THE VELOCITY OF SLOW NUCLEAR BURNING IN THE TWO-GROUP APPROXIMATION

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The velocity of slow nuclear burning was obtained in the two-group approximation. Two groups of neutrons were considered: the group of thermal (slow) neutrons and the group of fast neutrons; each group being described with its diffusion equation. It was shown that in the case of heavy moderators the obtained expression for the two-group velocity had the same structure as the one-group velocity studied by authors before if new effective diffusion and multiplication coefficients were introduced. The expressions for corresponding effective coefficients are presented.

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The simplest variant of many-group approximation is so named two-group approximation, which considers only two mutually connected neutron groups; namely the groups of thermal and fast neutrons. The neutrons with thermal energy belong to the thermal group and all the neutrons with the energy exceeding the thermal one belong to the fast neutron group. It worth noting, that such division is justified also from the physical consideration because cross-sections of reactions with thermal neutrons differ substantially from the cross-sections of fast neutrons. When we consider diffusion processes each of these groups is taken to be a "monoenergetic" one described by its own diffusion equation with constant coefficients not depending on energy. In other words it is supposed that the neutrons diffuse without energy loss within each group until some of the fast neutrons undergo a number of collisions necessary to decrease their energy down to the level of the lower thermal group; at that moment these neutrons immediately jump to the thermal group [1-3].

According to this picture, the diffusion equation that describes the neutron of the fast group takes the form

$$\frac{\partial n_2}{\partial t} = D_2 \Delta n_2 - \frac{n_2}{\tau_2} + \nu \frac{n_1}{\tau_1}, \tag{1}$$

where $n_2(\mathbf{r},t)$ is the neutron density of the fast group; D_2 is the neutron diffusion coefficient for the fast group; τ_2 is the fast neutron life-time during which it undergoes slowing down to the thermal energy range; ν is the average number of fast neutrons born at thermal fission; $n_1(\mathbf{r},t)$ is the density of thermal neutrons; and τ_1 is the thermal fission capture life-time.

The slow neutron density $n_1(\mathbf{r},t)$ satisfies the following diffusion equation

$$\frac{\partial n_1}{\partial t} = D_1 \Delta n_1 - \frac{n_1}{\tau_1} + \frac{n_2}{\tau_2}, \tag{2}$$

where D_1 is the thermal diffusion coefficient.

The diffusion coefficient can be estimated, if we use the relation $L_2^2 = D_2 \tau_2$, where L_2 is the diffusion length for the fast neutron group. In the case of graphite moderator $L_2^2 L_1^{-2} \approx 0.1$ [2], where L_1 is the thermal diffusion length and the ratio of fast and thermal neutron lifetimes $\tau_2 \tau_1^{-1}$ has the order of magnitude $\tau_2 \tau_1^{-1} \approx 10^{-2}$ (see, e.g., [2,3]). We obtain for the graphite moderator that $D_2 D_1^{-1} \approx 10$.

We shall solve the system of equations (1), (2) for the case of δ–shaped thermal source having the output $\sqrt{2\pi}\, n_0$ and placed at the initial moment in the $z = 0$ plane. Therefore the initial conditions for our problem are

$$n_1(z,0) = \sqrt{2\pi}\, n_0 \sin\frac{\pi}{a} x \sin\frac{\pi}{a} y\, \delta(z), \qquad n_2(z,0) = 0. \tag{3}$$

As for the physical picture, we can speak here about the propagation of an initial fluctuation of thermal neutron density.

The boundary conditions on the cylinder surface are assumed to be zero so we seek the solution of our system in the form $n_j(x,y,z,t) = n_j(z,t) \sin\frac{\pi}{a} x \sin\frac{\pi}{a} y$, $(j = 1,2)$. As is easy to see the existence of boundaries along *x* and *y* axes results in renormalization of quantities τ_j^{-1} $(j = 1,2)$,

$$\frac{1}{\tau'_{1,2}} = 2D_{1,2}\frac{\pi^2}{a^2} + \frac{1}{\tau_{1,2}}. \tag{4}$$

Finally the system of equations which describes the neutron diffusion along the reactor axis has the form

$$\begin{aligned} \frac{\partial n_1(z,t)}{\partial t} &= D_1 \frac{\partial^2 n_1(z,t)}{\partial z^2} - \frac{n_1(z,t)}{\tau'_1} + \frac{n_2(z,t)}{\tau_2}, \\ \frac{\partial n_2(z,t)}{\partial t} &= D_2 \frac{\partial^2 n_1(z,t)}{\partial z^2} - \frac{n_2(z,t)}{\tau'_2} + \nu \frac{n_1(z,t)}{\tau_1}. \end{aligned} \tag{5}$$

