COLLECTIVE EXCITATIONS AND SUPERFLUID PROPERTIES OF A TWO-DIMENSIONAL INTERACTING BOSE GAS IN AN EXTERNAL POTENTIAL

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We develop an approach for a description of collective excitations in a two-dimensional gas of interacting Bose particles in an external potential. We present a method of finding an approximate analytical solution for the spectra of collective excitations of a Bose gas in a linear potential and in a potential of the form $u(\mathbf{r}) = \mu - u_0 \cosh^2 x/l$, where μ is the chemical potential. Numerical study shows that the analytical solution corresponds to collective modes localized at the edge or at the low-density region. We investigate the influence of the external potential on a critical velocity of a superfluid flow. It is shown that the effect of strong suppression of the critical velocity takes place in a nonuniform Bose system. We discuss a possibility of Bose-Einstein condensation (BEC) in the systems under investigations at nonzero temperatures and find that in case of a finite number of the particles BEC can emerge.

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1. INTRODUCTION

The influence of an external potential on Bose-Einstein condensation (BEC) in two-dimensional (2D) Bose systems is one of challenge problems. In an ideal Bose gas an external potential may cause BEC at nonzero temperatures in two dimensions [1-3]. The question whether it is the case for interacting Bose gases has not been clearly understood yet. Discovery of BEC in alkali metal vapours confined in a trap [4-6] revives the interest to this question. Petrov *et al* [7] argued that at $T \neq 0$ a true condensate or a quasicondensate with a fluctuating phase can emerge in an interacting 2D Bose gas confined in a harmonic trap. A mean field study of a 2D interacting trapped gas was done by Bayindir and Tanatar [8]. Basing on the similarity in thermodynamic behavior of ideal 2D Bose systems and those with weak interactions they conclude that BEC takes place in systems with a finite numbers of the particles. Mullin [9,10] demonstrate the absence of BEC in trapped interacting Bose gases in 2D in the thermodynamic limit.

In the present paper we address this problem with reference on two specific forms of the external potential. We consider the potential $u(x,y) = \mu - u \cosh^2 \frac{x}{l}$ and the linear potential $u(x,y) = \mu - \alpha\, x$ (μ is the chemical potential). The first one corresponds to a situation when a low-density valley is formed in a Bose cloud. Since this potential pushes the particles away from the centre of the Bose cloud we call it an "anti-trap" one. The second potential models a Bose cloud with a linear increase of the density in a direction perpendicular to the edge. For such a choice of the potentials the analytical expressions for the low energy collective excitations can be found and macroscopic quantum properties of the systems can the investigated systematically. We should mention that the third example of the potential for which the analytical solution can be obtained is the harmonic trap potential. In 2D this problem was considered by Stringari [11].

In Sections 2 and 3 we outline our approach. In more details it was given in Ref. [12]. In Sec. 4 the influence of the external potential on the critical velocity of a superfluid flow is studied. In Sec. 5 we discuss the possibility of BEC in non-uniform systems.

2. BASIC EQUATIONS

We consider a Bose gas with a point interaction between the particles in an external potential $u(\mathbf{r})$. The Hamiltonian of the system has form

$$H = \int d^2r \{ \frac{\hbar^2}{2m} \nabla \hat{\Psi}^+ \nabla \hat{\Psi} + [u - \mu] \hat{\Psi}^+ \hat{\Psi} + \frac{1}{2} \gamma \hat{\Psi}^+ \hat{\Psi}^+ \hat{\Psi} \hat{\Psi} \}, \quad (1)$$

where $\hat{\Psi}$ is the boson field operator, m, the boson mass, γ, the interaction constant. To rewrite the Hamiltonian in terms of elementary excitation creation and annihilation operators we use the approach of Ref. [13]. We decompose the Bose field operators as

