THE SIMPLE SOLUTION OF RELATIVISTIC WAVE EQUATIONS FOR CHARGED PARTICLES IN CONSTANT ELECTRIC FIELD AND PAIR PRODUCTION

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We obtained the generalization of the simple solution of Dirac equations for the electron in constant electric field on the case of the Kemmer-Duffin-Proca equations for the charged vector boson with arbitrary gyromagnetic ratio g moving in constant electric field. The pair production of spin 1/2 and spin 1 particles is discussed.

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1. INTRODUCTION

There is a well known sad confession of A.Einstein: "All the fifty years of conscious brooding have brought me no closer to the answer to the question "what are light quanta?" Of course, today, every rascal thinks he knows the answer, but he is deluding himself."

A.I. Akhieser has taught his pupils do not change into such rascals and to distinguish understanding from a lack of understanding or from misunderstanding. A.I. Akhieser himself never was afraid of showing that he did not understand something and always took great pains to comprehend incomprehensible, and he disliked deeply those "human individuals who are able to fool us for a while into believing that they possess some understanding, when it finally emerges that indeed they possess none whatever!", [1].

A.I. Akhieser was passionately fond of science. He was not the man, who took to science "out of a joyful sense of superior intellectual power" or the man who "had offered the products of his brain on the altar for purely utilitarian purposes". The driving force of scientific research of A.I. Akhieser was "the devoted striving to comprehend a portion, be it ever so tiny, of the Reason that manifests itself in nature." (All quotations are taken from A. Einstein, who was the most favourite hero to A.I. Akhieser). A.I. Akhieser had wide interests in all theoretical physics, but quantum electrodynamics was the subject of permanent fixed attention of him. One of us (Yu. S.) has had frequently the lucky opportunities to discuss with A.I. Akhieser a lot of problems of quantum electrodynamics, such, for instance, as relativistic wave equations for arbitrary spin particles, the equivalence of different forms of relativistic wave equations, the description of states of polarizations of arbitrary spin particles, the different exact solutions of Dirac equation for electron, the production of particles and antiparticles by constant electric fields, black hole evaporation, and others. This report is the feeble echo of those discussions.

2. SIMPLE SOLUTION OF DIRAC EQUATIONS

Let us consider the Dirac equations for the electron moving in an electric field E with components (0, 0, E). Taking the no vanishing component of vector potential $A_3 = (z-t)/2$, scalar potential $\varphi = -E(z-t)/2$, and using the notations $e = |e|$, $z-t = \varsigma$, $z+t = \eta$, we obtain

$$\begin{aligned}&\Big[\frac{\partial}{\partial\eta}(\gamma_3 - i\gamma_4) + \frac{\partial}{\partial\zeta}(\gamma_3 + i\gamma_4) + i\frac{eE}{2}\zeta(\gamma_3 - i\gamma_4)\\ &+ \frac{\partial}{\partial x}\gamma_1 + \frac{\partial}{\partial y}\gamma_2 - (g-2)\frac{eE}{2m}\gamma_3\gamma_4 + m\Big]\psi = 0.\end{aligned} \tag{1}$$

(Other notations are as in the book [2], g is gyromagnetic factor for the electron). Put ψ in the form $\psi = e^{i\Pi\eta + ip_2 y}\Phi(\zeta)$. Then the equations (1) become

$$\begin{aligned}&\Big[i\Pi(\gamma_3 - i\gamma_4) + \frac{\partial}{\partial\zeta}(\gamma_3 + i\gamma_4) + i\frac{eE}{2}\zeta(\gamma_3 - i\gamma_4)\\ &+ ip_2\gamma_2 - (g-2)\frac{eE}{2m}\gamma_3\gamma_4 + m\Big]\Phi = 0.\end{aligned} \tag{2}$$

If we introduce the new variable $\xi = \sqrt{2/eE}\,(\Pi + eE\zeta/2)$ and put on ψ the polarization condition $i\gamma_5\gamma_1\Phi = \pm\Phi$ the Eqs. (2) reduce to

$$\Big[\left(\frac{\gamma_3 + i\gamma_4}{2}\right)\frac{d}{d\xi} + \left(\frac{\gamma_3 - i\gamma_4}{2}\right)i\xi + \gamma_3\gamma_4\tilde{p}_2 + \tilde{m}\Big]\Phi = 0, \tag{3}$$

where

$$\tilde{p}_2 = [\pm p_2 - (g-2)eE/2m]/\sqrt{2eE}, \quad \tilde{m} = m/\sqrt{2eE}.$$

