VERY INTENSE NON-STATIC MAGNETIC FIELDS AND PARTICLE PRODUCTION

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We present the calculation of the probability production of $e^+ - e^-$ pairs in a strong rotating magnetic field. By comparing this case with that in which the magnetic field changes only in strength, we conclude that for pair production the change of direction of the magnetic field is much more efficient than the change of its strength.

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I. INTRODUCTION

An amount of experimental data obtained through astrophysical observations indicate the presence of magnetic fields of extremely large strength ($\sim 10^{15}$ gauss) around compact objects like neutron stars or black holes [1,2]. Moreover, these magnetic fields are in general non-static because their sources are either rotating or collapsing. In view of these facts, the study of QED when the magnetic energy is comparable or larger than the electron rest mass is no longer a pure theoretical exercise but could be brought into correspondence with experimental processes, although not realizable in terrestrial laboratories.

Within this framework we compute here the probability of production of $e^+ - e^-$ pairs in the presence of a strong magnetic field varying both in strength and in direction. Since we are only interested in the effect of the electromagnetic field, no gravitational effects are considered so, in particular, the space-time is assumed to be flat. Moreover we consider a process taking place in absence of preexisting matter. Finally, this process could be interpreted as a first step in the production of bunches of gamma-rays which are the real experimental signals (gamma-ray bursts) [3-6].

II. DESCRIPTION OF THE MODEL

In the problem we are dealing with there are two scales of lengths and times: one related to the elementary particles (Compton wave-length of the electron) and the other related to the macroscopic source of the field. In this second case the order of magnitude of the length over which the field varies and of the typical time of variation are extremely larger than the previous ones. This allows us to treat the magnetic field as uniform in space and to take into account the time variation by means of the first-order adiabatic approximation [7].

A particular choice for the time dependence of the field has been made:

$$\mathbf{B}(t) = \begin{pmatrix} \mathrm{B}_x(t) \\ \mathrm{B}_y(t) \\ \mathrm{B}_z(t) \end{pmatrix} = \mathbf{B}_0 + \mathbf{b}\,t = \begin{pmatrix} 0 \\ \mathrm{b}_y(t) \\ \mathrm{B}_0 + \mathrm{b}_z(t) \end{pmatrix} \tag{1}$$

Maxwell equations ensure that with this choice the macroscopic electric current density $\mathbf{J}_e(\mathbf{r},\, t)$ vanishes. A form of the vector potential giving rise to the magnetic field (1) is (symmetric gauge)

$$\mathbf{A}(\mathbf{r},\, t) = -\frac{1}{2}\left[\mathbf{r}\times \mathbf{B}(t)\right] \tag{2}$$

With this choice we keep a cylindrical symmetry in the problem and, unless we also introduce a scalar potential, the electric field is completely determined ($\hbar = c = 1$ units are used)

$$\mathbf{E}(\mathbf{r},\, t) = -\frac{\partial\, \mathbf{A}(\mathbf{r},\, t)}{\partial\, t} = \mathbf{E}(\mathbf{r}) = \frac{1}{2}(\mathbf{r}\times \mathbf{b}) \tag{3}$$

The dynamics of the Dirac field $\psi(\mathbf{r},\, t)$ in the presence of this electromagnetic field is described by the Hamiltonian

$$H(t) = \int d\mathbf{r}\, \psi^{\dagger}(\mathbf{r},\, t)\, \mathcal{H}\,(\mathbf{r}, -i\nabla,\, t)\, \psi(\mathbf{r},\, t) \tag{4}$$

with (the charge of the electron is $-e$ ($e>0$))

$$\mathcal{H}\,(\mathbf{r}, -i\nabla,\, t) = \boldsymbol{\alpha}\cdot\left[-i\nabla + e\mathbf{A}(\mathbf{r},\, t)\right] + \beta\, m \tag{5}$$

In order to apply the adiabatic perturbation theory we have to determine the instantaneous eigenstates and eigenvalues of $H(t)$. To do this we remind some well-known results about the motion of a relativistic electron in a constant and uniform magnetic field which, for later convenience, has been chosen in the y-z plane: $\mathbf{B} = \mathrm{B}_y\,\hat{\mathbf{j}} + \mathrm{B}_z\,\hat{\mathbf{k}}$ [8].