$\hat{\Psi} = e^{i\hat{\varphi}}\sqrt{\rho_0 + \hat{\rho}_1}$, where $\hat{\varphi}$ is the phase operator and $\hat{\rho}_1$, the density fluctuation operator. Expanding the Hamiltonian into powers of $\nabla\hat{\varphi}$ and $\hat{\rho}_1$ we obtain

$$H = H_0 + H_1 + H_2 + \cdots \quad , \tag{2}$$

where H_0 is the operator independent part,

$$H_1 = \int d^2r\,\hat{\rho}_1[u - \mu + \gamma\rho_0 - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho_0}}{\sqrt{\rho_0}}], \tag{3}$$

$$H_2 = \int d^2r[\sqrt{\rho_0}\hat{\varphi}\hat{T}\sqrt{\rho_0}\hat{\varphi} + \frac{1}{4}\frac{\hat{\rho}_1}{\sqrt{\rho_0}}\hat{G}\frac{\hat{\rho}_1}{\sqrt{\rho_0}} + i\frac{\hbar^2}{2m}\{[\nabla\hat{\rho}_1,\nabla\hat{\varphi}] - \frac{\nabla\rho_0}{\rho_0}[\hat{\rho}_1,\nabla\hat{\varphi}]\} \quad . \tag{4}$$

In Eq.(4)

$$\hat{T} = \frac{\hbar^2}{2m}(\nabla^2 - \frac{\nabla^2\sqrt{\rho_0}}{\sqrt{\rho_0}}), \quad \hat{G} = \hat{T} + 2\gamma\rho_0 . \tag{5}$$

We require the vanishing of the linear term in the Hamiltonian (2). It yields the equation for ρ_0

$$u - \mu + \gamma\rho_0 - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{\rho_0}}{\sqrt{\rho_0}} = 0 \ . \tag{6}$$

The value of ρ_0 coincides with the density of the particles at T=0. One should note that our consideration is valid at temperature $T << \hbar^2\rho_0/m$.

To reduce the Hamiltonian into a diagonal form we rewrite the phase and density fluctuation operators as

$$\hat{\varphi} = \frac{1}{2i\sqrt{\rho_0}}\sum_\nu(\theta_\nu\hat{b}_\nu - \theta_\nu^*\hat{b}_\nu^+), \tag{7}$$

$$\hat{\rho}_1 = \sqrt{\rho_0}\sum_\nu(F_\nu\hat{b}_\nu + F_\nu^*\hat{b}_\nu^+) \ , \tag{8}$$

where $\hat{b}, \hat{b}^+$ are the Bose operators. The functions F and θ satisfy the equations

$$\hat{T}\theta_\nu = E_\nu F_\nu \ , \tag{9}$$

$$\hat{G}F_\nu = E_\nu\theta_\nu \ , \tag{10}$$

They are normalized by the condition $\int d^2r\theta_\nu F_\nu^* = 1$. After substitution Eqs. (7,8) into Eq. (4) the quadratic part of the Hamiltonian reads as

$$H_2 = const + \sum_\nu E_\nu\hat{b}_\nu^+\hat{b}_\nu \ . \tag{11}$$

Eq. (11) shows that the operators $\hat{b}_\nu^+, \hat{b}_\nu$ are the creation and annihilation operators of the elementary excitations. The energies of the excitations E_ν can be found from the solution of Eqs. (9,10) with the boundary conditions specified.

3. SPECTRUM OF THE EXCITATIONS

Let us consider an external potential

$$u(\mathbf{r}) = \mu - u_0\cosh^2\frac{x}{l} . \tag{12}$$

For the parameters satisfying the inequality $u_0 >> \hbar^2/2ml^2$ one can neglect the fourth term in the l.h.s. of Eq. (6). It yields $\rho_0 = (u_0/\gamma)\cosh^2(x/l)$. In the low-energy approximation one can also omit the operator part in the quantity $\hat{G}$ and reduce the eigenvalue problem to the following one

$$-\frac{\hbar^2 u}{m}\cosh^2\frac{x}{l}\left[\frac{d^2\theta_\nu}{dx^2} + \left(\frac{1}{l^2} - k^2\right)\theta_\nu\right] = E_\nu^2\theta_\nu \ . \tag{13}$$

Here we take into account that the system is uniform in y direction and put $\theta(\mathbf{r}) = e^{iky}\theta(x)$. Eq. (13) is reduced to a hypergeometric equation and their general solution is expressed through hypergeometric functions.

