ON A SIMPLE MODEL OF THE PHOTONIC OR PHONONIC CRYSTAL

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A model is proposed for a one-dimensional dielectric or elastic superlattice (SL) that relatively simply describes the frequency spectrum of electromagnetic or acoustic waves. The band frequency spectrum is reduced to mini-bands contracting with increasing frequency. A procedure is suggested for obtaining local states near a defect in a SL, and the simplest of these states is described. Conditions for the initiation of Bloch oscillations of a wave packet in a SL are discussed.
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1. By a photonic crystal is meant a macroscopic periodic structure composed of two spatially alternating dielectrics differing in dielectric constants (velocities of electromagnetic waves) [1]. Analogously, by a phononic crystal or acoustic superlattice (SL) is meant a periodic structure composed of two alternating elastic materials differing in elastic moduli and velocities of sound (the general acoustics theory of layered media is expounded in [2], and a useful bibliography on acoustic SLs is given in one of the last publications [3]). A great number of publications are devoted to studying the frequency spectrum of SLs. It is clear that, in the general case, this spectrum is extremely complicated and contains a system of both a great number of eigenfrequency bands and gaps corresponding to forbidden frequencies of eigenmodes. In order to characterize such spectra qualitatively and to illustrate their main quantitative features, it would be appropriate to use simple models that allow for these features. The well-known ID Kronig-Penney model [4] may serve as an example of such a model in the electronic theory of crystals. In this work, a model of a SL is proposed that provides an analytical description of the high-frequency part of its spectrum and suggests a possible implementation of an interesting acoustic SL.

Consider a SL in the form of alternating plane-parallel layers of two materials differing in either elastic or dielectric (depending on the implementation of interest) characteristics. Denote the layer thicknesses by d_1 and d_2; then, the SL period equals $d = d_1 + d_2$. The elastic or electromagnetic field inside each material, which is assumed to be isotropic, is described by the wave equation

$$\frac{\partial^2 u^\alpha}{\partial t^2} - c_\alpha^2 \frac{\partial^2 u^\alpha}{\partial x^2}, \quad \alpha = 1,2, \qquad (1)$$

where c_α is the wave velocity in the layer of the α type. The velocity of light in a dielectric equals $c_\alpha = c/\sqrt{\varepsilon}$ (c is the velocity of light in free space), and that in an elastic medium equals $c_\alpha = \sqrt{\mu_\alpha / \rho_\alpha}$; ε_α, μ_α and ρ_α (α=1,2) are dielectric constants,[1] elastic moduli, and mass densities, respectively.

Consider a wave propagating along the X axis perpendicular to the layers. In this case, waves of two possible polarizations do not interact, and it is possible to study scalar fields $u^{(\alpha)}$ (α = 1, 2).

The standard boundary conditions will be formulated as applied to the acoustic problem. The displacements $u^{(\alpha)}$ and stresses $\sigma^\alpha = \mu_\alpha(\partial u^{(\alpha)}/ \partial x)$ at the layer boundaries will be considered continuous. It is known that, by virtue of the periodicity of a structure with a period of d, eigenmodes can be characterized by a quasi-wave number k, considering that the field in a unit cell with the number n takes the form

$$u_n(x) = u_n(x-nd)e^{iknd}. \qquad (2)$$

The dispersion equations in this problem were obtained by Rytov for both electromagnetic field [5] and acoustics [6]

$$\cos(kd) = \cos(k_1 d_1)\cos(k_2 d_2) - \frac{1}{2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) \sin k_1 d_1 \sin k_2 d_2 \qquad (3)$$

where $k_1 = \omega/c_1$ and $k_2 = \omega/c_2$ (ω is frequency). Equation (3) determines the frequency as an implicit function of the quasi-wave number. It allows the

[1] Because I will be interested mainly in narrow frequency bands, the frequency dispersion of e can be neglected, and e can be related to the corresponding frequencies.

spectrum of long-wavelength vibrations $(kd << 1)$ to be described readily, for which a sound spectrum with averaged elastic moduli <μ> and density <ρ> is naturally obtained. It was shown [6] that

$$<\rho>d = \rho_1 d_1+\rho_2 d_2; \quad d/<\mu> = d_1/\mu_1+ d_2/\mu_2. \qquad (4)$$

The relationship for <μ>, which contains only d_α/μ_α ratios, is curious. A limiting case that is commonly of no interest in the dynamics of a quantum particle can be considered based on this relationship. Consider the limit

