cm^{-3}, the external magnetic field value is 3.4 T. The accuracy is built in Fig. 1. Although the requested accuracy can be achieved in the wide frequency range but it is not achievable near the lower hybrid frequency.

This region is zoomed in Fig. 2. As a sequence the FDTD scheme could issue a wrong result for the waves with the frequencies which are close to the lower hybrid one. And it is correct for any value of time step regardless of the selected (enough small to resolve the shortest wavelength) space step. Of course, there is a condition of the numerical stability for the studied scheme which provides the

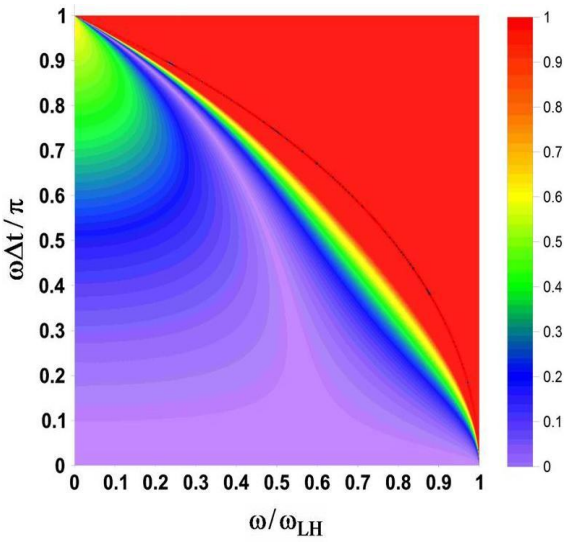

Fig. 1. The accuracy (10) of the numerical dispersion for the extraordinary wave in the low frequency range as a function of the normalized wave frequency and the normalized time step. Please note that for the convenience the large accuracy values are shown as 1 (absolutely wrong result)

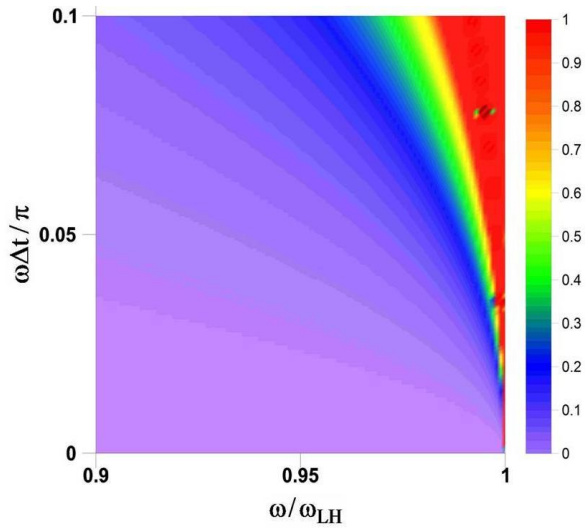

Fig. 2. The same as in Fig. 1 but with zoomed area near the lower hybrid frequency. The red area reaches the horizontal axis: numerical dispersion does not reproduce the analytical result for any chosen time step

acceptable relation between the space and time steps of the numerical grid. This condition could be even stronger than the condition on the numerical accuracy but it was not strictly defined yet for the magnetized plasmas. Also, although only the lower hybrid resonance has been considered here but the similar conclusion can be made for other plasma resonances: the nondissipative FDTD scheme cannot treat correctly the wave processes near the plasma resonances.

But it does not mean that the FDTD scheme can not be applied at all to simulate the processes near the plasma resonances (particularly near the lower hybrid resonance). A way to solve this problem is introducing some artificial damping of the plasma currents. In our case the collisional frequency of the plasma particles has been introduced to the current equations. The changes in the numerical accuracy can be seen from a comparison of the Fig. 2 and Fig. 3. Although the accuracy near the lower hybrid frequency depends still strongly on the time step value but there are time step values which keep the numerical accuracy in the selected limits.

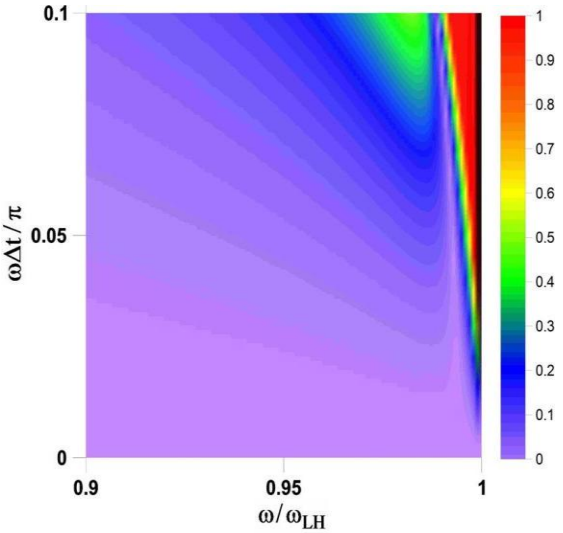

Fig. 3. The same as in Fig. 2 but with the artificial damping provided by the collisional frequency which is equal to 0.01 part of the wave frequency. The red area does not reach the horizontal axis: numerical dispersion reproduces the analytical result for some range of the time step values with the acceptable accuracy

