TRANSPORT EQUATIONS FOR LOW DENSITY SOLITON GAS

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We present the theory of transport phenomena in a gas of solitons for Sinus Gordon model system. The general case of relaxation phenomena is considered. A special attention is paid for a small density non-relativistic gas of breathers. Such kinetic coefficients as diffusion, thermal conductivity, and internal friction are found. It is shown, that diffusion coefficient and internal friction coefficient equal each other.
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1. INTRODUCTION

Many physical phenomena require for their explanation an exact solution of nonlinear equation of motion. Such examples as the domain boundaries in magnetic systems, tsunami, solitons both in plasma and non-linear optics are well known. Last years special attention was directed to the kinetic properties of solitons. At the moment there exists a theory of soliton diffusion. The authors attack the problem of construction of a general theory of kinetic properties of solitons like thermal conductivity, internal friction, etc. In the previous paper [1] authors presented a theory of kinetic equation for solitons. Here we develop the general method of kinetic coefficient calculation for a gas of the low density solitons.

2. KINETIC EQUATIONS

In the authors paper [1] a system of kinetic equations for low density soliton gas ($n\delta << 1$, n is the density of solitons, δ is the soliton characteristic size) were obtained. It was shown that the entropy production is connected with homogenization of soliton gas in the real coordinate space. In the cause of the homogenization process the establishment of a uniform density $n(x)$, hydrodynamic velocity $u(x)$ and temperature *T(x)* takes place. In another words, the soliton collisions are at the bottom of such processes as internal friction, thermal conductivity and diffusion of soliton gas. The kinetic equations are:

$$\frac{\partial f}{\partial t} + (v + \delta v_k(v))\frac{\partial f}{\partial x} = \mathrm{D}_k(v)\frac{\partial^2 f}{\partial x^2},$$

$$\frac{\partial B}{\partial t} + (v + \delta v_b(V))\frac{\partial B}{\partial x} + (\omega + \delta\omega_b(V))\frac{\partial B}{\partial \varphi} = \qquad (1)$$

$$= \mathrm{D}_b(V)\frac{\partial^2 B}{\partial x^2} + \mathrm{P}_b(V)\frac{\partial^2 B}{\partial \varphi^2} + 2\mathrm{K}_b(V)\frac{\partial^2 B}{\partial x \partial \varphi}.$$

Here

$$\delta v_k = \sum_i \delta v_{ki}, \qquad \mathrm{D}_k(v_k) = \sum_i \mathrm{D}_{ki},$$

$$\delta v_b = \sum_i \delta v_{bi}, \qquad \delta\omega_b = \sum_i \delta\omega_{bi};$$

$$\mathrm{D}_b = \sum_i D_{bi}; \quad \mathrm{P}_b = \sum_i \mathrm{P}_{bi}; \quad \mathrm{K}_b = \sum_i \mathrm{F}_{bi}, \qquad (2)$$

$$\mathrm{D}_k(v_k) = \frac{1}{2}\int |v| [\Delta x_{ki}]^2 F_i d2 \;; v_k \equiv v_1;\; v = v_k - v_2;$$

$$\mathrm{D}_{bi} = \frac{1}{2}\int |v| [\Delta x_{bi}]^2 F_{2i} d2;$$

$$\mathrm{P}_{bi} = \frac{1}{2}\int |v| [\Delta\varphi]^2 F_{2i} d2 \qquad (3)$$

$$\mathrm{K}_{bi} = \frac{1}{2}\int |v| \Delta x_{bi} \Delta\varphi_{bi} F_{2i} d2$$

In the above formulae $F \equiv \{f, B\}$ are the distribution functions of both kinks and breathers. The arguments of these functions are the center of mass coordinates x, velocities *v*, time *t*, internal frequencies ω and phase coordinate φ for breathers. For simplicity reasons we will use notations $1 \equiv \{x_1, v_1\}$ in the case of kinks, and $1 \equiv \{x_1, v_1, \omega_1, \varphi_1\}$ in the case of breathers. In equations (1) – (5) both Δx *and* $\Delta\varphi$ are the jumps of the center of mass coordinates and phase coordinates.

3. TRANSPORT EQUATION. GENERAL CASE

To obtain the transport equations from the kinetic equation one has to employ a usual method, i.e. to multiply the kinetic equation by 1, velocity *V*, and energy *E*, and integrate over all moments. This multiplier factor we will design as *a*. After direct calculation one can find the transport equations in the form of the local conservation laws. Here are the final results.