$d_2 \to 0$ and $\mu_2 \to 0$ at $d_2/\mu_2 = P = \text{const}$.[2]

In this case, $d_1 \to d$
and $k_2 d_2 = \omega d_2/c_2 = \sqrt{\rho_2 d_2}\omega\sqrt{d_2\mu_2} \to 0$

therefore, Eq. (3) is reduced to the following equation:

$$\cos(kd) = \cos(k_1 d\text{-}(1/2)P((\rho_2\mu_1/\rho_1 d)(k_1 d))\sin k_1 d \qquad (5)$$

It is useful to note that the dispersion law (Eq. 5) coresponds to an elastic SL composed of a chain of regularly repeating elements of length d with parameters μ_1 and c_1. The following boundary conditions are fulfilled at their joints: (1) continuity of the normal stresses $[\sigma]^+_- = 0$, which is equivalent to $[\partial u/ \partial x]^+_- = 0$ and (2) occurrence of a jump of displacements at a soft inter-layer determined by the stresses at the joint

$$[u]^+_- = M\sigma \equiv \mu_1 M(\partial u/ \partial x), \qquad (6)$$

where $M = P(\rho_1/\rho_2)$.

A set of such boundary conditions at a fixed M is used in describing capillary phenomena in solids [7] or planar defects in crystals [8]. If the parameter M is small, the system in hands is reduced to a periodic sequence of elastic sections weakly bound together. A chain of piezoelectric sections bound together by thin vacuum interlayers may serve, for example, as a possible implementation of such a system. Then, the coupling of elastic vibrations in neighboring sections would be accomplished through electromagnetic oscillations in vacuum gaps.

To illustrate the distribution of roots $\omega = \omega(k)$ of Eq. (5), this equation will be represented in the form

$$\cos kd = \cos z\text{-}Qz\sin z, \qquad (7)$$

where $z = k_1 d = \omega d/c_1$ and $Q = P(\rho_2\mu_1/2\rho_1 d)$. Consider the graphical construction in Fig. 1. The figure shows a plot of the right-hand side of Eq. (7). When it runs over values between ±1, the roots of the equation run over values within intervals marked off on the abscissa axis.

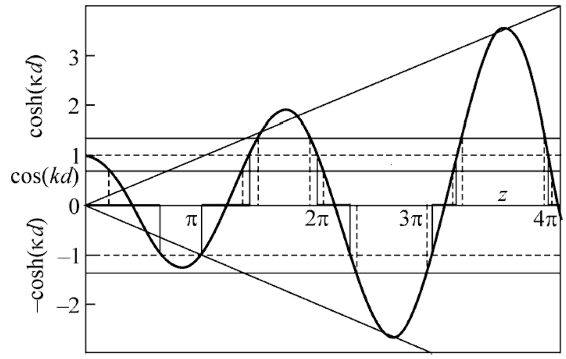

Fig. 1. *Graphical solution of Eq. (7). Eigenfrequency bands are shown in heavy lines on the z axis*

Note that the allowed frequencies are localized in contracting intervals at values $k_1 d = \pm m\pi$, where m is a large integer, as z increases.[3] Under the condition that $m^2Q>>1$, the dispersion laws in these intervals take the form

$$\omega = m\omega_0 + \frac{2\Omega}{m}\begin{cases}\sin^2(kd/2), & m = 2p,\\ \cos^2(kd/2), & m = 2p+1,\end{cases} \qquad (8)$$

where ($\omega_0 = \pi c/d$ and $\Omega. = c/\pi Qd$. It is clear that Eq. (8) gives the size quantization phonon spectrum in a layer of thickness d, whose levels are split into minibands because of low "transparency" of interlayer boundaries. An attempt to analyze the character of the SL band spectrum was made in [9], where the dispersion relation (Eq. (3)) was derived once again. However, their analysis is not satisfactory in a limiting case close to that considered in this work, because it leads to the conclusion that the miniband widths do not vary with increasing frequency.