For the system finite in the x direction with the rigid walls at $x = \pm L$ the flow through the walls should vanish. It determines the boundary conditions

$$\frac{d}{dx}\left(\frac{\theta}{\sqrt{\rho_0}}\right)\Bigg|_{x=\pm L} = 0 . \tag{14}$$

In the limiting case of $L \to \infty$ instead of Eq. (14) we should require the fluctuations be finite at $x \to \infty$. In this case the eigenvalue problem can be solved analytically. The spectrum has the form

$$E_{nk} = \omega_0\sqrt{\left(n + \sqrt{1 + k^2l^2}\right)\left(n + 1 + \sqrt{1 + k^2l^2}\right)}, \tag{15}$$

where $\omega_0 = \sqrt{\hbar^2 u_0/ml^2}$ and $n = 0,1,2\ldots$.

Comparing the analytical result with the solution obtained numerically for finite L we find that at $k > E_{nk}/(\hbar\bar{c})$ ($\bar{c}$ is the sound velocity in the uniform system with the same average density of the particles) Eq. (15) approximates the numerical solution with a good accuracy (Fig. 1).

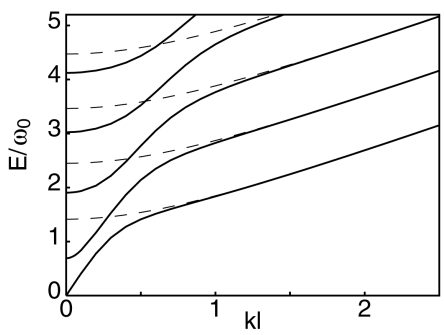

Fig. 1. *Spectrum of the excitations in the "anti-trap" potential. Solid curves – numerical solution for L/l=3, Dashed curves – analytical approximation (Eq. (15))*

The spatial dependence of the density fluctuation for the three lowest modes is shown in Fig. 2. One can see

that at small k the whole system is disturbed, while at large k the modes are localized at the low-density region. Analytical expression (15) describes the spectrum of the localized modes.

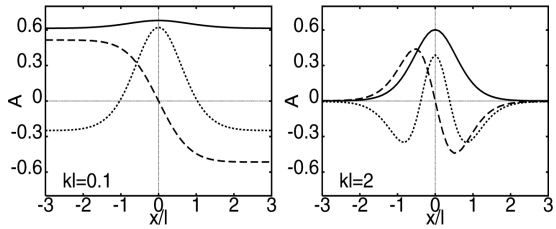

Fig. 2. *The amplitude (in relative units) of the density fluctuation for the low-energy collective modes in the "anti-trap" potential. Solid curves, n=0 mode; dashed curves, n=1, mode, dotted curves, n=2 mode*

For a linear potential $u(\mathbf{r}) = \mu - \alpha x$ the eigenvalue problem is reduced to a confluent hypergeometric equation

$$-\frac{\hbar^2\alpha}{m} x \left[\frac{d^2\theta_\nu}{dx^2} + \left(\frac{1}{4x^2} - k^2 \right)\theta_\nu \right] = E_\nu^2 \theta_\nu \quad . \tag{16}$$

If the system is confined in a region $0<x<L$ the solution should satisfy the boundary condition (14) at $x=+L$. The second boundary condition is the requirement for the solution to be finite at $x \to 0$.

The analytical expression for the excitation energies can be found in a limit $L \to \infty$. It has the form

$$E_{nk} = \sqrt{\frac{\hbar^2\alpha}{m} |k| (2n+1)} \tag{17}$$

($n = 0,1,2\ldots$). The spectra of the excitations given by the analytical expression and by the numerical solution are presented in Fig. 3.

