ON THE SYNCHROTRON RADIATION IN CURVED MAGNETIC LINES

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The general formulae that describe radiation of an ultrarelativistic electron moving along spiral trajectories with a small pitch angle in curved magnetic force lines are considered. These formulae can be identical to either synchrotron radiation or curvature radiation in certain parametric regions, but there are appreciable differences between new and old mechanisms when the velocity of Larmor's rotation is of the order of a centrifugal drift velocity. The case when the Larmor velocity is equal exactly to the velocity of centrifugal drift was not investigated earlier. There are points with zero curvature in the particle's trajectory, in that case. The spectral angular distribution, polarization characteristics in the neighborhood of these points have been obtained.

INTRODUCTION

Synchrotron radiation is widely used in different branches of science and technology. The history of synchrotron radiation in radio astronomy see in [3]. It is the main radiation mechanism to explain the power law spectra of nonthermal cosmic radio emission.

In radio galaxies and quasars, pulsars, Jupiter magnetosphere, *etc.* relativistic particles move in curved magnetic fields. However, the synchrotron radiation formulas in a homogeneous magnetic field [5, 12, 13] are used for description their radiation. The curvature radiation mechanism is also described by these formulae. It was necessary to take into account the curvature of magnetic field lines. The authors of [1] found that magnetic lines curvature modifies the properties of simple synchrotron radiation. Next step was made in [6, 9, 10]. The spectral angular distribution, polarization properties were derived in [9, 10], and the radiation spectrum was obtained in [6, 9, 10]. It was shown in [11] that the formulae, which describe the radiation spectrum in [6] and [9], can be transformed one another.

The case when the radiation pattern in given direction is gathered from many coils of a spiral trajectory winding around curved force line (undulator regime) was considered in [2, 7].

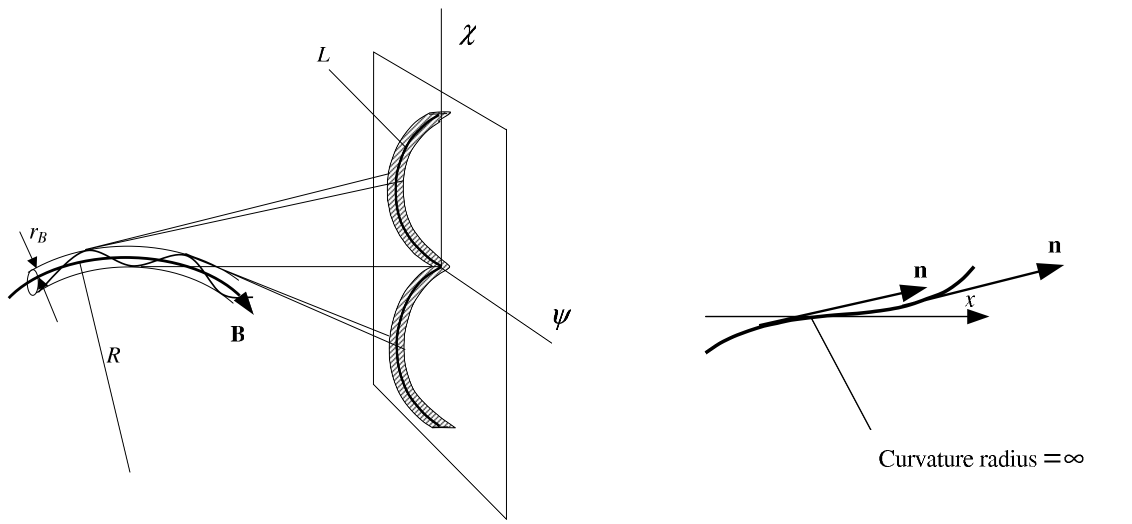
The *synchrotron* radiation mechanism for relativistic charged particles moving along the curved spiral trajectory within curved magnetic lines is considered. The curved magnetic field lines have been approximated by a circular magnetic field. Particles are in circular motion around the guiding center, which is moving along the drift trajectory. The velocity projection onto the magnetic field line is close to the velocity of light. The decrease in velocity caused by radiation is ignored. In the plane perpendicular to magnetic line the particle velocity is composed of the velocity of centrifugal drift v_D and Larmor velocity v_L. The curvature of magnetic field line ought to take into account when $v_D \sim v_L$ [9, 10].