The set of equations (5) can be solved by the Fourier transform. Taking the Fourier transform with respect to *z*,

$$N_j(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ikz} n_j(z,t)dz, \ \ (j = 1,2) \quad (6)$$

we obtain the system of ordinary differential equations for $N_j(k,t)$:

$$\frac{dN_1}{dt} = -(D_1 k^2 + \frac{1}{\tau'_1})N_1 + \frac{N_2}{\tau_2},$$
$$\frac{dN_2}{dt} = -(D_2 k^2 + \frac{1}{\tau'_2})N_2 + \nu \frac{N_1}{\tau_1}. \quad (7)$$

We seek the solution of equations (7) in the form $N_j = C_j e^{\lambda t} \ (j = 1,2)$, and after equating the determinant to zero we obtain the characteristic equation of the system

$$\left[\lambda + \left(D_1 k^2 + \frac{1}{\tau'_1}\right)\right]\left[\lambda + \left(D_2 k^2 + \frac{1}{\tau'_2}\right)\right] - \frac{\nu}{\tau_1 \tau_2} = 0 \quad . (8)$$

We are interested in multiplicative solutions, namely the solutions for which the neutron density increase exponentially with time at any point. As it was shown earlier [4,5] only such solutions describe a slow nuclear burning wave. Thus we can restrict ourselves only to one (positive) root of characteristic equation.

$$2\lambda_+ = -(D_1 + D_2)k^2 - (\frac{1}{\tau'_1} + \frac{1}{\tau'_2}) + \sqrt{\left\{\left[(D_1 - D_2)k^2 + (\frac{1}{\tau'_1} - \frac{1}{\tau'_2})\right]^2 + \frac{4\nu}{\tau_1 \tau_2}\right\}}, \quad (9)$$

though the coefficient $C_{1+}(k)$ is calculated accounting for both roots.

The problem of determination of slow nuclear burning velocity v_0 reduces itself to finding such velocity value $v = v_0$ that the asymptotic behavior of the thermal neutron density $n_1(z,t)$ for the case $z = vt$ and $t \to \infty$ undergoes a change when passing through this velocity value. This asymptotics can be obtained by the saddle-point method from the integral expression for $n_1(vt,t)$

$$n_1(vt,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk e^{(\lambda_+(k) - ikv)t} C_{1+}(k), \quad (10)$$

where $\lambda_+(k)$ is given by Eq.(9).

Asymptotic velocity v_0 in this case will be determined from the solution of two algebraic equations

$$iv = \frac{d\lambda_+(k)}{dk}, \quad (11)$$
$$\mathrm{Re}\left[ikv - \lambda_+(k)\right] = 0 .$$

The first of these equations determines the saddle-point $k_0(v)$, whereas the second represents the condition for the change in the asymptotics character of $n_1(vt,t)$ (from the exponential increase at $v < v_0$ to damped exponential at $v > v_0$).

Using the expression (9) for the $\lambda_+(k)$ we can write the equations (11) in explicit form

$$iv = -(D_1 + D_2)k + \frac{\left[(D_1 - D_2)k^2 + (\frac{1}{\tau'_1} - \frac{1}{\tau'_2})\right](D_1 - D_2)}{\sqrt{\left\{\left[(D_1 - D_2)k^2 + (\frac{1}{\tau'_1} - \frac{1}{\tau'_2})\right]^2 + \frac{4\nu}{\tau_1 \tau_2}\right\}}} k,$$

$$2ikv = -(D_1 + D_2)k^2 - (\frac{1}{\tau'_1} + \frac{1}{\tau'_2}) + \sqrt{\left\{\left[(D_1 - D_2)k^2 + (\frac{1}{\tau'_1} - \frac{1}{\tau'_2})\right]^2 + \frac{4\nu}{\tau_1 \tau_2}\right\}} . \quad (12)$$

The solution of system (12) can be easily obtained in the approximation of small multiplication $\gamma << 1$. As can be shown in this case $(D_1 - D_2)^2 k_0{}^4 \approx D^2 k_0^4$ has the order of magnitude γ^2 and we can expand the square root in (9) up to terms of the order of γ. Indeed, according to (11) characteristic k_0 have the order of $k_0 \approx vD^{-1}$ and the velocity of slow nuclear burning is the quantity of the order of $v_0 \approx \sqrt{RD}$ [4,5], where R is the characteristic coefficient of multiplication and D is the characteristic diffusion coefficient. In the case of small γ under consideration here, $R \approx \gamma\tau^{-1}$. Therefore, $k_0 \approx v_0 D^{-1} \approx \sqrt{\gamma (D\tau)^{-1}}$. Expanding the square root in Eq.(9) we obtain

$$\lambda_+ \approx -\frac{1}{2}\left[(D_1 + D_2) - \frac{(D_1 - D_2)(\tau'_2 - \tau'_1)}{\tau'_2 + \tau'_1}\right]k^2 + \frac{\gamma}{\tau'_2 + \tau'_1}. \quad (13)$$

