QUASI-SIMPLE SECONDARY WAVES IN A GAS OF QUASI-PARTICLES

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The equations describing simple secondary waves are obtained in gas dynamics with conserved and non-conserved number of quasi-particles at interactions. The non-linearity parameter in phonon and magnon gas dynamics is found, which appears to be in its value of an order of unity. The generalized Burgers equation is obtained describing the quasi-simple secondary waves.

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INTRODUCTION

A system of equations of gas dynamics of bose quasi-particles has been obtained in the work [1]. This system of equations is non-linear relative to the independent variables: the drift velocity u, the relative temperature $\theta = (T - T_0)/T_0$ (T_0 is the equilibrium temperature) and chemical potential μ, if the number of quasi-particles at interactions is kept constant. In the linear approximation, this system of equations describes, in certain circumstances, the weakly attenuating secondary waves [1, 2] being similar to the second sound waves in He II [3]. These waves relate to a hyperbolic type, i.e. their dispersion is absent. The paper deals with the non-linear secondary waves in a gas of quasi-particles, which are described by a system of non-linear equations of gas dynamics of quasi-particles in the second approximation. This system of equations is similar to the system of non-linear equations of gas dynamics of particles, describing the propagation of non-linear acoustic waves in a medium without dispersion [4-6]. Using this system of equations for the one-dimensional case, when there is no dissipation, we can derive an equation describing simple Riemann waves. Taking into account the dissipative processes one can find the generalized Burgers equation describing quasi-simple waves. These waves are featured by the distortion of their profiles, like, for example, in the case of a periodic wave - a saw-tooth profile is formed.

SIMPLE SECONDARY WAVE EQUATION

The simplest way is to obtain the simple secondary wave equation in isotropic gas of quasi-particles with the non-conserved number of quasi-particles. In the case of a plane along the *x*-axis, which can be realized in, for instance, an infinite plate of finite thickness, $u = u(x,t)$, $T = T(x,t)$. A system of gas dynamics equations in the second approximation, in terms of independent variables and kinetic equation will be written in the following form

$$\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial x} + 2P\frac{\partial u}{\partial x} + S_0\frac{\partial T}{\partial x} = \left(\frac{4}{3}\tilde{\eta} + \tilde{\zeta}\right)\frac{\partial^2 u}{\partial x^2} - ru,$$

$$\frac{C}{T}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x}\right) + \frac{\partial P}{\partial T}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + S\frac{\partial u}{\partial x} = \frac{\tilde{\kappa}}{T}\frac{\partial^2 T}{\partial x^2}, \tag{1}$$

where S, C are the densities local equilibrium entropy and heat capacity of a gas of quasi-particles, $P = \tilde{\rho}u$ is the pulse density, $\tilde{\rho}$ is the density of a gas of quasi-particles, $\tilde{\eta}$, $\tilde{\zeta}$ are the first and second viscosity kinetic coefficients, $\tilde{\kappa}$ is the hydrodynamic thermal conductivity at account of normal processes of quasi-particle interactions. r is the coefficient of external friction, as a result of the quasi-particle interaction effects leading to non-conservation of their momentum, e.g. the umklapp processes (U-processes). In the absence of dissipation the system of equations (1) describes simple secondary waves similar to Riemann waves [4-6]. In a simple wave, all quantities are the functions of one quantity. Let us assume that $\tilde{\rho}$ and T are the functions of the drift velocity u and $\tilde{\rho} = \tilde{\rho}(u, T(u))$; $T = T(u)$, $\frac{d\tilde{\rho}}{du} = \frac{\partial\tilde{\rho}}{\partial u} + \frac{d\tilde{\rho}}{dT}\frac{\partial T}{\partial u}$. By substituting these values into (1), we obtain:

$$\left(\tilde{\rho} + u\frac{\partial\tilde{\rho}}{\partial u} + u\frac{\partial\tilde{\rho}}{\partial T}\frac{dT}{du}\right)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \left(2u\tilde{\rho} + S\frac{dT}{du}\right)\frac{\partial u}{\partial x} = 0$$

$$\left(\frac{C}{T}\frac{dT}{du} + u\frac{\partial\tilde{\rho}}{\partial T}\right)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + S\frac{\partial u}{\partial x} = 0. \tag{2}$$

