NUMERICALLY STABLE MODELING OF RADIO FREQUENCY FIELDS IN PLASMA

V.E.Moiseenko

Institute of Plasma Physics, National Science Center "Kharkov Institute of Physics and Technology", 61108 Kharkov, Ukraine

The problem of numerically stable solving of Maxwell's equations written in terms of electric field in plasma is reviewed. The requirements for providing the numerical stability are discussed.

The problems arising from the stiffness of Maxwell's equations in plasma are analyzed. In this respect a newly proposed weighted residuals scheme with uniform trial functions is compared with numerically stable Galerkin scheme.

The numerical stability on two-dimensional mesh is addressed briefly. The specific version of staggered mesh method, which is stable on non-orthogonal mesh, is discussed.

Among other approaches a method consisting in discretization of Maxwell's equations written in other equivalent form and recently proposed local solution method are considered.

PACS:

INTRODUCTION

A number on calculations is performed for the boundary problem for Maxwell's equations in plasma:

$$\nabla \times \nabla \times \vec{E} - \frac{\omega^2}{c^2}\hat{\varepsilon} \cdot \vec{E} = \frac{4\pi i\omega}{c^2}\vec{j}_{ext}. \qquad (1)$$

Here $\hat{\varepsilon}$ is the dielectric tensor, j_{ext} is the density of external electric current. It was found that application of finite element or finite difference methods in standard their versions to this problem results in numerically unstable solutions that normally have some wiggles. The wiggles could be small and large. Sometimes the numerical solution is distorted completely. To explain this phenomenon we consider, following [1], the eigenvalue problem:

$$\nabla \times \nabla \times \vec{E} = \lambda \frac{\omega^2}{c^2}\hat{\varepsilon} \cdot \vec{E} \qquad (2)$$

with λ as an eigenvalue. The equation (2) has the multiple eigenvalue $\lambda = 0$. All vectors that could satisfy formula $E = \nabla\Gamma$ (Γ is an arbitrary scalar function) and boundary conditions are eigenvectors (eigenmodes) for this eigenvalue. In standard finite element or finite difference approach this eigenvalue is reproduced, but the degeneration of the problem is cancelled. For large-scale varying eigenmodes, λ is order of numerical error of the scheme used. For eigenmodes varying strongly at the mesh scale, λ can be large. If it is order of unity these spurious eigenmodes could be excited along with the true ones. The coupling of the correct solution with these eigenmodes is weak, but the resonance amplification of the eigenmode by the factor of $(1-\lambda)^{-1}$ comes to play. If the factor is large enough, the spurious solutions become visible.

The numerical stability could be provided if the following requirement formulated in four mostly equivalent forms is met

a) multiple eigenvalue $\lambda = 0$ is reproduced rigorously in numerical scheme;
b) the discretized Maxwell's term $\nabla \times \nabla \times E$ forms degenerate system of equations;
c) the finite elements used make it possible to satisfy $\nabla \times E = 0$ for non-trivial solutions;
d) the linear combination of test functions should form a gradient of the generating function.

The requirement in forms (a) and (b) is more general and relate to finite difference schemes too. Requirement (c) is for Galerkin method. Requirement (d) is introduced in [2] for weighted residuals schemes. The generating function mentioned there is normally the finite element function of leading order used in the scheme chosen.

WEIGHTED RESIDUALS METHOD WITH UNIFORM TRIAL FUNCTIONS

Galerkin method that meets the above-mentioned stability requirement is widely used for modeling the electromagnetic phenomena described by Maxwell's equations. However, it is much less well-to-do than the standard Galerkin method. It requires the different components of electric field to be represented by different trial functions. Moreover, as it follows from requirement (d) that is also valid for Galerkin method, the expression for trial function includes curvature terms if the geometry chosen is curvilinear. Thus, no possibility exists to use physical components of the electric field vector.

In plasma the typical situation is when the dielectric tensor is stiff. At low frequencies the parallel to the steady magnetic field B component of the tensor $\varepsilon_{//} = e_{//} \cdot \hat{\varepsilon} \cdot e_{//}$ (here $e_{//} = B/B$) is higher by several orders than other components. At the condition of the fundamental cyclotron resonance the component $\varepsilon_{+} = e_{+} \cdot \hat{\varepsilon} \cdot e_{+}$ increases sufficiently, where $\vec{e}_{+} = \frac{e - ie \times e_{//}}{\sqrt{2}}$ and e is the unit vector perpendicular to the steady magnetic field. In general, the electric field components $E_{//} = e_{//} \cdot E$ and $E_{+} = e_{+} \cdot E$ are represented as a sum of different finite element functions and, therefore, cannot be nullified. Thus, their representation contains numerical error that is amplified by big value of the corresponding component of dielectric tensor.

So, one can guess the Galerkin method cannot treat such stiffness efficiently. This suggestion is confirmed in [2] by numerical experiments in cylindrical geometry. They show that under conditions of stiffness the Galerkin method demonstrates unacceptably poor performance.

