SMOOTH TRAJECTORIES ON TOROIDAL MANIFOLDS

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In the recent paper trajectories of the natural motion on toroidal manifolds are considered. It is represented an information dealing with description of the rotation metric at the different methods of toroidal analytical fixing, description of geodesic integrals, links between global motion invariants of the trajectory with the toroidal manifolds parameters. Analytical representations of geodesics at the different values of the global invariants are given.

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1. INTRODUCTION

Progress in cyclic accelerators with high – current density makes up interest to investigate geodesic trajectories of charged particle beam in a stellatron [1,2].

One of example of manifolds where an infinite particle motion exists are toroidal manifolds. Electromagnetic potentials of these manifolds have to satisfy some requirements [3] that the particle motion will be in restrict volume. The same potentials are generated by the current rods distributed in one or another manner as a rule on the surface surrounding the volume in which particle motion goes on. One of the simplest methods of current distribution on the surface of the manifolds is the linear dependence between surface coordinates of the manifold [4-6]. Without external forces the linear dependence between surfaces coordinates leads to arising force acting in tangent to the trajectory plane. The same force is absent at the motion along geodesic trajectory. Motion in tangent plane goes on with time independent velocity. Besides the connection length between next periods of the geodesic are minimal one.

Geodesic trajectories equation in quasicylinder coordinates system has been obtained in [7,8].

In the mentioned coordinates system possessed for certain "good" properties division of variables is difficulty. The marked difficulty is absent in toroidal coordinates [9].

2. METRIC OF TOROIDAL MANIFOLDS

Let x_1, x_2, x_3 are the Cartesian coordinates of point P of toroidal surface, and r, ϕ, θ are its spherical coordinates. Let the plane P0x_3 crosses the circle $x_1^2 + x_2^2 = c^2$, $x_3 = 0$ in the points A and B. The radius of the marked circle is a supreme verge upon changing of distances from point P till axis x_3. Adding x_3 values differed from zero we get distances less than c. Let us denote the angle between directions under which from point P you will see points A and B as $\angle APB = \phi$ and $\ln(PA/PB) = \eta_0$ (Fig. 1). It follows from the ratio for the triangle APB area that

$$2cx_3 = AP \cdot PB \sin\phi ,$$

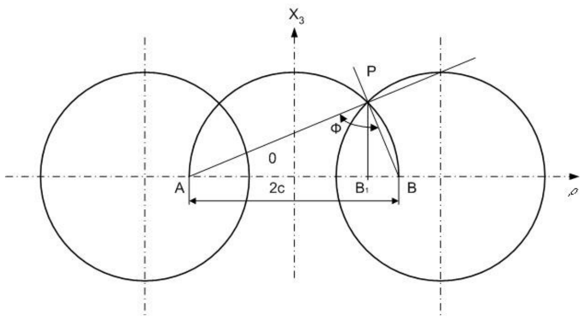

Fig. 1. *Definition for toroidal coordinates (η, ϕ, θ) of point P on given Cartesian coordinates (x_1, x_2, x_3) of point P*

and according in cosine theorem we have

$$4c^2 = AP^2 + PB^2 - 2AP \cdot PB \cos\phi .$$

That is why $PB^2 = 2c^2 e^{-\eta} / (ch\eta_0 - \cos\phi)$, and

$$x_3 = c \sin\phi / (ch\eta_0 - \cos\phi) .$$

For getting two rest connections linking Cartesian coordinates x_1, x_2 of point P u it's toroidal ones let us consider triangles APB_1 and B_1PB. If ρ is the distance from point P till axe x_3 then $AP^2 - x_3^2 = AB_1^2$, $PB^2 - x_3^2 = B_1B^2$, $AB_1 = c + \rho$, $B_1B = c - \rho$. Combining these connections we get

$$\rho = \frac{AP^2 - PB^2}{4c} = \frac{csh\eta_0}{ch\eta_0 - \cos\phi} .$$

Now representation φ of open set U on the 2 – dimensional manifold R^2, which is subset of set R^3 $\varphi : U \to R^2 \subset R^3$, can be written in the form

$$\varphi(\eta, \phi, \theta) = \frac{c}{ch\eta - \cos\phi}(sh\eta \cos\theta ,$$
$$sh\eta \sin\theta , \sin\phi).$$