In classical mechanics the electron performs a motion along an helix whose axis is parallel to **B**. We will call $R_\perp^2$ the square of the distance of the axis of the helix from the origin.

In first quantization, by working in the symmetric gauge, we can choose the Hamiltonian □ given by Eq. (5) (without the time dependence), the linear momentum parallel to $\mathbf{B}$ $□_{\|}$, the total angular momentum parallel to $\mathbf{B}$ $□_{\|}$ and $R_\perp^2$ as a complete set of commuting observables. If we call

$$\psi_{\pm,j}(\mathbf{r}) \qquad j = \{n_d, k_\|, \sigma_\|, n_g\} \tag{6}$$

the functions building up the common basis to that operators then

$$□\, \psi_{\pm,j}(\mathbf{r}) = \pm w_j\, \psi_{\pm,j}(\mathbf{r}), \tag{7}$$

$$□_\|\, \psi_{\pm,j}(\mathbf{r}) = k_\|\, \psi_{\pm,j}(\mathbf{r}), \tag{8}$$

$$□_\|\, \psi_{\pm,j}(\mathbf{r}) = \left(n_d - n_g + \frac{\sigma_\|}{2}\right)\psi_{\pm,j}(\mathbf{r}), \tag{9}$$

$$R_\perp^2\, \psi_{\pm,j}(\mathbf{r}) = \frac{2n_g + 1}{e\mathrm{B}}\psi_{\pm,j}(\mathbf{r}), \tag{10}$$

where

$$w_j = \sqrt{m^2 + k_\|^2 + e\mathrm{B}(2n_d + 1 + \sigma_\|)} \tag{11}$$

are the Landau levels and where the quantum numbers vary in the following ranges

$$n_d = 0,1,\ldots\ , \tag{12}$$

$$k_\| = -\infty,\ldots\ldots,\infty\ , \tag{13}$$

$$\sigma_\| = \mp 1 \tag{14}$$

$$n_g = 0,1,\ldots\ . \tag{15}$$

It is worth noting that the energies w_j do not depend on the quantum number n_g and that if $n_d = 0$ and $\sigma_\| = -1$ the corresponding energies w_j do not depend on **B**. We will call the corresponding states transverse ground states.

In order to go to second quantization we expand the Dirac field $\psi(\mathbf{r},t)$ in the previous basis

$$\psi(\mathbf{r},t) = \sum_j c_j \psi_{+,j}(\mathbf{r})\, e^{-iw_j t} + d_j^\dagger \psi_{-,j}(\mathbf{r})\, e^{iw_j t} \tag{16}$$

and the second quantized relevant operators become

$$H = \int d\mathbf{r}\, \psi^\dagger(\mathbf{r},t)\, □\, \psi(\mathbf{r},t) = \\ = \sum_j w_j\left(c_j^\dagger c_j + d_j^\dagger d_j\right), \tag{17}$$

$$P_\| = \int d\mathbf{r}\, \psi^\dagger(\mathbf{r},t)\, □_\| \psi(\mathbf{r},t) = \\ = \sum_j k_\|\left(c_j^\dagger c_j - d_j^\dagger d_j\right), \tag{18}$$

$$J_\| = \int d\mathbf{r}\psi^\dagger(\mathbf{r},t)\, □_\| \psi(\mathbf{r},t) = \\ = \sum_j \left(n_d - n_g + \frac{\sigma_\|}{2}\right)\left(c_j^\dagger c_j - d_j^\dagger d_j\right). \tag{19}$$

The minus signs between the operators $c_j^\dagger c_j$ and $d_j^\dagger d_j$ in $P_\|$ and in $J_\|$ are connected to the fact that $\psi_{-,j}(\mathbf{r})$ are not the charge conjugated functions of the $\psi_{+,j}(\mathbf{r})$ so the physical meaning of $k_\|$ and $\sigma_\|$ must be reversed and n_d and n_g must be exchanged for positrons.