For kinks

$$\frac{\partial}{\partial t} n_k \langle a_k \rangle + \frac{\partial}{\partial x}\left[U_k^r + U_k^m\right] = 0; \qquad (4)$$

For breathers

$$\frac{\partial}{\partial t} n_b \langle a \rangle + \frac{\partial}{\partial x}\left[U_b^r + U_b^m\right] + \frac{\partial}{\partial \varphi}\left[W_b^r + W_b^m\right] = 0, \qquad (5)$$

We use the following notations in equations (4) and (5). For kinks:

$$n_k \langle a \rangle = \int a(x,v) f(x,v,t) dv,$$

$$U_k^r = \int a(x,v)\left[v + \delta v_k\right] f(x,v,t)\, dv, \qquad (6)$$

$$U_k^m = -\frac{\partial}{\partial x}\int a(x,v)\, \mathrm{D_k}\, f(x,v,t) dv. \qquad (7)$$

For breathers:

$$n_b \langle a_b \rangle = \int a(x,v,\varphi,\omega) B(x,v,\varphi,\omega,t) dv d\omega,$$

$$U_b^r = \int a(x,v,\varphi,\omega)\left[v + \delta v_b\right] B(x,v,\varphi,\omega,t) dv d\omega, \qquad (8)$$

$$U_b^m = -\frac{\partial}{\partial x}\int a(x,v,\varphi,\omega)\, \mathrm{D_b}\, B(x,v,\varphi,\omega,t)\, dv\, d\omega - \frac{\partial}{\partial \varphi}\int a(x,v,\varphi,\omega)\, \mathrm{K_b} B(x,v,\varphi,\omega,t)\, dv\, d\omega, \qquad (9)$$

$$W_b^r = \int a(x,v,\varphi,\omega)\left[\omega + \delta\omega_b\right] B(x,v,\varphi,\omega,t) dv d\omega, \qquad (10)$$

$$W_b^m = -\frac{\partial}{\partial x}\int a(x,v,\varphi,\omega)\, \mathrm{P}_b\, B(x,v,\varphi,\omega,t)\, d v\, d\omega - \frac{\partial}{\partial \varphi}\int a(x,v,\varphi,\omega)\, \mathrm{K}_b B(x,v,\varphi,\omega,t)\, d v\, d\omega. \qquad (11)$$

The jumps of the center of mass coordinates at soliton-soliton collisions are presented below in the table.

The expressions for coordinates jumps Δx [1-4]

	K2	B2
K1	$(\delta_{1k}/2)\ln\lvert Z_{kk}\rvert$	$\delta_{1k}\ln\lvert Z_{kb}\rvert$
B1	$-(8/E_{1b})\ln\lvert Z_{bk}\rvert$	$(\delta_{1b}/2)\ln\lvert Z_{bb}\rvert$

The first line presents the jumps Δx of kink K1 coordinate due to collisions with a kink K2 and a breather B2; in the second line the same is shown for a breather B1. All the data in the table have to be multiplied by a sign of the relative velocity of solitons.

The $\tan\Delta\varphi$ of the phase jumps $\Delta\varphi$ due to soliton–soliton collisions [1 – 4] is

$$\tan\Delta\varphi_{bk} = -\omega_{1,b1}(1/v^2 - 1)^{1/2},$$

$$\tan\Delta\varphi_{bb} = -2\lvert v\rvert[1 - v^2]^{1/2}\omega^1\omega_1^1 [v^2 + (1 - v^2)\{(\omega_1^2)^2 - (\omega_1^1)^2\}]^{-1},$$

$$\omega_2 \equiv \omega,\ \omega_1 = \sqrt{1 - \omega^2}.$$

The quantities *Z* that enter the table are:

$$Z_{kk} = \frac{1 + \sqrt{1 - v^2}}{1 - \sqrt{1 - v^2}},\ ; Z_{kb} = \frac{1 + \omega_{2,b2}\sqrt{1 - v^2}}{1 - \omega_{2,b2}\sqrt{1 - v^2}}, \qquad (12)$$

$$Z_{bb} = \frac{1 - \sqrt{1 - v^2}C_1}{1 - \sqrt{1 - v^2}C_2}\cdot\frac{1 + \sqrt{1 - v^2}C_2}{1 + \sqrt{1 - v^2}C_1}. \qquad (13)$$

Here $C_{1,2} = \sqrt{(1 - \omega_1^2)(1 - \omega_1^2)} \mp \omega_1\omega_2$. From the above table one can see that the phase shifts are not small in general case. For slow (non relativistic solitons, V<<1) the phase shifts are $\pi/2$.