Let us introduce now two bispinors that are determined by the conditions

$$i\gamma_4\gamma_3 u_+ = u_+,\ i\gamma_5\gamma_1 u_+ = \pm u_+;\quad u_- = \left(\frac{\gamma_3 - i\gamma_4}{2}\right)u_+ \tag{4}$$

and put $\Phi(\xi) = \phi(\xi)u_+ + \chi(\xi)u_-$, introducing two new scalar functions, ϕ and χ. From (3) we obtain the equations

$$(S^{+}\frac{d}{d\xi}+S^{-}i\xi+2iS_{3}\widetilde{p}_{2}+\widetilde{m})\Psi=0, \tag{5}$$

where $\Psi=\begin{pmatrix}\phi\\ \chi\end{pmatrix}$ and S^{+}, S^{-} and S_3 are the usual spin matrices for spin ½. We can rewrite (5) in the form

$$\begin{pmatrix}\widetilde{m}+i\widetilde{p}_{2} & \frac{d}{d\xi}\\ i\xi & \widetilde{m}-i\widetilde{p}_{2}\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix}=0, \tag{6}$$

and finally we obtain the simple solution of the Dirac equations (1)

$$\psi=e^{i\Pi\eta+ip_{2}y}\left(\xi\right)^{-i[\frac{m^{2}+p_{2}^{2}}{2eE}\mp(g-2)\frac{p_2}{2m}+(g-2)^{2}\frac{eE}{8m^{2}}]}\times\left(i\frac{m-i\left(\pm p_{2}-(g-2)eE/2m\right)}{\xi\sqrt{2eE}}u_{+}+u_{-}\right). \tag{7}$$

Replacing in the solution (7) Π by $\Pi=(p_3-\varepsilon)/2$, where $\varepsilon=\sqrt{m^{2}+p_{2}^{2}+p_{3}^{2}}$, we can verify after some simple calculations that in the limit $eE\to 0$, ($\xi\neq 0$) our solution (7) passes into ordinary plane wave solution

$$\psi=e^{-i\varepsilon t+ip_{2}y+ip_{3}z}u, \tag{8}$$

where u satisfies the equations

$$\left(ip_{2}\gamma_{2}+ip_{3}\gamma_{3}-\varepsilon\gamma_{4}+m\right)u=0. \tag{9}$$

The solution (7) for $g = 2$ is well known [3,4]. It can also be obtained as the generalization of the solution, obtained in [5,6]. But our method of derivation of this solution may be easily generalized on the case of the more complex problem of the charged vector boson with arbitrary g moving in constant electric field.

3. SIMPLE SOLUTION OF KEMMER-DUFFIN-PROCA EQUATIONS

There are very many ways to describe vector boson. The corresponding equations were invented by A. Proca (1938), R.J. Duffin (1938), N. Kemmer (1939), W. Pauli and M. Firz (1939), V, Bargman and E.P. Wigner (1948), H.S. Green (1949) (see the book [6] for review). All these equations are equivalent one to another if the vector boson is free. But it is not the case for vector boson interacting with electromagnetic field.

Let us consider the Kemmer-Duffin equations (that are completely equivalent to the Proca equations) for the negative charged vector boson with arbitrary gyromagnetic ratio g in the form similar to (1),

$$\begin{aligned}&\{\frac{\partial}{\partial\eta}[(\gamma_{3}-i\gamma_{4})^{(1)}+(\gamma_{3}-i\gamma_{4})^{(2)}]+\frac{\partial}{\partial\zeta}[(\gamma_{3}+i\gamma_{4})^{(1)}+(\gamma_{3}+i\gamma_{4})^{(2)}]+i\frac{eE}{2}\zeta[(\gamma_{3}-i\gamma_{4})^{(1)}+(\gamma_{3}-i\gamma_{4})^{(2)}\\&+\frac{\partial}{\partial x}[\gamma_{1}^{(1)}+\gamma_{1}^{(2)}]+\frac{\partial}{\partial y}[\gamma_{2}^{(1)}+\gamma_{2}^{(2)}]-(g-1)\frac{eE}{2m}[(\gamma_{3}\gamma_{4})^{1}+(\gamma_{3}\gamma_{4})^{2}]+2m\}\psi=0.\end{aligned} \tag{10}$$

Here ψ is symmetrical spin-tensor, $\psi_{\alpha\beta}=\psi_{\beta\alpha}$, $\alpha,\beta = 1,2,3,4$, and the Dirac matrices with superscript (1) and (2) act on the first or second indices of $\psi_{\alpha\beta}$ correspondingly. After transformations similar to the Dirac case we obtain instead of (3)

$$\begin{aligned}&\{[\left(\frac{\gamma_{3}+i\gamma_{4}}{2}\right)^{(1)}+\left(\frac{\gamma_{3}+i\gamma_{4}}{2}\right)^{(2)}]\frac{d}{d\xi}+[\left(\frac{\gamma_{3}-i\gamma_{4}}{2}\right)^{(1)}+\left(\frac{\gamma_{3}-i\gamma_{4}}{2}\right)^{(2)}]i\xi\\&+[(\gamma_{3}\gamma_{4})^{(1)}+(\gamma_{3}\gamma_{4})^{(2)}]\widetilde{p}_{2}+2\widetilde{m}\}\Phi=0,\end{aligned} \tag{11}$$

where

$$\widetilde{p}_{2}=[\pm p_{2}-(g-1)eE/2m]/\sqrt{2eE},\quad \widetilde{m}=m/\sqrt{2eE}.$$