If $\partial u/\partial x \neq 0$, then from the conditions of existence of a solution it follows that:

$$\frac{dT}{du} = \sqrt{\frac{T\tilde{\rho}}{C}}\left[-M \pm \sqrt{1 + M^2\left(1 - 2\frac{S}{C}\frac{\partial\ln\tilde{\rho}}{\partial\ln T}\right) + \frac{u}{\tilde{\rho}}\frac{\partial\tilde{\rho}}{\partial u}}\right], \tag{3}$$

where $M = \frac{u}{W_{II}}$ is the number analogue to the Mach number in acoustics $(M << 1)$ $W_{II} = \left(TS_0^2/C\tilde{\rho}\right)^{1/2}$ is the quasi-equilibrium velocity of secondary waves. By substituting (3) into any of the equations (2), we obtain the following simple wave equation in the second approximation by the Mach number M

$$\frac{\partial u}{\partial t} \pm W_{II0}\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\left(2 + \overline{\frac{S}{C}\left(\frac{\partial \ln(W_{II}/\tilde{\rho})}{\partial \ln T}\right)}\right) = 0, \quad (4)$$

where $W_{II0}^2 = \frac{T_0\bar{S}^2}{\tilde{\rho}_0\bar{C}}$ is the equilibrium velocity of secondary waves. The line signifies the equilibrium values of the quantity are taken.

It is conveniently to deal with the variables $x = x'$ and $\tau = t - (x/W_{II0})$, when solving the boundary-value problems. Then $\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}$; $\frac{\partial}{\partial x} = -\frac{1}{W_{II0}}\frac{\partial}{\partial \tau} + \frac{\partial}{\partial x'}$ and from (4) it follows (omitting the prime x):

$$\frac{\partial u}{\partial x} - \frac{\varepsilon}{W_{II0}^2}u\frac{\partial u}{\partial \tau} = 0, \quad (5)$$

where the non-linearity parameter equals

$$\varepsilon = 2 + \overline{\frac{S}{C}\frac{\partial \ln(W_{II}/\tilde{\rho})}{\partial \ln T}}.$$

If the number of quasi-particles is conserved, then a simple secondary wave equation can be derived similarly. Assuming the quantities T, μ and $\tilde{\rho}$ to be the functions of the drift velocity u: $T = T(u)$, $\mu = \mu(u)$, $\tilde{\rho} = \tilde{\rho}(u, T(u), \mu(u))$ and $\frac{d\tilde{\rho}}{du} = \frac{\partial\tilde{\rho}}{\partial T}\frac{dT}{du} + \frac{\partial\tilde{\rho}}{\partial \mu}\frac{d\mu}{du}$, in the absence of dissipation, one can obtain

$$\left(\tilde{\rho} + u\frac{\partial\tilde{\rho}}{\partial u} + u\frac{\partial\tilde{\rho}}{\partial T}\frac{dT}{du} + u\frac{\partial\tilde{\rho}}{\partial \mu}\frac{d\mu}{du}\right)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) +$$
$$+\left(2u\tilde{\rho} + S\frac{dT}{du} + n\frac{d\mu}{du}\right)\frac{\partial u}{\partial x} = 0,$$
$$\left(\frac{\alpha}{T}\frac{dT}{du} + \frac{\partial n}{\partial u} + \frac{\beta}{T}\frac{d\mu}{du}\right)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + n\frac{\partial u}{\partial x} = 0, \quad (6)$$
$$\left(\frac{C}{T}\frac{dT}{du} + u\frac{\partial\tilde{\rho}}{\partial T} + \frac{\alpha}{T}\frac{d\mu}{du}\right)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + S\frac{\partial u}{\partial x} = 0.$$