We would like to emphasize that the topology charges are constant for all type of collisions. The processes which can change the topology charge of solitons are the processes that destroy the full integrability of the system. In other words, as soon as we are within the framework of the fully integrable model, the numbers of kinks, antikinks and breathers are constant. This numbers are determined by the initial conditions. In this paper we will study only fully integrable systems.

Continuity equations (*a=1*) can be written as:

$$\frac{\partial n_k}{\partial t} + \frac{\partial}{\partial x}\left(j_k^r + j_k^m\right) = 0 \qquad (14)$$

$$\frac{\partial n_b}{\partial t} + \frac{\partial}{\partial x}\left(j_b^r + j_b^m\right) + \frac{\partial}{\partial \varphi}\left(i_b^r + i_b^m\right) = 0, \qquad (15)$$

Here the standard notations were used for both kink and breather currents instead of *U* and *W*.

Hydrodynamic equations (*a*=*v* for kinks and *a*=(*v*, *ω*) for breathers) have the following form:

$$\frac{\partial}{\partial t} n_k u_k + \frac{\partial}{\partial x}\left(P_k^r + P_k^m\right) = 0, \qquad (16)$$

$$\frac{\partial}{\partial t} n_b u_b + \frac{\partial}{\partial x}\left(P_b^r + P_b^m\right) + \frac{\partial}{\partial \varphi}\left(\Pi_b^r + \Pi_b^m\right) = 0, \qquad (17)$$

$$\frac{\partial}{\partial t} n_b w_b + \frac{\partial}{\partial x}\left(Q_b^r + Q_b^m\right) + \frac{\partial}{\partial \varphi}\left(R_b^r + R_b^m\right) = 0. \qquad (18)$$

Here u_i is the hydrodynamic velocity of solitons, w_b is the hydrodynamic velocity in *φ*-space, P_i^r and P_i^m, Q_i^r and Q_i^m are the pressures of kinks and breather correspondingly, Π_b^r and Π_b^m, R_i^r and R_i^m are the pressures of breathers in *φ*-space. These quantities are defined with equations (9) – (12).

Energy transport equations can be obtained by putting $a = E(v)$ for kinks, and $a = E(v,\omega)$ for breathers. They are:

$$\frac{\partial}{\partial t} n_k T_k + \frac{\partial}{\partial x}\left(U_k^r + U_k^m\right) = 0, \qquad (19)$$

$$\frac{\partial}{\partial t} n_b T_b + \frac{\partial}{\partial x}\left(U_b^r + U_b^m\right) + \frac{\partial}{\partial \varphi}\left(W_b^r + W_b^m\right) = 0, \qquad (20)$$

where T_i means average energy of a soliton. If $u_i = w_i = 0$, T_i is the local soliton temperature.

We would like to emphasize the important property of equations (15) – (20). It is easy to see that any uniform in real space distribution functions

$$f = f(v,t), \quad B = B(v,\omega,t) \qquad (21)$$

satisfy both kinetic equations and transport equations.

4. TRANSPORT EQUATIONS IN THE CASE OF SMALL SOLITON VELOCITIES

If the soliton velocity satisfies the condition $v \ll 1$, than the phase change is the same for both breather-breathers and breather-kink collision, and equals to $\pi/2$ in absolute value. The values of *Z*, which determine the center of mass coordinates jumps, are:

$$Z_{kk} = \ln(4/v^2); \quad Z_{kb} = ((1+\omega)/(1-\omega)) \qquad (22)$$

$$Z_{bb} = \frac{1-C_1}{1-C_2}\frac{1+C_2}{1+C_1};\ if\ 1 >> (\omega_1 - \omega_2)^2 >> v^2;$$

$$Z_{bb} = \frac{4}{v^2}\frac{\omega^2}{1-\omega^2};\ if\ (\omega_1 - \omega_2)^2 << v^2 << 1$$

We would like to remind that the center of mass coordinates jumps are connected with Z by relation $\Delta x = (1/2)\delta_\kappa \ln(4/v^2)$ for kinks, and by relation $\Delta x = (1/2)\delta_b \ln Z_{bb}$ for breathers.

Using these values it is easy to calculate the relaxation time in both real space and phase space. For this purposes we will use the relation between distance and time for the diffusion process,

$x^2 = Dt$ or $t = x^2/D$.