Since the radiation in the given direction is going from a small length, this trajectory segment can be replaced by a circle that has radius equal curvature radius of the trajectory at this point. We have the characteristic frequency corresponding the curvature radius. The curvature radius is different at distinct trajectory points, so synchrotron radiation mechanism in curved magnetic field exhibits a range of characteristic frequencies instead of that one in homogeneous magnetic field. In the case of exact equality $v_L = v_D$ the trajectory has points at which the curvature becomes equal to zero. The approximation of inscribed circle is failed at these points, and this case is not investigated yet.

In this paper, the spectral angular distribution and spectrum for the case $v_L \sim v_D$ is discussed and the spectral angular distribution for the case $v_L = v_D$ is derived.

PROPERTIES OF RADIATION MECHANISM

Let us assume, that magnetic force lines look like a circle of radius R, and the magnitude of magnetic field is B_0. Select a Cartesian coordinates system with (x, y)-axes in the plane of magnetic field lines, and z-axis coinciding with the axis of cylindrical magnetic surface. The magnetic field vector can be expressed as



a) curved spiral trajectory and picture plane b) trajectory in the vicinity of inflection point

Figure 1. Magnetic field, trajectory, and directional radiation pattern

$$\mathbf{B} = B_0 \left(\sin\varphi\,\mathbf{i} - \cos\varphi\,\mathbf{j}\right), \tag{1}$$

where φ is the polar angle in $(x,\ y)$-plane, $\mathbf{i}$, $\mathbf{j}$ are the basis vectors of Cartesian frame. A particle with the Lorentz-factor $\gamma = \left(1 - v^2/c^2\right)^{-1/2} \gg 1$ is moving along the magnetic force line. The angular velocity Ω corresponding this motion ($\Omega \equiv\ v_{||}/R$, where $v_{||}$ is the velocity of the guiding center along the magnetic line with curvature radius R) is much less than the frequency of rotation around magnetic force line ω_B, $\Omega \ll |\omega_B|$. The radius of Larmor circle r_B is much less than R, $r_B \ll R$ (Fig. 1a).

The equations of a charged particle motion in magnetic field (1) are integrated in quadratures. The solution is expressed through elliptic integrals of the 1st and 3rd kind. The asymptotic expansion of the trajectory radius-vector, in which the terms proportional $(r_B/R)^2 \ll 1$ are dropped, has the form [9, 10]

$$\begin{aligned}\mathbf{r} \;=\; & \left[2\delta r_B \sin\omega_B t \cos\Omega t + (R - r_B\cos\omega_B t)\sin\Omega t\right]\mathbf{i} \\ & + \left[-2\delta r_B \sin\omega_B t\, \sin\Omega t + (R - r_B\cos\omega_B t)\cos\Omega t\right]\mathbf{j} + (v_D t + r_B \sin\omega_B t)\;\mathbf{k}\,, \end{aligned} \tag{2}$$

where $\omega_B = e_\alpha B_0/m_\alpha c\gamma$, $\delta = \Omega/\omega_B \ll 1$, $v_D = -\Omega^2 R/\omega_B$ is the drift velocity, e_α and m_α are the charge and mass of a particle of sort α, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the basis vectors of Cartesian frame. (As compared with the expression for the trajectory radius-vector in [9, 10], here the trajectory has the maximal curvature radius at the time $t = 0$.)

The modulus of v remains constant. The curvature radius at the time t is

$$r_C = k^{-1} = v^2/\Omega^2 R \left(1 + q^2 - 2q\cos\omega_B t\right)^{1/2}, \tag{3}$$

where the curvature is denoted by k, the parameter $q = \omega_B^2 r_B/\left(\Omega^2 R\right)$ is equal to the ratio of Larmor's velocity $v_L = |\omega_B| r_B$ to the magnitude of drift velocity. As it can be seen from (3), the curvature radius belongs to the range from $r_{Cmin} = R/\left(1+q\right)$ to $r_{Cmax} = R/|1-q|$. The characteristic radiation frequencies that is proportional $\sim (c/r_C)\gamma^3$ are changed accordingly.

Further, we shall consider the case, for which the projection of a particle velocity on magnetic lines is close to speed of light, $v_{||} \to c$. Thus, the Lorentz-factor corresponding to the motion along magnetic field lines, $\gamma_{||} = \left(1 - v_{||}^2/c^2\right)^{-1/2} \gg 1$. In the case of *synchrotron* radiation a particle has to be relativistic in the frame moving with velocity $v_{||}$ along field line. This requirement is fulfilled when $\gamma^2 \gg \gamma_{||}^2$ [10, 11].