Consider Eq. (8) from another point of view: Eq. (8) describes the spectrum of a pseudo-quantum particle for which the Schrödinger equation within the tight-binding model takes the form (for $m = 2p$)

$$i\partial\psi/\partial t = m\omega_0\psi_n\text{-}(\Omega/m)(2\psi_n\text{-}\psi_{n+1}\text{-}\psi_{n-1}) \qquad (9)$$

$$i\partial\psi/\partial t = m\omega_0\psi_n+(\Omega/2m)(2\psi_n+\psi_{n+1}+\psi_{n-1}) \qquad (9a)$$

Actually, Eqs. (9) and (9a) are equations for the envelop curve of SL vibrations taken at discrete points (at joints). As usual, the order of derivative with respect to time decreases in such equations. These equations describe analytically the dynamics of a wave packet corresponding to the allowable high frequencies. Using the explicit form of the dispersion laws (Eq. (8)) and

[2] A more general case $d_2 \to 0$ and $c_2 \to 0$ at d_2/c_2= const could be considered; however, no new results arise in this case.

[3] The contraction of bands with increasing frequency was also noted previously; in particular, this was mentioned in [3].

simple Eqs. (8) and (9), the passage of wave packets through the system under study can be described readily, and explicit relationships can be proposed for comparison with possible experimental results.

Nonlinear effects in optical SLs associated with the dependence of the refraction coefficient (that is, the velocity of light c and the parameter ω_0) or the characteristic of joints Q on the field strength ψ_n can be readily taken into account using Eqs. (9) and (9a) much as it was done in [10] when describing optical solitons in such SLs.

The frequencies of forbidden bands correspond to displacements of the type $u_n \sim e^{\pm knd}$ (when $k = i\kappa$) or $u_n \sim (-1)^n e^{\pm knd}$ (when $k = i\kappa + \pi$), which drop (grow) with increasing displacement number n. The frequency dependence of the parameter κ for solutions of the first type can be found from the relationship

$$\cosh\kappa d = \cos z - Qz\sin z > 1, \quad (10)$$

and, for solutions of the second type, from the relationship

$$-\cosh\kappa d = \cos z - Qz\sin z < -1 \quad (10a)$$

It is clear that such states have a physical meaning only in the x semiaxis under the condition that a solution vanishing at infinity and corresponding to certain boundary conditions at the origin is selected. Solutions of the first and the second types correspond to frequencies in the intervals

$(2p - 1)\pi < z < 2p\pi$

and

$2p\pi < z < (2p + 1)\pi$, respectively, (see Fig. 1).

The necessity of using exponentially decreasing solutions arises in describing displacements in the vicinity of a local SL defect.

2. Assume that the boundary conditions at one of the joints (let its number $n = 0$) differ from those described above, or more specifically, these conditions differ in the parameter M: $M^* \neq M$. The local vibration frequency is essentially determined by the difference $M^* - M = \xi M$ Calculations show that the boundary condition at the defect leads to the relationship

$$\sinh\kappa d = \xi Qz\sin z \quad (11)$$

which, along with Eq. (10) or (l0a) (depending on the sign ξ) gives the local vibration frequency. The local frequencies are determined by the intersection points of plots of the right-hand sides of Eqs. (10) and (l0a) with a plot of the function

$f(z) = \sqrt{1 + \sinh^2 z} = [1 + (\xi Qz)^2 \sin^2 z]^{1/2}$,

which is determined by Eq. (11). Because $\kappa > 0$, the solutions correspond to the frequencies (values of z) determined by the equation (see Fig. 2)

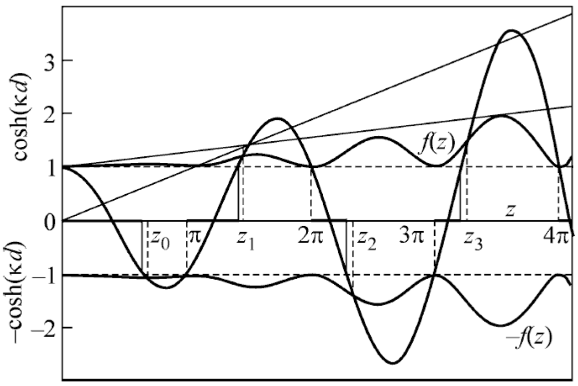

Fig. 2. *Determining a series of roots of Eq.(12) graphically: roots z_1, z_2, ... and z_1, z_3 ... correspond to two types of vibrations*

$$\cos z - Qz\sin z = \\ = \mathrm{sgn}\{\xi Qz\sin z\}\sqrt{1 + (\xi Qz)^2 \sin^2 z}\,. \quad (12)$$

The local vibration frequencies corresponding to different signs of ξ are located in alternating intervals between $z = 2p\pi$ and $z = (2p + 1)\pi$ ($p = 0, 1, 2, \ldots$):$\omega = \omega_s$., $s = 1, 2, 3, \ldots$. The corresponding solutions can be presented in the standard form $u_n(x,t) = w_n(x)\exp\{-i\omega t\}$, where $w_n(x)$ is an odd function $w_{-n}(-x) = w_n(x)$ of the following form (see Fig. 3):