In [2] it is proved that for weighted residuals method the requirement (d) relates only to test functions and does not restrict the choice of trial function. Therefore, the trial functions could be chosen uniform. If so, the electric field components $E_{//}$ and E_{+} could be nullified separately and, the stiffness of the problem would make less effect. This is confirmed in [2] by test calculations. Moreover, the weighted residuals method with uniform trial functions has the performance better than Galerkin method even in the case without stiffness.

TWO-DIMENSIONAL STAGGERED MESH METHOD

In two dimensions there is another problem. It arises from the term $T = \nabla \times f$, where f is one of the trial functions. In process of discretization that in finite element approach is done by integrating the product of the left and right-hand sides of the equation and test function, this term appears from the integration in part of the term $f \cdot \nabla \times \nabla \times E$. Since the requirement (d) is met the trial function could be written as

$$f = ee \cdot \nabla G, \tag{3}$$

where $e = f / f$ and G is the generating function, normally two-dimensional finite element function. The term $T = \nabla \times (ee \cdot \nabla G)$ clearly contains the second order cross derivative. It could produce delta-function at the edge of finite element if the mesh is not orthogonal.

For this reason among low order schemes the staggered mesh scheme, the allied method for numerically stable finite element method, seems more attractive. The technology of its application to the reduced order problem for Maxwell's equations is presented in [3]. Remind here that reduced order approach is realized setting $E_{//} = 0$. So, only two projections of Maxwell's equations are necessary to use.

We explain this technology using the simple one-dimensional problem in cylindrical geometry assuming $\frac{d}{dz} = 0$. There the components of the vector $\nabla \times \nabla \times E$ read:

$$\left(\nabla \times \nabla \times \vec{E}\right)_r = \frac{m^2}{r^2} E_r + \frac{im}{r^2} \frac{d}{dr}\left(rE_\varphi\right), \tag{4}$$

$$\left(\nabla \times \nabla \times \vec{E}\right)_\varphi = im \frac{d}{dr}\left(\frac{E_r}{r}\right) - \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(rE_\varphi\right)\right]. \tag{5}$$

Following [3], before the discretization this system should be written in the following form:

$$\left(\nabla \times \nabla \times \vec{E}\right)_r = \frac{im}{r^2}\left(H - imE_r\right), \tag{6}$$

$$\left(\nabla \times \nabla \times \vec{E}\right)_\varphi = -\frac{d}{dr}\left(\frac{H - imE_r}{r}\right), \tag{7}$$

$$H = \frac{d}{dr}\left(rE_\varphi\right). \tag{8}$$

Note that on the staggered mesh E_r and E_φ are specified on different mesh nodes. Since H should be specified at the same mesh nodes as E_r, it is not a problem to conserve the degeneracy of the system (6-7) after the discretization. The quantity H could be easily eliminated afterwards because the discretized formula (8) is the explicit expression for it.

ALTERNATE FORM OF MAXWELL'S EQUATIONS

The Maxwell's equations in terms of electric field could be written in another equivalent form (see [4]):

$$\nabla \times \nabla \times \vec{E} - \frac{\omega^2}{c^2}\hat{\varepsilon} \cdot \vec{E} + \nabla \alpha \nabla \cdot \hat{\varepsilon} \cdot \vec{E} = \\ = \frac{4\pi i\omega}{c^2}\vec{j}_{ext} + 4\pi \nabla \alpha \rho_{ext} \tag{9}$$

Here $\alpha(r)$ is a free parameter. This equation is obtained from equation (1) by applying the operator

$$\hat{M} = 1 - \frac{c^2}{\omega^2}\nabla(\alpha \nabla \cdot \tag{10}$$

to both its sides. The equation (9) is no more degenerate if $\omega = 0$. Besides solutions of equation (1), equation (9) has additional solutions. The equation for them we obtain taking divergence from equation (9)

$$\Delta \alpha D - \frac{\omega^2}{c^2} D = \frac{4\pi i\omega}{c^2}\left(-i\omega \rho_{ext} + \nabla \vec{j}_{ext}\right). \tag{11}$$

If the charge continuity law is met, the right-hand side of the equation nullifies. So, as is expected, in analytical theory the additional solution is not excited. But in discretized system the excitation of the additional solutions could occur because of error of numerical approximation. Note also that for positive α the equation (11) has only evanescent solutions. Thus, for real ω they cannot form eigenmodes. So, the danger that the numerical error could be amplified by the resonance factor does not exist here.

The equation (9) could be discretized with the standard finite element or finite difference technique without restriction (a)-(d) and, therefore, could take all advantages that the cancellation of the restrictions gives, i.e. both test and trial functions could be normal finite element functions, the appropriate components of the electric field could be chosen, etc.