At the same time, the beamwidth $\sim 1/\gamma$ is much smaller than the angle between particle's velocity and the drift trajectory.

Fourier component of the electric field has the form

$$\mathbf{E}(\omega) = \frac{-i\omega\, e_\alpha}{cR_0} \exp\{\frac{i\omega R_0}{c}\} \int\limits_{-\infty}^{+\infty} [\mathbf{n}[\mathbf{n}, \boldsymbol{\beta}]] \exp\{i\omega(t - \mathbf{n}\mathbf{r}/c)\} dt\,. \tag{4}$$

Here R_0 is the distance up to the observer, $\mathbf{n}$ is the unit vector pointing to the observer, $\boldsymbol{\beta} = \mathbf{v}/c$, $\mathbf{v}$ is the velocity, $\mathbf{r}$ is the particle radius-vector (2). The particle is at the point $\mathbf{r}(t_0)$, when the time is t_0. To describe radiation in given direction we use the frame of natural trihedral at $\mathbf{r}(t_0)$. Denote the tangent, normal, and binormal by $\boldsymbol{\tau} = \mathbf{v}(t_0)/v$, $\boldsymbol{\nu}(t_0)$, $\mathbf{b}(t_0)$. The instant t_0 is taken from the condition $\mathbf{n}\nu(t_0) = 0$. Then [9, 10]

$$\mathbf{n} = \cos\chi\,\tau + \sin\chi\,\mathbf{b}, \tag{5}$$

where χ is the angle between the vectors $\boldsymbol{\tau}$ and $\mathbf{n}$.

The polarization unit vectors $\mathbf{e}_\pi$, $\mathbf{e}_\sigma$ on the plane that is perpendicular to the line of sight are

$$\mathbf{e}_\sigma = \boldsymbol{\nu}, \qquad \mathbf{e}_\pi = \sin\chi\,\boldsymbol{\tau} - \cos\chi\,\mathbf{b}, \tag{6}$$

Expanding the radius-vector $\mathbf{r}(t)$ into a Taylor series about $(t - t_0)$, then substituting in (4), we obtain the energy $\mathcal{E}$ emitted by a charged particle in the solid angle between o and $o+do$, and the interval of frequencies between ω and $\omega + d\omega$ [9, 10]

$$\frac{d\mathcal{E}_\pi}{do\,d\omega} = \frac{e_\alpha^2\omega^2}{3\pi^2\,c}\frac{\beta^2}{k^2v^2\gamma^4}\psi^2\left(1+\psi^2\right)K_{1/3}^2(\eta)\,, \qquad \frac{d\mathcal{E}_\sigma}{do\,d\omega} = \frac{e_\alpha^2\omega^2}{3\pi^2\,c}\frac{\beta^2}{k^2v^2\gamma^4}\left(1+\psi^2\right)^2 K_{2/3}^2(\eta)\,, \tag{7}$$

where $\psi = \gamma\chi$, $\eta = (1/2)(\omega/\omega_c)\left(1+\psi^2\right)^{3/2}$, $\omega_c = (3/2)\gamma^3 kv$, kv is given by (3); $K_{1/3}(x)$ and $K_{2/3}(x)$ are modified Bessel functions. These are the synchrotron radiation formulae that contain the trajectory curvature radius corresponding to the given direction.

Integrating Eqs. (7) over the solid angle, dividing by the time interval of radiation, and summing over polarization, we find the radiation spectrum [9–11]. The power emitted in the frequency range is given by

$$\frac{dP}{d\omega} = \frac{P_C}{\omega_C} f(y_C, q)\,, \tag{8}$$

$$f(y_C, q) = \frac{9\sqrt{3}}{16\pi} y_C \left\{ \int\limits_{\frac{y_C}{|1-q|}}^{\infty} K_{5/3}(x)\,dx + \frac{1}{\pi}\int\limits_{\frac{y_C}{1+q}}^{\frac{y_C}{|1-q|}} K_{5/3}(x)\left(\frac{\pi}{2} + \arcsin\frac{1+q^2 - y_C^2/x^2}{2q}\right)dx \right\},$$

where $P_C = \left((2e_\alpha^2)/(3c)\right)\gamma^4\beta_{||}^2\Omega^2$ is the total power emitted by a charged particle moving with a velocity $v_{||}$ along a circular orbit of the radius R, $y_C = \omega/\omega_C$, $\omega_C = (3/2)\gamma^3\Omega$.