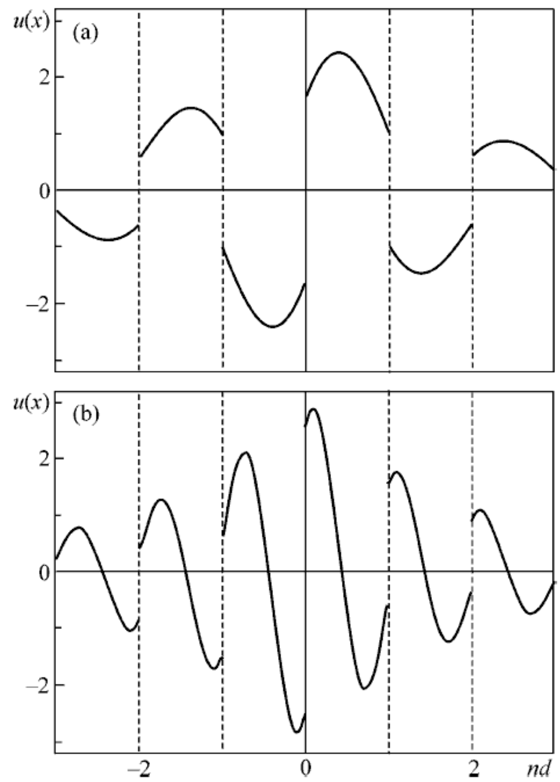

Fig. 3. *Coordinate dependence of displacements of a SL in the vicinity of a defect for two types of vibrations consistent with roots in Fig. 2: (a) corresponds to z_0, z_2 ... roots (antiphase vibrations of unit cells) and (b) corresponds to z_1, z_3, ... roots (in-phase vibrations of unit cells)*

$$w_0^{(s)}(x) = a\cos(k_1^{(s)}x - \theta_s), \qquad 0<x<d;$$

$$w_n^{(s)}(x) = a_1\cos(k_1^{(s)}x - \theta_s)e^{-\kappa nd}, \qquad (13)$$

$$nd<x<(n+1)\,d, \quad n\geq 1;$$

where θ_s is the constant phase corresponding to the eigenfrequency ω_s. The function $w_1(\xi)$ depends harmonically on the argument and can be found easily.

In this case, the local vibrations for which $(2p-1)\pi<z<2p\pi$ (points z_2 and z_4 in Fig. 2) are described by a monotonic function decreasing with increasing number of the unit cell, and the vibrations with frequencies

$(2p-1)\pi<z<2p\pi$

(points z_1 and z_3 in Fig. 2) are described by a function proportional to $(-l)^n e^{-\kappa nd}$. It is essential that a local vibration may arise at any sign of the perturbation ξ.

A local vibration with an even eigenfunction cannot arise at a defect localized at one boundary at any sign of ξ. Assume that this is a joint with $n = 0$; at this joint,

$[u]_-^+ = 0$ and $\sigma_0 = 0$;

therefore, an excitation in the form of a standing wave with an even dependence on the x coordinate is not sensitive to the value of the parameter Q at the joint $n = 0$ and does not differ from the vibration of the free SL boundary passing along this joint.

The free SL boundary corresponds to a section through the joint $n = 0$. This is equivalent to the condition $\sigma_0 = 0$, which is obtained in the given model at $\xi = \infty$ ($M^* = \infty$).

It follows from Eq. (13) that only uniform vibrations ($\kappa = 0$) are possible in this case at frequencies

$$\omega = (c/d)\pi m, \; m = 0, 1, 2, ...$$

Hence, no localized wave exists at the free SL end. This means that vibrations of the even type are impossible if the defect is lumped at one joint. Such localized excitations arise upon variation (perturbation) of the parameter M at least at two neighboring joints. As in the case of an odd solution, the regions of occurrence of such local vibrations with in-phase and antiphase displacements of neighboring unit cells alternate, depending on the sign of ξ, with the period $\Delta z = \pi$.

3. It is interesting to discuss the possibility of occurrence and experimental observation of Bloch oscillations of a wave packet in the SL under consideration. Bloch oscillations of an optical pulse in a different situation were described and observed experimentally [11,12]. Therefore, this discussion is not groundless.

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