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The coordinate surfaces in R^3 are: tori $\eta = \eta_0 = const$

$$\left(\sqrt{x_1^2 + x_2^2} - ccth\eta_0\right)^2 + x_3^2 = c^2/sh^2\eta_0 ,$$

spheres $\phi = \phi_0 = const$

$$x_1^2 + x_2^2 + \left(x_3 - cctg\phi_0\right)^2 = c^2/\sin^2\phi_0 ,$$

planes $\theta = \theta_0 = const \;\; tg\theta_0 = x_2/x_1$. To clear the geometrical meaning of scale factor c we need other than η, ϕ, θ parameterization for toroidal manifold

$$\varphi(r,\upsilon,\theta) = \left((R_0 - r\cos\upsilon)\cos\theta ,\; (R_0 - r\cos\upsilon)\sin\theta ,\sin\upsilon \right)$$

image of which is Descartes' product of two circles, that are: the circle of radius R_0 in (x_1,x_2) – plane (this circle is named azimuthally one, axis x_3 is the toroidal axis) and in (ρ,x_3) – plane, radius of which is equal r. The last circles lie in θ=const plane and are toroidal meridians (Fig. 2). In this parameterization the coordinate surfaces in R^3 are: tori $r=r_0=const$,

$$\left(R_0 - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 = r_0^2 ,$$

$\upsilon = \upsilon_0 = const$ are cones

$$x_1^2 + x_2^2 - \left(R_0 - x_3 cth\upsilon_0\right)^2 = 0 ,$$

vertexes of which have coordinates $(0, 0, R/ctg\upsilon_0)$.

Therefore it is valid correlations

$$R_0 = ccth\eta_0 ,\; r_0 = c/sh\eta_0 ,$$

and so the scale number $c = \sqrt{R_0^2 - r_0^2}$ equals digitally length of tangent in the plane θ=const to meridian led from the torus center. From the other side,

$ch\eta_0=R_0/r_0,$ $\eta_0 = \ln\left(R_0/r_0 + \sqrt{\left(R_0^2/r_0^2\right)^2 - 1}\right)$.

Just as hyperbolic cosine is the quantity greater the unity then the values $0<\eta<\eta_0$ correspond the area outside of a torus and values $\eta>\eta_0$ – the inside of a torus. The value $\eta\to\infty$ corresponds to the azimuthally axe of a torus on which z=0, ρ=c. Besides at $r_0>R_0$ hyperbolic cosine will be less than the unity that makes impossible single – valued attribution for every point coordinates η, ϕ, θ at ρ<c. Some more property differing coordinates η, ϕ, θ from r, υ, θ consists in that that the parabolic lines of a torus correspond the quantities $\upsilon=\pm\pi/2$ and in toroidal coordinates

$$\phi = \pm 2arctg\sqrt{\frac{ch\eta_0 - 1}{ch\eta_0 + 1}} .$$

In the table numerical relations for some ϕ and υ so and typical values x_3 and ρ are given for a torus. We have for the metric

$$g = d\rho^2 + \rho^2 d\theta^2 + dx_3^2$$

following expression in coordinates η, ϕ, θ:

$$g = \frac{c^2}{\left(ch\eta - \cos\phi\right)^2}\left(d\eta^2 + d\phi^2 + sh^2\eta\, d\theta^2\right).$$

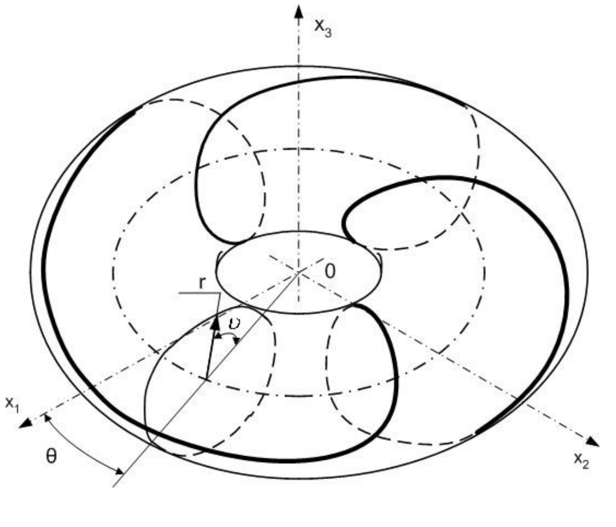

Fig. 2. *Closed geodesic with p=1, q=3*

ϕ	υ	z	ρ
0	π	0	$c\sqrt{\frac{ch\eta_0 + 1}{ch\eta_0 - 1}}$
$2arctg\sqrt{\frac{ch\eta_0 - 1}{ch\eta_0 + 1}}$	$\frac{\pi}{2}$	$c/sh\eta_0$	$ccth\eta_0$
$\frac{\pi}{2}$	$2arctg\sqrt{\frac{R_0 - r_0}{R_0 + r_0}}$	$c/ch\eta_0$	$cth\eta_0$
π	0	0	$c\sqrt{\frac{ch\eta_0 - 1}{ch\eta_0 + 1}}$