III. RESULTS AND CONCLUSIONS

The first order transition amplitude from the vacuum state $|0\rangle$ to the $e^- - e^+$ pair state $|j\,j'\rangle$ is given by [7]

$$\gamma_{jj'}(t) = \int_0^t dt' \frac{\langle j\,j'|\nabla_{\mathbf{B}} H(t')|0\rangle \cdot \mathbf{b}}{w_j(t') + w_{j'}(t')} \times \\ \times \exp\left\{ i\int_0^{t'} dt''\left[w_j(t'') + w_{j'}(t'')\right]\right\}. \tag{20}$$

The transition matrix elements are

$$\langle j\,j'|\nabla_{\mathbf{B}} H(t)|0\rangle \cdot \mathbf{b} = \\ = \frac{e}{2}\int d\mathbf{r}\psi_{+,j}^\dagger(\mathbf{r},t)(\mathbf{r}\times\boldsymbol{\alpha})\psi_{-,j'}(\mathbf{r},t)\cdot\mathbf{b} = \\ = \frac{e}{2}\int d\mathbf{r}\psi_{+,j}'^\dagger(\mathbf{r},t)\, □^\dagger(t)(\mathbf{r}\times\boldsymbol{\alpha})\, □(t)\, \psi_{-,j'}'(\mathbf{r},t)\cdot\mathbf{b} \tag{21}$$

where $□(t) = e^{-i\vartheta(t)J_x}$ is the operator which performs a rotation around the x-axis of an angle $\vartheta(t)$ defined by

$$\tan\vartheta(t) = \frac{\mathrm{B}_y(t)}{\mathrm{B}_z(t)} \tag{22}$$

and where $\psi_{\pm,j}'(\mathbf{r},t)$ form a basis of the Hilbert space in the case in which $\mathbf{B}(t) = \mathrm{B}(t)\hat{\mathbf{k}} = \sqrt{\mathrm{B}_y^2(t) + \mathrm{B}_z^2(t)}\,\hat{\mathbf{k}}$. After some easy manipulation the matrix elements (21) can be rewritten as

$$\langle j\,j'|\nabla_{\mathbf{B}} H(t)|0\rangle \cdot \mathbf{b} = T_{jj'}'(t)\cdot\mathbf{b}'(t) \tag{23}$$

where

$$\mathbf{b}'(t) = \begin{pmatrix} 0 \\ \cos\vartheta(t)\mathrm{b}_y - \sin\vartheta(t)\mathrm{b}_z \\ \sin\vartheta(t)\mathrm{b}_y + \cos\vartheta(t)\mathrm{b}_z \end{pmatrix} \tag{24}$$

$$T_{jj'}'(t) = \frac{e}{2}\int d\mathbf{r}\psi_{+,j}'^\dagger(\mathbf{r},t)\,(\mathbf{r}\times\boldsymbol{\alpha})\psi_{-,j'}'(\mathbf{r},t). \tag{25}$$

In this way all the effects of the rotation of the magnetic field have been put into $\mathbf{b}'(t)$ which is the constant vector **b** seen from the rotated frame whose *z*-axis is instantaneously parallel to **B**(*t*). Finally, the form of the matrix elements (25) suggests the usual one particle interpretation of the pair creation as a jump of an electron from a negative energy level to a positive energy one. This interpretation makes easier to write down the useful selection rules we will obtain below.

One of us has already calculated all the transition probabilities in the case in which **B**(t) changes only in strength [9]. Then, now we want to stress the new effects due to the rotation of the magnetic field. If $\mathbf{B}(t) = \mathrm{B}(t)\hat{\mathbf{k}}$ the only transition operator is $\frac{e}{2}(\mathbf{r}\times\boldsymbol{\alpha})_z$ and then only $\Delta J_{||} = 0$ transitions are allowed. This is the quantum counterpart of the classical fact that in the same physical situation the angular momentum of a classical electron is an adiabatic invariant. Instead, if $\mathbf{B}(t) = \mathrm{B_y}(t)\hat{\mathbf{j}} + \mathrm{B_z}(t)\hat{\mathbf{k}}$ also $\Delta J_{||} = \pm 1$ transitions are allowed because of the presence of the transition operator $\frac{e}{2}(\mathbf{r}\times\boldsymbol{\alpha})_y$. Last but not least, from the anticommutation