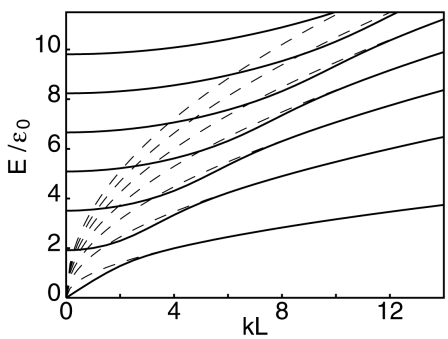

Fig. 3. *The spectrum of the excitations in the linear potential* ($\varepsilon_0 = \sqrt{\hbar^2\alpha / mL}$). *Solid curves – numerical solution, dashed curves – analytical approximation (Eq. (17))*

Here as in the previous case the analytical solution is valid at $k > E_{nk}/(\hbar\bar{c})$. The spatial dependence of the density fluctuations for three lowest modes is shown in Fig. 4. One can see that analytical solution (17) corresponds to the modes localized at the edge.

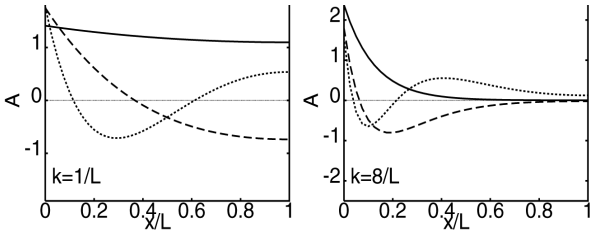

Fig. 4. *The amplitude of the density fluctuation for the low enegry modes in the linear potential. Solid, dashed and dotted curves – n=0,1,2 modes, correspondingly*

Eq. (16) was derived in a linear approximation for the function $\rho_0(x)$. This approximation is correct at $x > x_0 = \sqrt[3]{\hbar^2 / m\alpha}$, when the fourth term in the l.h.s. of Eq. (6) can be neglected. The solutions of Eq. (16) singular at $x \to 0$ can be omitted at $kx_0 << 1$. It yields the condition

$$k < k_c = \left(\frac{\alpha m}{\hbar^2} \right)^{1/3} \quad . \tag{18}$$

Inequality (18) establishes the validity of the analytical as well as the numerical solution.

4. CRITICAL VELOCITIES

In this section we consider the influence of the external potential on a critical velocity of a superfluid flow. We specify the system infinity in x direction with a nonuniform density area of a finite width. Let the external potential has the form

$$u = \begin{cases} \mu - u_0 \cosh^2 \dfrac{x}{l} & \text{at } x \le L \\ U & \text{at } x \ge L \end{cases}, \tag{19}$$

where $U = \mu - u_0 \cosh^2(L/l)$. In the potential (19) a Bose cloud has a low-density valley of the width $2L$ aligned along the y direction. The valley separates two uniform density areas. In this case the analytical solution (15) corresponds to the modes localized in the valley. The extended modes have higher energies at the same k ($E_{ext} = \hbar\bar{c}\sqrt{k^2 + k_x^2}$). Using the Landau criterium we determine the critical velocity for the superfluid flow along the y direction as

$$v_c = \min\left(\frac{E_{nk}}{\hbar k} \right). \tag{20}$$

Since the n=0 localized mode has the lowest energy one should substitute Eq. (15) with n=0 into Eq. (20), taking into account that Eq. (15) is valid at $k_1<k<k_2$. Here k_1 is given by the equation $E_{0k_1} = \hbar\bar{c}k_1$. Since the low-energy approximation was used in Eq. (15), the upper restriction on k emerges. This approximation requires $\hbar^2 k^2 / 2m < 2\gamma\rho_0(0)$. It yields $k_2 = (2/L_c)\cosh^{-1}(L/l)$, where $L_c = \hbar / m\bar{c}$ is some

length[1]. At $k_1 < k_2$ the critical velocity is equal to $E_{0k_2}/\hbar k_2$. If the opposite inequality $k_1 > k_2$ is satisfied the critical velocity is determined by the extended modes. The last ones yield $v_c = \bar{c}$. The general formula for v_c is the following

$$v_c = \bar{c}\min\left[1, f\left(\frac{l}{L_c}, \frac{L}{L_c}\right)\right], \tag{21}$$

where

$$f(x,y) = \frac{1}{2x}\left(1 + 4x^2\,\mathrm{sech}^2\frac{y}{x}\right)^{\frac{1}{4}} \times \left(1 + \sqrt{1 + 4x^2\,\mathrm{sech}^2\frac{y}{x}}\right)^{\frac{1}{2}}. \tag{22}$$

At $l/L_c, L/L_c >> 1$ we find $v_c = \bar{c}\cosh^{-1}(L/l)$. In this limit the presence of the low-density valley results in a strong suppression of the critical velocity. When the ratio L/L_c becomes smaller the critical velocity grows up. It is illustrated in Fig. 5.