Let us introduce now three quantities u_+, u_0, and u_- that are determined by the conditions

$$\begin{aligned}&i(\gamma_{4}\gamma_{3})^{(1,2)}u_{+}=u_{+},\ i(\gamma_{5}\gamma_{1})^{(1,2)}u_{+}=\pm u_{+};\\&u_{0}=[\left(\frac{\gamma_{3}-i\gamma_{4}}{2}\right)^{(1)}+\left(\frac{\gamma_{3}-i\gamma_{4}}{2}\right)^{(2)}]u_{+},\\&u_{-}=[\left(\frac{\gamma_{3}-i\gamma_{4}}{2}\right)^{(1)}+\left(\frac{\gamma_{3}-i\gamma_{4}}{2}\right)^{(2)}]u_{0}\end{aligned} \tag{12}$$

and put

$$\Phi\left(\xi\right)=\phi\left(\xi\right)u_{+}+\chi\left(\xi\right)u_{0}+\theta\left(\xi\right)u_{-},$$

introducing three new scalar functions, ϕ, χ and θ. From (11) we obtain the equations

$$(S^{+}\frac{d}{d\xi}+S^{-}i\xi+2iS_{3}\widetilde{p}_{2}+2\widetilde{m})\Psi=0, \tag{13}$$

where $\Psi=\begin{pmatrix}\phi\\ \chi\\ \theta\end{pmatrix}$ and S^{+}, S^{-} and S_3 are the usual spin matrices for spin 1. We can rewrite (13) in the form

$$\begin{pmatrix} \tilde{m}+i\tilde{p}_2 & \frac{1}{\sqrt{2}}\frac{d}{d\xi} & 0 \\ \frac{1}{\sqrt{2}}i\xi & \tilde{m} & \frac{1}{\sqrt{2}}\frac{d}{d\xi} \\ 0 & \frac{1}{\sqrt{2}}i\xi & \tilde{m}-\tilde{p}_2 \end{pmatrix}\begin{pmatrix}\phi\\ \chi\\ \theta\end{pmatrix}=0 \tag{14}$$

and finally we obtain the simple solution of the Kemmer-Duffin equations (10)

$$\psi = \frac{1}{\sqrt{\xi}}e^{i\Pi\eta+ip_2y}\left(\xi\right)^{-i[\frac{m^2+p_2^2}{2eE}\mp(g-2)\frac{p_2}{2m}+(g-1)(g-3)\frac{eE}{8m^2}]}$$
$$\times\begin{pmatrix}[i\dfrac{m-i(\pm p_2-(g-1)eE/2m)}{2\xi\sqrt{eE}}+\dfrac{\sqrt{eE}}{2\xi m}]u_+ \\ +u_0-i\xi\dfrac{\sqrt{eE}}{m-i(\pm p_2-(g-1)eE/2m)}u_-\end{pmatrix}. \tag{15}$$

Of course, one should to find the solution for the third state of polarization of vector boson that is determined by condition

$$i[(\gamma_5\gamma_1)^{(1)}+(\gamma_5\gamma_1)^{(2)}]\psi=0.$$

This solution must be similar to (15) but to obtain it one needs to make somewhat more complicated calculations than in considered case.

Replacing in the solution (15) Π by $\Pi=(p_3-\varepsilon)/2$, where $\varepsilon=\sqrt{m^2+p_2^2+p_3^2}$, we can verify after some calculations that our solution (15) in the limit $eE\to 0$, ($\xi\neq 0$) also passes into ordinary plane wave solution

$$\psi = e^{-i\varepsilon t+ip_2y+ip_3z}u, \tag{16}$$

where u satisfies the equations similar to (9)

$$\left(ip_2\gamma_2^{(1,2)}+ip_3\gamma_3^{(1,2)}-\varepsilon\gamma_4^{(1,2)}+m\right)u=0. \tag{17}$$

4. THE DEPENDENCE OF PAIR PRODUCTION FROM POLARIZATIONS

The solutions (7) and (15) have the remarkable features, they have branching point at ξ=0. At this point the value of the wave function amplitude springs and multiplies on the factor

$$e^{-\pi[\frac{m^2+p_2^2}{2eE}\mp(g-2)\frac{p_2}{2m}+(g-2)^2\frac{eE}{8m^2}]} \tag{18}$$

for electron and on the factor

$$e^{-\pi[\frac{m^2+p_2^2}{2eE}\mp(g-2)\frac{p_2}{2m}+(g-1)(g-3)\frac{eE}{8m^2}]} \tag{19}$$

for vector boson. It is known that [4,5] that the squares of these factors determine the probabilities of pair production by constant electric field. We see that these probabilities do not depend from the polarizations of the particles only if g = 2. (It agrees with the results of [8] where the case of charged vector boson with g = 2 is considered).

This dependence from the polarizations is due to the fact that if the classical magnetic dipole μ moves and the direction of motion of magnetic dipole is orthogonal to its orientation then the electric dipole $d=\mu p/m$ arises. This electric dipole interacts with electric field, and this interaction depends from the magnetic dipole orientation. We see that only (g-2) part of quantum magnetic dipole manifests itself as the classical magnetic dipole.

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