$$\left[(\mathbf{r}\times\boldsymbol{\alpha})_z, \sigma_z\right]_+ = 0 \tag{26}$$

it follows that the selection rule

$$\sigma_{||i} + \sigma_{||f} = 0 \tag{27}$$

holds for the eigenstates of σ_z. Since the transverse ground states

$$\psi'_{+,g}(\mathbf{r},t) = g_{n_g,k_{||}}(\mathbf{r},t)\begin{pmatrix} 0 \\ w_g + m \\ 0 \\ k_{||} \end{pmatrix}, \tag{28}$$

$$\psi'_{-,g'}(\mathbf{r},t) = g'_{n'_g,k'_{||}}(\mathbf{r},t)\begin{pmatrix} 0 \\ -k'_{||} \\ 0 \\ w_{g'} + m \end{pmatrix} \tag{29}$$

are eigenstates of σ_z with $\sigma_{||} = \sigma'_{||} = -1$ then only the rotation of **B**(t) allows the transitions from a transverse ground state to another transverse ground state or, in second quantization language, only the rotation of **B**(t) allows the creation of a pair in which both the electron and the positron are in a transverse ground state.

This is an important observation because if $P_0(t) = P\left(|0\rangle \to |gg'\rangle\right)$ is the probability transition to a state in which both e^- and e^+ are in a transverse ground state and $P(t) = P\left(|0\rangle \to |j\,j' \neq g\,g'\rangle\right)$ then

$$\frac{P(t)}{P_0(t)} \sim \left(\frac{m^2}{e\mathrm{B}_0}\right)^{3/2} << 1. \tag{30}$$

and we are allowed at first approximation to calculate only $P_0(t)$. Actually, we want to calculate the transition probability to a normalized state and we consider the state

$$\left|e^-\, e^+\right\rangle = \left|e^-\right\rangle\left|e^+\right\rangle \tag{31}$$

with

$$\left|e^-\right\rangle = \int_{-\infty}^{\infty} dk \sqrt{\frac{a}{\sqrt{2\pi}}}\, \mathrm{e}^{-\frac{\alpha^2}{4}(k-k_0)^2}\left|0,k,-1,n_g\right\rangle. \tag{32}$$

$$\left|e^+\right\rangle = \int_{-\infty}^{\infty} dk \sqrt{\frac{a}{\sqrt{2\pi}}}\, \mathrm{e}^{-\frac{\alpha^2}{4}(k-k_0)^2}\left|0,k,-1,n_g+1\right\rangle, \tag{33}$$

where the quantum numbers n_d and $\sigma_{||}$ are fixed to 0 and –1 respectively and where we have put $k = k_{||}$. This state represents an electron with mean longitudinal momentum k_0 and a positron with opposite mean momentum because of the opposite meaning of $k_{||}$ for positrons. The widths of the two Gaussian wave packets are both equal to a. Finally, we note that the final probability will be multiplied by two to take into account the contribution of the states with $n_g + 1$ for electron and n_g for the positron.

Since this transition is an effect of the rotation of **B**(t), we put for simplicity $\mathrm{b}_z = 0$ and $\mathrm{b}_y = \mathrm{b} = \frac{\mathrm{B}_0}{\tau}$, then the transition amplitude is

$$\gamma_{n_g,k_0,a}(t) = \sqrt{\frac{n_g+1}{32}\frac{e\mathrm{B}_0}{m^2}}\int_{-\infty}^{\infty} dk \times \frac{a}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\alpha^2}{2}(k-k_0)^2}\frac{m^2}{\varepsilon(k)^2}\int_0^{t/\tau} d\eta\, \frac{\mathrm{e}^{2i\varepsilon(k)\tau\eta}}{\left(1+\eta^2\right)^{3/4}} \tag{34}$$