Taking into account that $< v^2 > \approx (T/m)$ one can find that average value of the diffusion coefficient for kinks is

$$D_k = < v(\Delta x)^2 > \approx (1/4)\delta_k^2 (T/m_k)^{1/2} \ln^2(T/4m_k)$$
$$\approx \delta_k^2 (T/m_k)^{1/2}.$$

For breathers one has:

$$D_b = < v(\Delta x)^2 > \approx$$
$$\approx (1/4)\delta_b^2 (T/m_b)^{1/2} \ln^2((1-\omega^2)T/4\omega^2 m_b)$$
$$\approx \delta_b^2 (T/m_b)^{1/2}.$$

The final results for diffusion coefficients we present here with logarithmic accuracy. From the formulae we see that the diffusion coefficients in real space and phase space are of the same order. Taking into account that the average distance between solitons in real space is $l \approx (1/n)$, where n is the soliton density, one can find that the relaxation time τ is of order:

$$\tau \approx (l^2/D) \approx (m_i/T)^{1/2} / (n\delta)^2 \quad . \tag{23}$$

In this formula m_i is the kink or breather mass. They have the same order of value $(m_b/m_k) = 2\omega$. Average value of the breather diffusion coefficient in phase space is

$$D_{b\varphi} = < v(\Delta\varphi)^2 > \approx < v\pi^2 > \approx \pi^2 (T/m)^{1/2}.$$

As the characteristic distance in the phase space is π, for relaxation time τ in phase space one finds

$$\tau_\varphi \approx (\pi^2 / .D_{b\varphi}) \approx (m_b/T)^{1/2}) \tag{24}$$

Comparing these two formulae for relaxation times in real and phase spaces one can see that

$$\tau_\varphi << \tau \quad , \tag{25}$$

because $n\delta << 1$ (approximation of low density gas).

This means that the relaxation proceeds in two steps. First of all at the end of the first step of relaxation the uniform distribution in phase space establishes. After that the much more slow processes of homogenization of density, hydrodynamic velocities and temperature in coordinate space take place.

5. KINETIC COEFFICIENTS OF BREATHERS

Now we are going to evaluate the fluxes of number-of-particle density, momentum density and energy density. We will not consider the re-normalization of velocities and pressures. The kink's fluxes were investigated in paper [1]. Here we consider the breather's fluxes.

Continuity equations for the breathers are

$$\frac{\partial n_b}{\partial t} + \frac{\partial j_b^m}{\partial x} = 0, \quad j_b^m = -\frac{\partial}{\partial x} D_b \tag{26}$$

where the diffusion coefficient D_b equals to

$$D_b = (1/2)(\Delta x_{bk})^2 n_b n_k (2T/\pi\mu)^{1/2} + (\Delta x_{bb})^2 n_b n_b (T/\pi m_b)^{1/2} \tag{27}$$

Thermal conductivity equation is

$$\frac{\partial}{\partial t} n_b T_b + \frac{\partial}{\partial x} U_b^m = 0. \tag{28}$$

Here U_b^m is a dissipative energy flux. This flux is related to the temperature gradient,

$$U_b^m = -\frac{\partial \kappa_b}{\partial x} \tag{29}$$

where κ_b, as a function of temperature, is:

$$\kappa_b = \kappa_{b1} + \kappa_{b2};$$
$$\kappa_{b1} = (3/2)T(T/\pi m_b)^{1/2} n_b n_b (\Delta x_{bb})^2,$$
$$\kappa_{b2} = (1 + (m_k/M))T(T/2\pi\mu)^{1/2} n_b n_k (\Delta x_{bk})^2$$

The hydrodynamic equation for breathers takes the form

$$\frac{\partial}{\partial t} n_b u_b + \frac{\partial}{\partial x}\left(P_b^r + P_b^m\right) = 0 \tag{30}$$

where P_b^r and P_b^m are the pressure density in the breather gas and friction pressure density caused by breather collisions,

$$P_b^r = -\frac{\partial u_b}{\partial x}\eta = -\frac{\partial u_b}{\partial x} D_b. \tag{31}$$

The quantity η is the internal friction coefficient. This coefficient is equal to the diffusion coefficient D_b. The relation

$$D_b = \eta \tag{32}$$

is an analog of the Einstein relation between diffusion and friction coefficients for ordinary particles. For kinks this relation was derived in [1].

With this paper we commemorate our teacher Professor A.I. Akhiezer.

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