CONCLUSIONS

The numerical dispersion of the transverse extraordinary wave for one of the widely used FDTD scheme is compared with the analytical dispersion of this wave in the low frequency range. The accuracy of the numerical dispersion is calculated as function of the wave frequency and the time step of numerical grid. It is shown that numerical dispersion issues wrong result in the frequency range near the lower hybrid frequency for any value of time step even if the space step is enough small for well resolution of the shortwavelength perturbations. Intuitively this result follows from vacuum Courant condition [2] of the numerical stability. But numerical accuracy and numerical stability are important but different characteristics of the FDTD numerical scheme. In general one of them does not follow from another. Moreover there is not a proof for Courant condition of the magnetized plasma: many authors suppose and motivate that vacuum condition is still valid for the magnetized plasmas but the conclusion is based on statistical data only. Here we have proved that the accuracy of scheme without current dissipation is not enough to simulate

waves near the plasma resonances regardless of the numerical stability.

The accuracy of the FDTD scheme near the plasma resonances can be improved by introducing some artificial dissipation (for example, the artificial damping through the collisions). In this case there is a possibility to choose the time step value to reach the selected numerical accuracy but the price of this possibility is a nonconservative system. Amount of power which is "lost" in the system due to this artificial damping is the subject of separate discussion, especially for the scenarios with the mode conversion processes. So, the artificial collisions allow to treat the plasma resonances by the FDTD method reproducing the space- time distribution of the electromagnetic field of the different plasma modes but the power conservation low for these modes should be corrected on the "lost" power.

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ТОЧНОСТЬ МЕТОДА КОНЕЧНЫХ РАЗНОСТЕЙ ВО ВРЕМЕННОЙ ОБЛАСТИ ПРИ МОДЕЛИРОВАНИИ ХОЛОДНОЙ МАГНИТОАКТИВНОЙ ПЛАЗМЫ

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Метод конечных разностей во временной области (КРВО) применяется для описания распространения поперечных электромагнитных волн через магнитоактивную плазму. Численное дисперсионное уравнение получено в приближении холодной плазмы. Точность численной дисперсии посчитана как функция частоты распространяющейся волны и величины временного шага численной сетки. Показано, что при отсутствии механизмов диссипации в системе численный метод не воспроизводит аналитические результаты возле плазменных резонансов для любого выбранного временного шага. Это значит, что метод КРВО нельзя применять прямолинейно для моделирования задач, в которых плазменные резонансы играют ключевую роль (например, в задачах по изучению конверсии мод). Но точность численной схемы можно улучшить через включение механизма затухания для плазменных токов. Хотя в этом случае часть энергии волны теряется в системе, но численная схема описывает волновые процессы в соответствии с аналитическими предсказаниями.

ТОЧНІСТЬ МЕТОДУ КІНЦЕВИХ РІЗНИЦЬ У ЧАСОВІЙ ОБЛАСТІ ПРИ МОДЕЛЮВАННІ ХОЛОДНОЇ МАГНІТОАКТИВНОЇ ПЛАЗМИ

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Метод кінцевих різниць у часовій області (КРЧО) застосовується для вивчення поширення поперечних електромагнітних хвиль через магнітоактивну плазму. Числове дисперсійне рівняння отримано в наближенні холодної плазми. Точність числової дисперсії обрахована як функція частоти електромагнітної хвилі та величини часового кроку числової сітки. Показано, що за відсутністю механізмів дисипації в системі числовий метод не відтворює аналітичні результати біля плазмових резонансів для будь- якого вибраного часового кроку. Це означає, що метод КРЧО не можна застосовувати прямолінійно для моделювання задач, у яких плазмові резонанси відіграють ключову роль (наприклад,у задачах з вивчення конверсії мод). Але точність числової схеми можна покращити через включення механізму загасання плазмових струмів. Хоча в цьому випадку частина енергії хвилі втрачається в системі, але числова схема описує хвильові процеси у відповідності до аналітичних передбачень.