where

$$\varepsilon(k) = \sqrt{m^2 + k^2}\,. \tag{35}$$

Since τ is a typical evolution time of the magnetic field ($\tau \sim 10^5 s$) then $\tau\varepsilon(k) >> 1$ the integral in η can be substituted by the first term of its asymptotic series:

$$\gamma_{n_g,k_0,a}(t) \cong \sqrt{\frac{n_g+1}{128}\frac{e\mathrm{B}_0}{m^2}}\frac{i}{m\tau}\int_{-\infty}^{\infty} dk \times \frac{a}{\sqrt{2\pi}}\frac{\mathrm{e}^{-\frac{\alpha^2}{2}(k-k_0)^2}}{\left[1+(k/m)^2\right]^{3/2}}\left\{1 - \frac{\mathrm{e}^{2i\varepsilon(k)t}}{\left[1+(t/\tau)^2\right]^{3/4}}\right\} \tag{36}$$

If we look for times t such that tm>>1 then $t\varepsilon(k)$>>1 and if we also assume that the width of the Gaussians is much larger than the Compton wave-length (ma>>1) we obtain the approximated transition amplitude

$$\gamma_{n_g,k_0,a}(t) \cong \sqrt{\frac{n_g+1}{128}\frac{e\mathrm{B}_0}{m^2}}\frac{i}{m\tau}\frac{1}{\left[1+(k_0/m)^2\right]^{3/2}} \times$$

$$\times \left\{1-\frac{3}{2}\frac{1}{(ma)^2}\frac{1-4(k_0/m)^2}{\left[1+(k_0/m)^2\right]^2}\right\} \tag{37}$$

By squaring the modulus of $\gamma_{n_g,k_0,a}(t)$ we have

$$P_{n_g,k_0,a}(t) \cong \frac{n_g+1}{128}\frac{e\mathrm{B}_0}{m^2}\frac{1}{(m\tau)^2}\frac{1}{\left[1+(k_0/m)^2\right]^3} \times \tag{38}$$

$$\times \left\{1-3\frac{1}{(ma)^2}\frac{1-4(k_0/m)^2}{\left[1+(k_0/m)^2\right]^2}\right\}$$

We can transform the sum on n_g into an integral by using the operator relation $R_\perp^2 = \frac{2N_g+1}{e\mathrm{B}(t)}$ (see Eq. (10)):

$$\sum_{n_g} \to \frac{e\mathrm{B}(t)}{2}\int_0^{R_{\perp M}^2} dR_\perp^2 \tag{39}$$

and the final result in c.g.s. units is

$$\frac{dP_{k_0,a}(t)}{dt} \overset{\frac{ct}{\bar{\lambda}}>>1}{\cong} \left(\frac{e\mathrm{B}_0\bar{\lambda}}{mc^2}\right)^3\left(\frac{R_{\perp M}^2}{16\bar{\lambda}c\tau}\right)^2 \times \tag{40}$$

$$\times \frac{t}{\tau^2}\frac{1}{\left[1+(k_0\bar{\lambda})^2\right]^3}\left\{1-3\left(\frac{\bar{\lambda}}{a}\right)^2\frac{1-4(k_0\bar{\lambda})^2}{\left[1+(k_0\bar{\lambda})^2\right]^2}\right\}$$

where $\bar{\lambda} = \frac{\hbar}{mc}$ is the Compton wave-length of the electron. In this expression we observe a plane wave contribution ($a \to \infty$) and the first order wave packet width contribution. Finally, the unusual dependence on $R_{\perp M}^4$ is an effect due to the presence of the electric field $\mathbf{E}(\mathbf{r})$ which necessarily depends on $\mathbf{r}$.

By resuming, we have calculated the production probability of a pair e^- - e^+ in a strong rotating magnetic field. We have seen that the probability of creating both e^- and e^+ in those particular states with energies independent of the magnetic field and that we called transverse ground states is different from zero only if the direction of the magnetic field changes with time and that this probability is much larger than the other probabilities. This allows us to conclude that for pair production in very intense magnetic fields the change of the direction of $\mathbf{B}(t)$ is much more important than the variation of its strength.

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