Thus, the universal function of synchrotron radiation for a relativistic electron moving in circular orbit [5, 12, 13]

$$f(y) = \frac{9\sqrt{3}}{8\pi} y \int\limits_y^\infty K_{5/3}(x)\,dx \tag{9}$$

is replaced by the expression (8) for a relativistic electron moving along the spiral trajectory in the circular magnetic field.

Integrating in (8) with respect to frequency, we obtain the total emitted power

$$P = \frac{2}{3}\frac{e_\alpha^2}{c}\gamma^4\beta_{||}^2\Omega^2\left(1+q^2\right). \tag{10}$$

The same form has the expression for power losses of a relativistic electron moving along the circular trajectory of the effective radius $R/\sqrt{1+q^2}$.

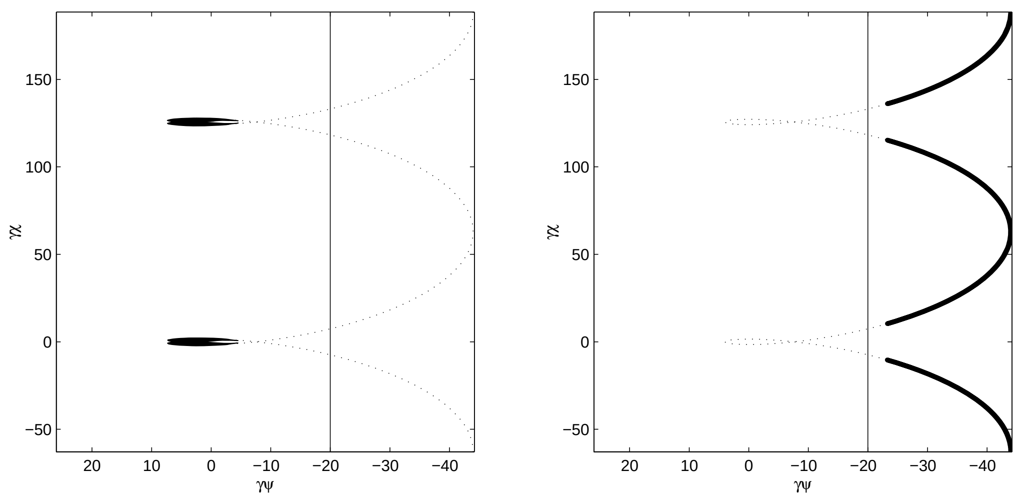
Let us consider the directional radiation pattern (7). As the radiation of an ultrarelativistic particle is concentrated into narrow cone along the particle velocity, the radiation beam will be within a narrow solid angle along the velocity directions (line L in Fig. 1).

Eqs. (7) describe the directional radiation pattern in the frame of reference (coordinates t_0, χ) that is connected with trajectory (2). To examine the spectral angular distribution in (x, y, z) coordinates, we put in angular coordinates ψ, χ such that χ is the angle between $\mathbf{n}$ and (x, z)-plane, ψ is the angle between x-axis and the projection of vector $\mathbf{n}$ onto (x, z)-plane. Then the line L in picture plane (for the line of sight along x-axis, Fig. 1a) is described by

$$\chi \approx v_y/v = -\frac{\Omega}{\omega_B}\left(\omega_B t_0 - q\sin\omega_B t_0\right), \qquad \psi \approx v_z/v = -\frac{\Omega}{\omega_B}\left(1 - q\cos\omega_B t_0\right). \tag{11}$$

The form of the radiation pattern depends on the relation $q \equiv |v_L/v_D|$ between the velocity of centrifugal drift and Larmor velocity $|v_L| = |\omega_{\rm B}|r_B$. In the case $q \gg 1$, the radiation pattern resembles the radiation cone (with the apex angle $\sim 1/\gamma_{||}$ and angular width $\sim 1/\gamma$ for the cone wall) in the straight magnetic field. In the case $q \ll 1$, we have the limit of curvature radiation. In both cases $q \gg 1$ and $q \ll 1$, the curvature radius does not depend on the time t_0 so that the profiles of radiation pattern remain constant.