From the solvability conditions of the system (6), we can find the relation between dT/du and $d\mu/du$ and the following expression for dT/du in zero approximation by the Mach number M

$$\frac{dT}{du} = \pm\frac{TS^*}{W_{II}^*C^*} \quad (7)$$

where $S^* = S - (n\alpha/\beta)$; $C^* = C - (\alpha^2/\beta)$; $W_{II}^{*2} = \frac{S^*T}{\tilde{\rho}C^*}\left(S^* + \frac{C^*n^2}{S^*\beta}\right)$, the hydrodynamic quantities n, α, β are determined in [1].

Proper substitution of variables gives the following simple secondary wave equation in gas dynamics with the conserved number of quasi-particles

$$\frac{\partial u}{\partial x} - \frac{\varepsilon^*}{(W_{II0}^*)^2}u\frac{\partial u}{\partial \tau} = 0, \quad (8)$$

where the non-linearity parameter equals

$$\varepsilon^* = 2 + \overline{\frac{S^*}{C^*}\frac{\partial \ln(W_{II}^*/\tilde{\rho})}{\partial \ln T}} + \overline{\frac{C^*n - \alpha S^*}{C^*}\frac{\partial \ln(W_{II}^*/\tilde{\rho})}{\partial \ln \mu}}. \quad (9)$$

Because secondary waves are essentially the thermal waves, it is more convenient to deal with the equations with more natural variable θ rather than with for the drift velocity. In the linear approximation, the drift velocity u and θ, as it follows from (3), (7) are related to each other in gas dynamics with the non-conserved number of quasi-particles by the relationship $u = \theta\overline{W_{II0}C/S}$, and by the relationship $u = \theta\overline{W_{II0}^*C^*/S^*}$ in the case of the conserved number. By substituting these values into (5) and (8), we obtain the simple secondary wave equation in terms of θ:

$$\frac{\partial\theta}{\partial x} - \nu\theta\frac{\partial\theta}{\partial \tau} = 0, \quad (10)$$

where $\nu = \varepsilon\overline{\frac{C}{SW_{II0}}}$ ($\nu = \varepsilon^*\overline{\frac{C^*}{S^*W_{II0}^*}}$).

Let us present the values of the non-linearity parameters ε and ε^* for some specific gases of quasi-particles. In gas dynamics of phonons, using the values of the necessary thermodynamic quantities given in [1], we obtain $\varepsilon = 2/3$ [7]. In gas dynamics of magnons in ferromagnetics with magnetic anisotropy of the "light axis" type in the case when the number of magnons is not conserved and energy of magnon activation is small ($\varepsilon_a << T$) and using the given in [1] values of thermodynamic quantities characteristic of magnon gas, we have $\varepsilon = 4/3$. If an interchange magnon scattering appears to be decisive at a number of magnons being constant, one can easily verify that in this case the non-linearity parameter equals as well $\varepsilon^* = 4/3$.

In gas dynamics of magnons in anti-ferromagnetics with magnetic anisotropy of the "light plane" type, in the low temperature range, a three-magnon interaction [1] appears to be decisive; in this case the number of phonons is not conserved. In this case the value of the non-linearity parameter will be of order of unity $\varepsilon \sim 1$. Solutions of the equation (10) are well known [4 - 6]. As the simple wave front spins, the importance of dissipation coefficients is increasing. Further evolution of the non-linear secondary waves will be then described by the system (1).