At $L < L_c$ Eq. (21) predicts no suppression of the critical velocity. Physically, such a case corresponds to two weakly coupled semi-infinite Bose clouds. Thus, the quantity L_c plays a role of a critical parameter for the width of the low-density valley. If this width becomes large then $2L_c$ the critical velocity decreases.

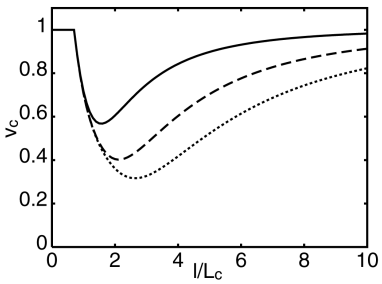

Fig. 5. *Critical* velocity (in units of $\bar{c}$) in the "anti-trap" potential. *Solid curve – L/L_c=3; dashed curve – L/L_c=5; dotted curve – L/L_c=7*

Let us then consider the potential of the form

$$u = \begin{cases} \mu - \alpha x & \text{at } x \le L \\ \mu - \alpha L & \text{at } x \ge L \end{cases}. \tag{23}$$

In such a potential the density of the particles increases linearly from x=0 to x=L and then at x>L becomes uniform. For this geometry Eq. (17) yields the spectrum of collective modes localized at the edge. These modes have lower energies then extended ones and determine the critical velocity. The value of the critical velocity is given by the formula (20) with the energy (17). The values or k are restricted by the condition (18). They should also satisfy the inequality $\hbar k\bar{c} > E_{ok}$. If $\hbar k_c\bar{c} < E_{ok_c}$ (this inequality is equivalent to $L < L_c$) the localized modes do not exist and the spectrum of the extended modes should be put into Eq. (20). The dependence of the critical velocity on the parameters of the system has the form

$$v_c = \bar{c}\begin{cases} \left(\dfrac{L}{L_c}\right)^{1/3} & \text{at } L > L_c \\ 1 & \text{at } L < L_c \end{cases}. \tag{24}$$

One can see from Eq. (24), that a suppression of the critical velocity takes place at L large then the critical length L_c.

We should note, that Eqs. (21,24) should be understood as estimate expressions. They describe the situation more qualitatively then quantitatively. Rigorous consideration should be based on the solution of Eqs. (6,9,10) without the approximations used in Sec. 3.

5. BOSE-EINSTEN CONDENSATION

A temperature dependence of the density of the Bose-Einstein condensate can be extracted from the asymptotic behavior of the one particle density matrix $\langle\Psi^+(\mathbf{r})\Psi(\mathbf{r}')\rangle$. If this quantity remains finite at $|\mathbf{r}-\mathbf{r}'|\to\infty$ then the Bose condensate exists and its density $n_c(\mathbf{r})$ is given by the equation

$$\lim_{|\mathbf{r}-\mathbf{r}'|\to\infty}\langle\Psi^+(\mathbf{r})\Psi(\mathbf{r}')\rangle = \left(n_c(\mathbf{r})n_c(\mathbf{r}')\right)^{1/2}. \tag{25}$$