In the case $q \sim 1$, the spectral angular distribution of radiation has specific features as compared with the synchrotron radiation mechanism in homogeneous magnetic field. For $q \sim 1$, the drift velocity is approximately equal to Larmor velocity $|v_D| \cong |\omega_{\rm B}|r_B \cong c/(\sqrt{2}\gamma_{||})$ so that the curvature radius varies with time t_0. There appears the range of characteristic frequencies from $\omega \sim \gamma^3\Omega|1-q|$ up to $\omega \sim \gamma^3\Omega(1+q)$ instead of one characteristic frequency ($\omega \sim \gamma^3 q\Omega$ for an emission in straight magnetic field lines, or $\omega \sim \gamma^3\Omega$ for a curvature radiation).



a) minimal characteristic frequency b) maximal characteristic frequency

Figure 2. Radiation peaks at different frequencies; $\gamma\delta = 20$, $q = 1.2$

In Fig. 2, the black regions in picture plane depict directions, in which the total intensity ($\sigma + \pi$-polarization) is greater than the maximum value multiplied by 0.8.

At the minimal characteristic frequency (Fig. 2a) the maximal intensity corresponds to directions in the vicinity of points $\psi = 0$, $\chi = 2\pi\delta\, n$, $n = 0, \pm1, \pm2, \ldots$. As the frequency increases the direction of radiation beam changes. At the maximal characteristic frequency the radiation is directed along the tangent to trajectory (2) at the points with minimal curvature radius (Fig. 2b). For intermediate frequencies, the radiation is concentrated in the vicinity of the line L (dotted in Fig. 2). If we take the curvature radiation mechanism, the isophote will be located parallel to the vertical solid line in Fig. 2.

Note that in the case of the radiation from many particles, the isophotes will be placed parallel to χ-axis, and each frequency has its own value ψ.

RADIATION FROM THE NEIGHBORHOOD OF INFLECTION POINT

In the case of exact equality $q = \omega_B^2 r_B/(\Omega^2 R) = 1$ (or $\delta^2 = r_B/R$) the trajectory (2) has points with a zero curvature. To describe radiation that generates in vicinity of these points we can not use the computing technique of classic synchrotron radiation mechanism. Approximation of the inscribed circle does not take place. In addition, a wave generated before the inflection point interferes with the wave generated after the point (Fig. 1b).

Let us take the coordinates χ and ψ to describe the direction to the observer. Then (compare with (5))

$$\mathbf{n} = \left(\cos\chi\cos\psi,\ \sin\chi,\ \cos\chi\sin\psi\right), \tag{12}$$

$$\mathbf{e}_2 = \operatorname{sign} e_\alpha \frac{[\mathbf{n}_1, \mathbf{n}]}{|[\mathbf{n}_1, \mathbf{n}]|}, \quad \mathbf{e}_1 = [\mathbf{e}_2, \mathbf{n}], \quad \mathbf{n} = [\mathbf{e}_1, \mathbf{e}_2], \tag{13}$$

where $\operatorname{sign} e_\alpha = e_\alpha/|e_\alpha|$ is the sign of a charge and $\mathbf{n}_1 = (\cos\psi, \quad 0, \quad \sin\psi)$.

Expanding radius-vector (2) into a Taylor series up to fifth-order terms in respect of $|\omega_B t| \ll 1$, taking into account that $\psi \ll 1$, $\chi \ll 1$, we obtain spectral angular distribution in the neighborhood of point $\psi = 0$, $\chi = 0$ (call it as A) that corresponds to the time $t_0 = 0$

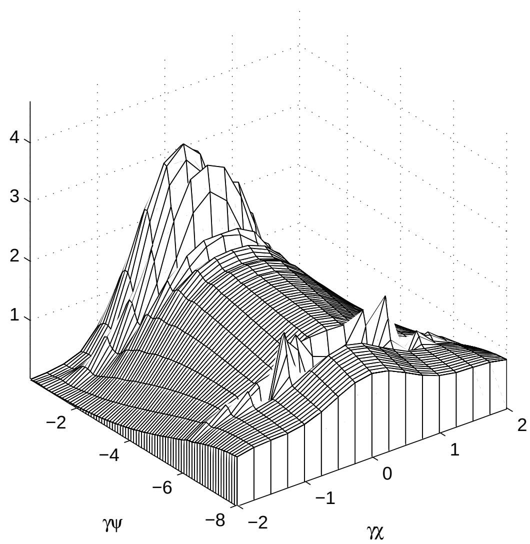
$$\frac{d\mathcal{E}_1}{d\omega do} = \frac{e_\alpha^2\beta^2}{\pi^2 c}\gamma^2 f^{8/5}\left|\gamma\chi V_0^4\left(\mathbf{y}\right) + i\frac{2}{3}\frac{1}{\sqrt{2\gamma|\delta|}}\frac{V_3^4\left(\mathbf{y}\right)}{f^{3/5}}\right|^2, \tag{14}$$