3. LOCAL EGUATION FOR GEODESIC

According known metric we can write local equation for geodesic on toroidal manifold $\eta=\eta_0=const$ [10] as a function arc length s

$$\frac{d\phi}{ds} = \frac{ch\eta_0 - \cos\phi}{c}\cos\alpha ,$$

$$\frac{d\theta}{ds} = \frac{ch\eta_0 - \cos\phi}{csh\eta_0}\sin\alpha ,$$

$$\frac{d\alpha}{ds} = \frac{\sin\phi}{c}\sin\alpha .$$

The set of equations for geodesic has two integrals. The first one, which composes Clairaut's theorem content $\sin\alpha = h\left(ch\eta_0 - \cos\phi\right)$ describes variation of angle α of geodesic inclination to a meridian η=const, θ=const. Later we miss zero indexes at η.

Angle α between geodesic and meridian is maximum $\sin\alpha_{\max} = h(ch\eta + 1)$ at the crossing by geodesic of torus inner equator. Minimal angle value equals $\sin\alpha_{\min} = h(ch\eta - 1)$ and attained at passing by geodesic through outer torus equator. Integration constant h along geodesic has finite magnitude according to Clairaut's theorem. The first integral permits to lead geodesic through two neighboring points.

Representation about geodesic behavior in a whole can be received by using the second integral of set of equations for geodesics [11]

$$\theta(\phi) = \int \frac{h(ch\eta - \cos\phi)d\phi}{sh\eta\sqrt{1 - h^2(ch\eta - \cos\phi)^2}} .$$

Depending on value of integration constant h geodesic will be tight or periodic on a toroidal manifold. Since it is examined the smooth geodesic then the periodicity means and closeness of geodesic.

To receive presentation about geodesic behavior on a toroidal manifold let us consider some partial quantities of integrate constant values.

At $h = (ch\eta - 1)^{-1}$ Clairaut's theorem is valid only at $\phi = 0$ and the second integral does not belong to the set R of the real – valued functions. Hence in this case there are not smooth geodesics.

Let Clairaut's constant is a small value. Then the second integral of geodesics has a form

$$\theta(\phi) \approx h(\phi\, cth\eta - \frac{\sin\phi}{sh\eta}).$$

In this case geodesic gets cover tightly a toroidal surface. The smallness of h means that inclination angle of geodesic to meridian of torus is small.

Another possibility for geodesic to be sometimes in toroidal surface region where Gauss curvature is positive so and in a region where it is negative consists in that that the angle α at the crossing by geodesic the inners equator of torus differs slightly from the right one. Integration constant in this case is equal

$$h \cong (ch\eta + 1)^{-1} - \delta .$$

For the estimation of integral for geodesics it is handy to introduce a new variable $\xi = tg(\phi/2)$ and rewrite it in the form

$$\theta(\xi) = \frac{2h}{sh\eta}(ch\eta + 1)(T_1 + T_2) .$$

The first integral

$$T_1(\xi) = \int \frac{d\xi}{\sqrt{1 - h(ch\eta - 1) + (1 - h(ch\eta + 1))\xi^2}} \cdot \frac{1}{\sqrt{1 + h(ch\eta - 1) + (1 + h(ch\eta + 1))\xi^2}}$$

we expand in a series on degrees of small δ and abandoning only the first term has

$$T_1(\xi) \cong \frac{\sqrt{ch\eta + 1}}{2} F(\varphi, k) .$$

$F(\varphi,k)$ is an elliptic function [12], its argument

$$\varphi = arctg\left(\sqrt{\frac{ch\eta + 1}{ch\eta - 1}}\xi\right),$$

and square of additional module is equal

$$k'^2 = 1 - k^2 \cong \frac{\delta}{2}(ch\eta + 1)ch\eta .$$

The second integral

$$T_2(\xi) = \frac{-2}{ch\eta + 1} \cdot \int \frac{(1 + \xi^2)^{-1} d\xi}{\sqrt{1 - h(ch\eta - 1) + (1 - h(ch\eta + 1))\xi^2}} \cdot \frac{1}{\sqrt{1 + h(ch\eta - 1) + (1 + h(ch\eta + 1))\xi^2}}$$

we wrote down in a series on δ abandoning only the first term

$$T_2(\xi) = \frac{-1}{\sqrt{ch\eta + 1}} \int \frac{(1 + \xi^2)^{-1} d\xi}{\sqrt{\frac{ch\eta}{ch\eta + 1} + \xi^2}} = -\frac{1}{2}\ln\left|\frac{\sqrt{ch\eta + (ch\eta + 1)\xi^2} + \xi}{\sqrt{ch\eta + (ch\eta + 1)\xi^2} - \xi}\right|.$$