GENERALIZED BURGERS EQUATION FOR QUASI-SIMPLE SECONDARY WAVES

At small (but finite) amplitudes and small dissipation coefficients, there is a solution of the system (1), which can be considered as an analogue of the simple waves propagating along one-way direction [5, 6]. Such waves are called quasi-simple. To obtain the equations describing these waves, we shall assume - in much the same way as it was made in [8] - that all quantities are the functions of one of them with an accuracy of some arbitrary small function, i.e.

$$\tilde{\rho} = \tilde{\rho}(u, T); \quad T = T(u) + \psi(x,t) \quad (11)$$

We are seeking such a form of this function when the corresponding solution is the most close to a simple wave. Let us consider the function $\psi(x,t)$ to be a quantity of the second order smallness. Obviously, it will satisfy the secondary wave equation up to the second order terms, when the waves are propagating along the positive direction of the x axis

$$\frac{\partial \psi}{\partial t} + W_{II0}\frac{\partial \psi}{\partial x} = 0\,. \qquad (12)$$

Substituting (11) and (12) in (1), we received in the second approximation a following system:

$$\left(\tilde{\rho} + u\frac{\partial \tilde{\rho}}{\partial u} + u\frac{\partial \tilde{\rho}}{\partial T}\frac{dT}{du}\right)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) +$$

$$+\left(2u\tilde{\rho} + S\frac{dT}{du}\right)\frac{\partial u}{\partial x} + S\frac{\partial \psi}{\partial x} = \left(\frac{4}{3}\tilde{\eta} + \tilde{\zeta}\right)\frac{\partial^2 u}{\partial x^2} - ru\,,$$

$$\left(\frac{C}{T}\frac{dT}{du} + u\frac{\partial \tilde{\rho}}{\partial T}\right)\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + S\frac{\partial u}{\partial x} + \frac{C}{T}\frac{\partial \psi}{\partial t} = \tilde{\kappa}\frac{dT}{du}\frac{\partial^2 u}{\partial x^2} \qquad (13)$$

Using expression (3) to within the linear members on u and expressing from equation (12) $\frac{\partial \psi}{\partial t}$ through $\frac{\partial \psi}{\partial x}$, from (13) we received the value $\frac{\partial \psi}{\partial x}$:

$$\frac{\partial \psi}{\partial x} = \frac{\tilde{\rho}}{2S}\left[\frac{1}{\tilde{\rho}}\left(\left(\frac{4}{3}\tilde{\eta} + \tilde{\zeta}\right)\frac{\partial^2 u}{\partial x^2} - ru\right) - \frac{\tilde{\kappa}}{C}\frac{\partial^2 u}{\partial x^2}\right]. \qquad (14)$$

If we substitute (14) into the first equation in (1), in which the function ψ is taken into account, we find the following generalized Burgers equation for quasi-simple waves propagating along the positive directions of the x axis

$$\frac{\partial u}{\partial t} + W_{II0}\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x}\left(2 + \frac{\bar{S}}{C}\left(\frac{\partial \ln(W_{II}/\tilde{\rho})}{\partial \ln T}\right)\right) =$$

$$= \frac{1}{2}\left[\left(\frac{\tilde{\kappa}}{C} + \frac{1}{\tilde{\rho}}\left(\frac{4}{3}\tilde{\eta} + \tilde{\zeta}\right)\right)\frac{\partial^2 u}{\partial x^2} - u\frac{r}{\tilde{\rho}}\right]. \qquad (15)$$

Proceeding to the variables τ and x' and assuming that the wave profile is changed slowly, we can neglect the derivatives with respect to x assuming that they increase the order of smallness. With account of the above, we shall obtain the following generalized Burgers equation.

$$\frac{\partial u}{\partial x} - \frac{\varepsilon}{W_{II0}}\frac{\partial u}{\partial \tau}u = \frac{1}{2}\left[\left(\frac{\tilde{\kappa}}{C} + \frac{1}{\tilde{\rho}}\left(\frac{4}{3}\tilde{\eta} + \tilde{\zeta}\right)\right)\frac{\partial^2 u}{\partial \tau^2} - u\frac{r}{\tilde{\rho}}\right] \qquad (16)$$