$$\frac{d\mathcal{E}_2}{d\omega do} = \frac{e_\alpha^2\beta^2}{\pi^2 c}\gamma^2 f^{8/5}\left|\gamma\psi\ V_0^4\left(\mathbf{y}\right) - \frac{e_\alpha}{|e_\alpha|}\frac{V_2^4\left(\mathbf{y}\right)}{f^{2/5}} - \frac{e_\alpha}{|e_\alpha|}\frac{1}{6\gamma|\delta|}\frac{V_4^4\left(\mathbf{y}\right)}{f^{4/5}}\right|^2, \tag{15}$$

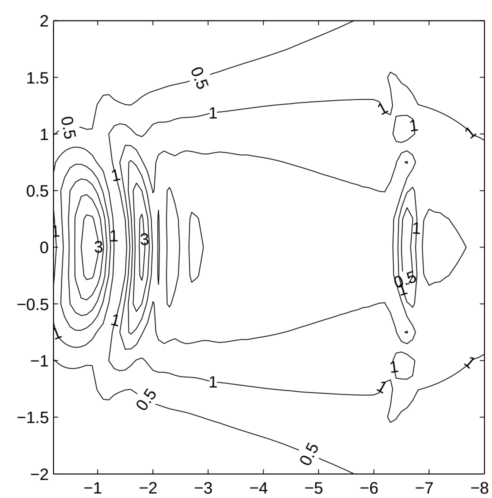
where $V_m^N\left(\mathbf{y}\right)$ is the special functions of wave catastrophes [4]

$$V_m^N\left(\mathbf{y}\right) = \int\limits_{-\infty}^{\infty}\left(i\xi\right)^m \exp\left\{i\sum_{k=1}^{N+1} y_k \xi^\kappa\right\} d\xi,$$

$$\mathbf{y} = \left(f^{4/5}\left(1+\gamma^2\chi^2+\gamma^2\psi^2\right),\ 0,\ \frac{2}{3}\frac{e_\alpha}{|e_\alpha|}f^{2/5}\gamma\psi,\ \frac{1}{3\sqrt{2\gamma|\delta|}}f^{1/5}\gamma\chi,\ \frac{1}{5}\right), \quad f = \frac{\omega}{2\gamma^2\sqrt{\gamma|\delta|/2}|\omega_B|}. \tag{16}$$



a) minimal characteristic frequency



b) isophotes in ψ-χ plane

Figure 3. Radiation pattern forming in the neighborhood of zero curvature points

Formulae (14), (15) replace Eqs. (7) for radiation in the direction corresponding to the particle velocity at the time $t_0 = 0$. Fig. 3 gives the directional radiation pattern plotted according equations (14), (15) at the frequency $f \sim 1$. Note that the radiation pattern has interference form.

Plotting total intensity ($\mathbf{e}_1 + \mathbf{e}_2$-polarization), we drop factors $e_\alpha^2\beta^2\gamma^2\pi^{-2}c^{-1}$ in (14), (15). In these units the maximum value in (7), Fig. 2, is equal to 1.47. In Fig. 3b we see isophotes that have a label 3. The intensity increasing takes place because the radiation in given direction is coherent sum of contributions from points before and after the inflection point, Fig. 1b. In other directions at the line L the radiation pattern, as before, is described by Eqs. (7). As the frequency increases, the intensity in former direction decreases, the radiation peak displaces, and we has the picture shown in Fig. 2.

CONCLUSION

Thus, the synchrotron radiation in curved magnetic field lines has additional features as compared with synchrotron radiation in straight magnetic field lines: i) there exists a range of characteristic frequencies instead of the single characteristic frequency; ii) at the same time, the direction of radiation beam varies correspondingly; iii) if the trajectory has inflection points, then the directional radiation pattern in the direction corresponding this point is described by special functions of wave catastrophes; iv) and in this case, the peak of radiation beam at the minimal characteristic frequency is several times greater than the peaks at other characteristic frequencies.

As it was shown in [1, 10] there are regions in pulsar magnetosphere where the discussed radiation mechanism can take place. In that case the above-mentioned properties may be useful in investigating the fine structure of radiation pulses, or in analyzing of the radius to frequency mapping hypothesis.

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