The numerical experiments on solving the equation (9) made in [4] show normal convergence and absence of numerical pollution. To provide the stability α could be chosen in vide range. The choice $\alpha \sim |\varepsilon_\perp|^{-1}$ seems to be optimum.

LOCAL SOLUTION METHOD

Local solution method [5] is the stand-alone method different from the standard methods. Its principles we explain on one-dimensional boundary problem for Helmholz equation

$$\frac{d^2y}{dx^2} + G(x)y = R(x) \qquad (12)$$

with appropriate boundary conditions. At the mesh interval $x \in (x_i, x_{i+1})$ the solution of the equation (12) could be written as

$$y = C_{i,1}y_{i,1}(x) + C_{i,2}y_{i,2}(x) + y_{i,R}(x), \qquad (13)$$

where $y_{i,1}$ and $y_{i,2}$ are linear independent solutions of uniform Helmholtz equation and, $y_{i,R}$ is a special solution of equation (12).

The algorithm of the local solution method is the following. First the approximations of the local solutions $y_{i,1}$, $y_{i,2}$ and $y_{i,R}$ should be found at each mesh interval. The coefficients in formula (13) $C_{i,1}$ and $C_{i,2}$ are considered as unknowns. The equations for them are obtained matching local solutions (13) and their derivatives at the mesh nodes. The missing two equations should be added using the boundary conditions. So, the linear problem for the coefficients $C_{i,1}$ and $C_{i,2}$ is well defined and, they could be calculated solving the matrix equation obtained. After this, with formula (13) the solution of equation (12) could be found in any point of the domain.

The local solution method gives much more freedom than other numerical methods for the choice of the form in which the approximate local solutions could be represented. In [5] the polynomial-exponential form of solutions is used $y_{i,(1,2)} = A_{i,(1,2)}(x)\exp[B_{i,(1,2)}(x)]$ in that the coefficients of the polynomials $A_{i,(1,2)}$ and $B_{i,(1,2)}$ are calculated from the minimization of residual of the uniform Helmholtz equation. With such local solutions this method always show better performance than the finite element method of the corresponding order. The advantage of it becomes noticeable when the solutions oscillate rapidly.

For the local solutions method, the problem of numerical stability lays in other plane: the numerical stability is provided if the number of local solutions is correct and they are reproduced well.

SUMMARY AND DISCUSSION

The numerically stable version of Galerkin method for Maxwell's equations written in terms of electric field pays too much for its stability. The major its disadvantages result from non-uniform finite element functions set and from the fact that these functions are obtained by means of differentiating. Recently proposed weighted residuals method with uniform trial functions [2] does not solve all the problems of Galerkin method. But the efficiency of it is higher and, unlike the Galerkin method, does not decrease dramatically in the case of stiffness of the problem. The technique of application of both methods is similar and, the area of their usage is the same.

There are evident problems in usage of Galerkin and of the parent method, the staggered mesh method, on two-dimensional non-orthogonal mesh. The successful attempt to solve them on staggered mesh for the reduced-order Maxwell's equations in plasma is done in [3]. In the special case of PLFEM-S code that models radio-frequency heating in axisymmetrical open traps and in that this approach is realized, the non-orthogonal mesh is highly preferable because it could be both aligned along magnetic field lined and made dense in the zone of cyclotron resonance or near the antenna.

The alternate form of Maxwell's equations proposed for discretization in [4] allows one to use standard finite element or finite difference technique. This is clearly demonstrated for one-dimensional problem. Maybe this approach is less suitable for hot plasma because it involves the differentiation of the dielectric tensor.

The local solution method is a stand-alone method that gives wide freedom for choice of the basic functions. The exponential-polynomial form of them allows this method to reproduce oscillating solutions on very coarse mesh. This feature is important in the case when, together with fast waves, short-scale slow waves and kinetic waves can propagate in plasma. The problem of numerical stability for this method is the problem of appropriate choice of local solutions that seems less difficult.

REFERENCES

[1] R.Gruber and J.Rappaz *Finite Element Methods in Linear Ideal Magnetohydrodynamic* Springer-Verlag, Berlin, 1985.

[2] V.E.Moiseenko *Modelling of Maxwell's Equations Using Uniform Finite Elements,* in this issue.

[3] V.E.Moiseenko, V.V.Pilipenko *Numerically Stable Reduced-Order ICRF Field Modeling in an Open Trap*, Transactions of Fusion Science and Technology, in press.

[4] V.E.Moiseenko, A.I.Zhukov, E.Tennfors, T.Hellsten In: Physics Computing (Proc. of 8th Joint EPS-APS International Conference, Krakow, 17-21 Sept. 1996), Ed. P.Borcherds, M.Bubak, A.Maksymowicz, Academic Computer Centre CYFRONET, Krakow, 1996, p.222.

[5] V.E.Moiseenko, V.V.Pilipenko Computer Physics Communications (141)**3** (2001) p. 342.