After substitution T_1 and T_2 the second integral of geodesics takes the form

$$\theta(\xi) \cong \frac{1}{sh\eta}\left(\frac{2}{\pi}\sqrt{ch\eta + 1}\ln\frac{4}{k'} \cdot arctg\left(\sqrt{\frac{ch\eta + 1}{ch\eta - 1}}\xi\right) - \ln\left|\frac{\sqrt{ch\eta + (ch\eta + 1)\xi^2} + \xi}{\sqrt{ch\eta + (ch\eta + 1)\xi^2} - \xi}\right|\right).$$

4. CLOSED GEODESICS

Abovementioned Clairaut's constant values allows to describe geodesics which gets cover tightly a torus surface. Here we link Clairaut's constant with the global invariants for the closed geodesics.

Let any geodesic which is closed performs p rounds around torus axis (Fig. 2) and round a toroidal surface q times around the azimuthally circle axis. Then according Darboux's theorem [13] equality

$$\int_0^{\pi} \frac{h(ch\eta - \cos\phi)d\phi}{sh\eta\sqrt{1 - h^2(ch\eta - \cos\phi)^2}} = \frac{p}{q}\pi$$

is fulfilled at possible h (p and q are integral numbers). At $h(ch\eta - \cos\phi) << 1$ equation for the closed geodesics can be written in the form

$$\theta \cong \frac{p}{q}\left(\phi - \frac{\sin\phi}{ch\eta}\right).$$

Clairaut's constant as it is followed from closeness conditions is equal

$$h \cong \frac{p}{q} sh\eta .$$

Hence the value

$$\frac{p}{q} th\eta\,(ch\eta + 1) << 1$$

should be small.

The marked condition is satisfied slightly when p<<q. It is not valid at p=q=1.

The closed geodesic at the small inclination to the meridian angle makes up to closeness the revolution number around azimuthally axis of a torus, which is significantly greater than the geodesic round number of toroidal axis.

Some weaker looks limitation on the ratio p and q when the closed geodesic trajectory goes on the inners equator of a torus under the angle of near to zero degrees. In this case the small magnitude is

$$\delta\,(ch\eta + 1) = \frac{32}{ch\eta}\exp\left\{-2\left(\frac{p}{q}\pi\sqrt{ch\eta - 1} + \right.\right.$$

$$\left.\left. + \frac{1}{\sqrt{ch\eta + 1}}\ln\frac{\sqrt{ch\eta + 1} + 1}{\sqrt{ch\eta + 1} - 1}\right)\right\} << 1.$$

The closed geodesic trajectory itself

$$\theta\,(\phi) \cong 2\left(\frac{p}{q} + \frac{1}{\pi\, sh\eta}\ln\frac{\sqrt{ch\eta + 1} + 1}{\sqrt{ch\eta + 1} - 1}\right)$$

$$arctg\left(\sqrt{\frac{ch\eta + 1}{ch\eta}}tg\frac{\phi}{2}\right) - \frac{1}{sh\eta}\cdot$$

$$\cdot\ln\left|\frac{\sqrt{ch\eta + \sin^2\frac{\phi}{2}} + \sin\frac{\phi}{2}}{\sqrt{ch\eta + \sin^2\frac{\phi}{2}} - \sin\frac{\phi}{2}}\right|$$

is a smooth curve without any self-crossings.

5. RESULTS

1. Parameterization for a toroidal manifold in form of Cartesian product of two circles with radii R_0 and r_0 makes possible the link between scale number of toroidal coordinate system and radii of formed torus circles.
2. To the parabolic lines of toroidal manifold correspond different values of parameters υ and *ϕ* (see Table).
3. Toroidal coordinate system with rotation metric permits to write in evident forms the equation of smooth geodesic trajectory.
4. The angle between torus meridian and geodesic trajectory is maximum when geodesic goes on the inner equator of torus and minimum at crossing the outer one.
5. At the small angle between closed geodesic trajectory and meridian of torus loxodrome makes more rounds around torus axis till closing than geodesics.
6. When closed geodesic trajectory crosses the inner torus equator at the small angle geodesic is a smooth curve without any self - crossings.

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