In 2D uniform systems at $T \ne 0$ the quantity (25) is equal to zero. The destruction of the coherence is caused mainly by the thermally excited phase fluctuations. Taking into account only the phase fluctuation we obtain

$$\langle\Psi^+(\mathbf{r})\Psi(\mathbf{r}')\rangle = \left(\rho_0(\mathbf{r})\rho_0(\mathbf{r}')\right)^{1/2} e^{-\frac{\Phi(\mathbf{r},\mathbf{r}')}{2}}, \tag{26}$$

where $\Phi(\mathbf{r},\mathbf{r}') = \left\langle\left(\hat{\varphi}_{\mathbf{r}} - \hat{\varphi}_{\mathbf{r}'}\right)^2\right\rangle$. It is convenient to direct the vector $\mathbf{r}-\mathbf{r}'$ along the x axis. Then

$$\Phi(x,y;x,y') = \frac{1}{\rho_0(x)}\sum_{nk}|\theta_{nk}(x)|^2 \times \left(1 - \cos k(y-y')\right)\left(n_B\left[\frac{E_{nk}}{T}\right] + \frac{1}{2}\right). \tag{27}$$

Let us consider a Bose cloud of a rectangle shape $L\times L_y$ in the "anti-trap" potential and evaluate the value (27) at x=0. The answer at $|y-y'| > l$, $T >> \omega_0$ is

$$\Phi(0,y;0,y') = \frac{T}{2\pi T_0}\ln\frac{l}{\xi_T} + \frac{T}{2\pi\bar{T}_0}\ln\frac{|y-y'|}{\bar{\xi}_T} + \Phi_0, \tag{28}$$

where $T_0 = \hbar^2\rho_0(0)/2m$, $\bar{T}_0 = \hbar^2\bar{\rho}_0/2m$, $\xi_T = \sqrt{\hbar^2\gamma\rho_0(0)/mT^2}$ $\bar{\xi}_T = \sqrt{\hbar^2\gamma\bar{\rho}_0/mT^2}$. The first term in the r.h.s. of Eq. (28) is caused by the localized modes, the second one – the acoustic mode and the third

[1] If the density of the particles in the uniform part is fixed, this length remains constant under variation of the parameters of the external potential.

one - the zero point fluctuations. The value of Φ_0 does not depend of the temperatures and determine the density of the condensate n_{c0} at T=0. While the quantity (28) depends on $|y-y'|$, for the system with $L \sim L_y >> l$ this dependence can be neglected. In this case the density of the condensate at $T \neq 0$ is equal to

$$n_c(0) = n_{c0}(0)\left(\frac{\xi_T}{l}\right)^{\frac{T}{4\pi T_0}} \qquad (29)$$

It is important to note that our result does not contradict the general theorems [14,15] about the absence of true Bose-Einstein condensation in two-dimensional systems in the thermodynamic limit. In this limit one should put the total number of the particles N tends to infinity, keeping the average density unchanged. Then the quantity L also tends to infinity while the ratio L/l remains constant. Correspondingly, if such a definition of the thermodynamic limit is implied, the parameter l tends to infinity as well. One can see from Eq. (29) that the density of the condensate approaches to zero at $l \to \infty$. It is obvious result, since at $l \to \infty$ the system becomes locally uniform at all scales.

In case of linear potential similar derivation yields

$$n_c(x) = n_{c0}(x)\left(\frac{\xi_T(x)}{x}\right)^{\frac{T}{4\pi T_0(x)}}, \qquad (30)$$

where $T_0(x) = \hbar^2 \rho_0(x)/2m$, $\xi_T(x) = \sqrt{\hbar^2 \gamma \rho_0(x)/mT^2}$, and the inequality $\xi_T(x) < x$ is implied. In the thermodynamic limit we are interested in a density of the condensate at x_c, satisfying the condition $\rho_0(x_c) = const$ at $N \to \infty$. Substituting into Eq. (30) $x = x_c$ and taking into account that in the thermodynamic limit $x_c \to \infty$, we find that the density of the condensate tends to zero.

One can see, that for the potential considered (as well as for the harmonic trap potential [9,10]) the definition of the thermodynamic limit implies that the form of the potential depends on the total number of the particles. In practice, the form of the potential is fixed and under variation of the total number of the particles the average density is changed. Since the density cannot be infinitely large, the number of the particles should remain finite. Thus, our conclusion about the existence of the Bose-Einstein condensate at nonzero temperatures is formally applicable only to the systems with a finite number of the particles. But, in practice, this limitation is not important, because just such systems are used in experimental studies of BEC.

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