Using Eq.(13) we shall put the system (11) into the form

$$iv = -\left[(D_1 + D_2) - \frac{(D_1 - D_2)(\tau'_2 - \tau'_1)}{\tau'_2 + \tau'_1}\right]k,$$
$$2ikv = -\left[(D_1 + D_2) - \frac{(D_1 - D_2)(\tau'_2 - \tau'_1)}{\tau'_2 + \tau'_1}\right]k^2 + \frac{2\gamma}{\tau'_2 + \tau'_1}. \quad (14)$$

Solving these equations with respect to v one obtains

$$v_0 = 2\sqrt{\frac{D_{\mathrm{eff}} A_{\mathrm{eff}}}{\tau'_2 + \tau'_1}}, \quad (15)$$

where D_{eff} is the effective diffusion coefficient

$$D_{\mathrm{eff}} = \frac{1}{2}\left[(D_1 + D_2) - \frac{(D_1 - D_2)(\tau'_2 - \tau'_1)}{\tau'_2 + \tau_1}\right] \quad (16)$$

and $A_{\rm eff}$ is the effective coefficient of multiplication

$$A_{\rm eff} = \gamma = \frac{\nu\tau'_1\tau'_2}{\tau_1\tau_2} - 1 . \qquad (17)$$

The role of the effective lifetime plays the quantity $(\tau'_1 + \tau'_2)$ Therefore the concept of slow nuclear burning velocity allows generalization for the case of two neutron groups. In the case of heavy moderator, as we saw before, there exists the relation $\tau'_2(\tau'_1)^{-1} << 1$ between lifetimes τ'_1 and τ'_2. In this case it follows from Eq.(16)

$$D_{\rm eff} \cong D_1 . \qquad (18)$$

Therefore, up to the terms of the order of γ Eq.(15) obtained in the two-group approximation turns into the expression for the velocity which was obtained earlier [4,5] in the one-group approximation

$$v_0 = 2\sqrt{\frac{D_1\widetilde{A}^*}{\tau_1}} \qquad (19)$$

and the quantity $\widetilde{A}^*$ equals to

$$\widetilde{A}^* = \frac{\nu\tau'_2}{\tau_2} - \frac{\tau_1}{\tau'_1} . \qquad (20)$$

We consider now the solution of the system (12) in the general case of heavy moderators for which we assume the validity of the following conditions $\zeta = D_1/D_2$, $\tau_2/\tau_1 << \zeta << 1$, $\tau_2 D_2/\tau_1 D_1 << 1$, (note that for graphite $D_1/D_2 \propto 0.1$, $\tau_2/\tau_1 \propto 0.01$). As for quantity $\nu\tau_2/\tau_1$ we have $\nu\tau_2/\tau_1 \le \zeta$ (for U^{235} $\nu = 2.5$).

Using these relations we consider now the radicand in (9) which after taking the quantity $(\tau'_2)^{-1}$ out of radical we can write in the form

$$\frac{1}{\tau'_2}\left\{\left[D_1\tau'_2(1-\zeta)k^2 + \left(1 - \frac{\tau'_2}{\tau'_1}\right)\right]^2 + \frac{\nu\tau'^2_2}{\tau_1\tau_2}\right\}^{1/2} . \qquad (21)$$

As it is easy to see two summands under the radical are comparable by magnitude only in the case when the dimensionless parameter $D_2\tau'_2k^2$ is close to –1. Indeed, assuming $D_2\tau'_2k^2 = -1 + y$ we obtain that the condition $y + \zeta \propto (\nu\tau_2/\tau_1)^{1/2}$ must be fulfilled .For all other values of k^2 the first term exceeds considerably the second one and the quantity $\lambda_+(k)$ has correspondingly the form

$$\lambda_+(k) = -\left[D_1 - D_2\frac{\nu\tau'^2_2}{\tau_1\tau_2}\right]k^2 + \frac{\gamma}{\tau_1} ; \qquad (22)$$

here γ is given by Eq.(17).

Substituting the obtained expression for the $\lambda_+(k)$ into the system (11) we find for the velocity of slow nuclear burning the expression

$$v_0 = \sqrt{\frac{D_{\rm eff}\widetilde{A}^*}{\tau_1}} , \qquad (23)$$

where

$$D_{\rm eff} = D_1(1-\Delta), \qquad \Delta = \frac{D_2\tau'_2}{D_1\tau'_1}\frac{\nu\tau'_2\tau'_1}{\tau_2\tau_1} \qquad (24)$$

(for graphite $\Delta \approx 25\%$).The quantity $\widetilde{A}^*$ is given by formula (20).

Therefore, according to Eq. (23) in this case we can also speak about the slow nuclear burning velocity the expression for which has a characteristic structure similar to the structure of the slow nuclear burning velocity in the one-group approximation [4,5].

According to the above discussion we proved that both in the one-group and the two-group approximations we could say about the slow nuclear burning velocity. This velocity is proportional to the square root from the product of diffusion and multiplication coefficients.

Finally, the account of the neutrons with the energy above the thermal one as a separate neutron group in the framework of two-group approximation doesn't change the result qualitatively though it leads to not so large quantitative renormalization of the thermal diffusion coefficient in the case of heavy moderators.

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