Using the relationships between the drift velocity u and θ, we shall write the Burgers equation (16) in the more convenient form in order to carry out further investigations in terms of θ:

$$\frac{\partial \theta}{\partial x} - \nu\theta\frac{\partial \theta}{\partial \tau} = \delta\frac{\partial^2 \theta}{\partial \tau^2} - \gamma\theta\,, \qquad (17)$$

where $\delta = \frac{1}{2W_{II0}^2}\left[\frac{\tilde{\kappa}}{C} + \frac{1}{\tilde{\rho}}\left(\frac{4}{3}\tilde{\eta} + \tilde{\zeta}\right)\right]$, $\gamma = \frac{r}{\tilde{\rho}_0 W_{II0}^2}$. If one employs the condition of existence ("window") of secondary waves [1], then $\delta >> \gamma$. In this case, the last term in (17) could be neglected and we come to the conventional Burgers equation [5,6,8,9], being one of the most comprehensively studied evolution equations in the non-linear wave theory

$$\frac{\partial \theta}{\partial x} - \nu\theta\frac{\partial \theta}{\partial \tau} = \delta\frac{\partial^2 \theta}{\partial \tau^2}\,. \qquad (18)$$

As it was shown by Hopf [10] and Cole [11], by substituting the variables $\theta = \frac{2\delta}{\nu}\frac{\partial \ln\varphi}{\partial \tau}$ and performing the one-fold integration, the equation (18) takes the form of the linear equation of heat conductivity

$$\frac{\partial \varphi}{\partial x} = \delta\frac{\partial^2 \varphi}{\partial \tau^2} \qquad (19)$$

which is capable of being precisely integrated. This gives a rare opportunity to precisely solve the wave problem for a dissipation medium. If we know boundary condition $\theta(0,\tau) = \theta_0(\tau)$ the solution of equation (19) can be written in the following from:

$$\varphi(x,\tau) = \frac{1}{\sqrt{4\pi\delta\, x}}\int_{-\infty}^{\infty}\exp\left\{-\frac{(\tau - \tau')^2}{4\delta\, x} - \frac{1}{2\delta}\int_0^{\tau'}\theta_0(\tau'')d\tau''\right\}d\tau'$$

REFERENCES

1. A.I. Akhiezer, V.F. Aleksin, V.D. Khodusov. Gas dynamics of quasi-particles // *Low Temp. Phys.* 1994, v. 20, №12, p. 939-970.
2. A.I. Akhiezer, V.F. Aleksin, V.D. Khodusov. For theory of secondary waves // *Ukr. Fiz. Zh.* 1985,v. 30, №8, p. 1248-1262.
3. L.D. Landau. The theory of superfluid helium II. // *Zh. Eksp. Teor. Fiz.* 1941, v. 11, №5, p. 592-614.
4. L.D. Landau and E.M. Lifshits. *Fluid Dynamics*. Moskow: "Nauka", 1988, 736 p.
5. K.A. Naugol'nykh, L.A. Ostrovsky. *Nonlinear waves processes in acoustics.* Moscow: "Nauka", 1990, 235 p.
6. O.V. Rudenko and S.I. Soluyan. *Theoretical foundations of nonlinear acoustics*. Moscow: "Nauka", 1975, 287 p.
7. V.D. Khodusov. Second sound waves of finite amplitude in solids // *AIP Conference Proceeding*. 2000, v. 524, p. 261-264.
8. V.I. Karpman. *Nonlinear waves in dispersive media*. Moscow: "Nauka", 1973, 175 p.
9. J.M. Burgers Application of a model illustrate some points of statistical theory of free turbulence // *Proc. Roy. Neth. Acad. Sci. (Amsterdam),* 1940, v. 43, p. 2-12.
10. E. Hopf The partial differential $u_t + u \cdot u_x = \mu \cdot u_{xx}$ // *Comm. Pure Appl. Math.* 1950, v. 3, p. 201-230.
11. J.D. Cole On a quasilinear parabolic equation occurring in aerodynamics // *Appl. Math.* 1951, v. 9, p. 225-236.