

ON THE THEORY OF SCALAR PAIR PRODUCTION BY A POTENTIAL BARRIER

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The problem of the scalar pair production by a one-dimensional vector-potential $A_\mu(x_3)$ is reduced to the S-matrix formalism of the theory with an unstable vacuum. Our choice of in- and out-states does not coincide with that of other authors and we argue extensively in favor of our choice. We show that the norm of a solution of the wave equation is determined by one of the amplitude of its asymptotic form for $x_3 \rightarrow \pm\infty$. For the constant electric field we obtain the scalar particle propagator in terms of the stationary states and show that with our choice of in- and out-states it has the form dictated by the general theory.

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1. INTRODUCTION AND THE CHOICE OF IN- AND OUT-STATES

Pair production by an external field can be treated either in the framework of S-matrix formalism [1-5], or equivalently by the Feynman method using the propagators [1,6-8]. For the stationary potential the field is not switched off for $t \rightarrow \pm\infty$. So the reduction to the S-matrix formalism requires choosing the in- and out-states. How to do this is briefly shown in [1]. Another choice is made in [9] and accepted in later literature [10,11]. The correct choice is especially important in dealing with higher order processes, when the answer is not known in advance from some other considerations. In this paper we argue extensively in favor of our choice. It is reasonable to consider the case of scalar particle separately, because the complications due to the spin are absent here. Besides it is useful to have all the stages of a more simple case before eyes, when treating the spinor case.

We consider at first the one-dimensional potential $A^0(x_3)$ and assume for the beginning that the corresponding electrical field $E_3 = -\frac{\partial A_0}{\partial x_3}$ disappears for $x_3 \rightarrow \pm\infty$. We use the metric

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1). \quad (1)$$

It is useful to introduce the kinetic energy $\pi^0(x_3)$ and momentum $\pi_3(x_3)$ of a classical particle defined by the expressions

$$\begin{aligned} \pi^0(x_3) &= p^0 - eA^0(x_3), \quad \pi_3(x_3) = \sqrt{\pi_0^2(x_3) - m_1^2}, \\ m_1^2 &= m^2 + p_1^2 + p_2^2. \end{aligned} \quad (2)$$

The first relation in (2) merely expresses the total energy conservation. We also use the notation

$$\begin{aligned} \pi^0(x_3)|_{x_3 \rightarrow \pm\infty} &= \pi^0(\pm) \\ \pi_3(x_3)|_{x_3 \rightarrow \pm\infty} &= \pi_3(\pm) = \sqrt{\pi_0^2(\pm) - m_1^2}, \end{aligned} \quad (3)$$

In contrast to [1] and [5] we assume here that the charge of a scalar particle $e = -|e|$ in order the analogy with the electron would be closer. We are interested here mainly in the states that can be created by the field (Klein region). Assuming for definiteness $E_3 > 0$, we have in this region

$$\pi^0(-) > m_1^2, \quad \pi^0(+ < -m_1^2, \quad (4)$$

i.e. large positive x_3 are accessible only to antiparticles.

For brevity reasons we write only the wave function factor depending on x_3 . Outside the field the particle is free and we first classify the states by their asymptotic form

$$\begin{aligned} {}_+f_p|_{x_3 \rightarrow -\infty} &= [2\pi_3(-)]^{-1/2} \exp[\pm i\pi_3(-)x_3] \\ {}_+f_p|_{x_3 \rightarrow \infty} &= [2\pi_3(+)]^{-1/2} \exp[\pm i\pi_3(+)x_3] \end{aligned} \quad (5)$$

The normalization factors are chosen in that way that the density current along the third axis is equal to unity up to a sign. Two sets of functions in (5) are connected by the relations

$$\begin{aligned} {}_+f_p &= c'_{1p} {}_+f_p + c'_{2p} {}_-f_p, \\ {}_-f_p &= c'^*_{2p} {}_+f_p + c'^*_{1p} {}_-f_p. \end{aligned} \quad (6)$$

The second relation can be obtained from the first one by complex conjugation. The current conservation along the third axis gives (in Klein region)

$$\left|c'_{1p}\right|^2 - \left|c'_{2p}\right|^2 = 1. \quad (7)$$

From (6) and (7) we find the reversed relations

$$\begin{aligned}
{}^+f_p &= c_{1p}^* f_p - c_{2p} f_p, \\
{}^-f_p &= -c_{2p}^* f_p - c_{1p} f_p.
\end{aligned} \tag{8}$$

Now we have to classify solutions as in- and out-states. Our choice is [1]

$$\begin{aligned}
{}^-\psi_p &\equiv {}^-\psi_{pout} = {}^+f_p, & {}^+\psi_p &\equiv {}^+\psi_{pout} = {}^+f_p, \\
{}^-\psi_p &\equiv {}^-\psi_{pin} = -\frac{c_{2p}^*}{c_{2p}} {}^-\psi_p, & {}^-\tilde{\psi}_p &= -f_p, \\
{}^+\psi_p &\equiv {}^+\psi_{pout} = {}^-f_p.
\end{aligned} \tag{9}$$

Here the \pm indexes before ψ -functions indicate the sign of frequencies.

The heuristic argument in favor of this choice was based on the fact that the description of a scattering process in terms of pure states (unlimited vectors) is only a way to a more realistic description by means of wave packets. For the wave packets the field is effectively switched off, when they leave the field region. Consider, for example, the process described by ${}^+f_p$, see the first relation in (6). Initially we have the antiparticle current with amplitude c_{1p}^* moving from the region of large positive x_3 towards the field region near $x_3=0$. In the opposite direction from the region of large negative x_3 moves the particle current with unity amplitude. It annihilates completely in the field region. As a result we have the diminished antiparticle current reflected from the barrier. So for $t \rightarrow \infty$ there is only antiparticle packet, i.e. ${}^+f_p = {}^-\psi_p$. We would like to remind you here that the momentum of the negative-frequency wave function is opposite to the antiparticle velocity.

In terms of in- and out-states the relations (6) and (8) take the form

$$\begin{aligned}
{}^+\psi_p &= c_{1p} {}^+\psi_p + c_{2p} {}^-\psi_p, \\
{}^-\psi_p &= c_{2p}^* {}^+\psi_p + c_{1p}^* {}^-\psi_p;
\end{aligned} \tag{10}$$

$$\begin{aligned}
{}^+\psi_p &= c_{1p}^* {}^+\psi_p - c_{2p} {}^-\psi_p, \\
{}^-\psi_p &= -c_{2p}^* {}^+\psi_p + c_{1p} {}^-\psi_p;
\end{aligned} \tag{11}$$

$$\begin{aligned}
|c_{1p}|^2 - |c_{2p}|^2 &= 1, c_{1p} = -\frac{c_{1p}^*}{c_{2p}}, \\
c_{2p} &= \frac{1}{c_{2p}^*}.
\end{aligned} \tag{12}$$

We note that in [1] the present f -functions were denoted as ψ and it was explained that after using the transformation indicated in (9) and (12) we get relations (10-11) with (new) ψ -functions. The latter relations coincide with those for the non-stationary solutions. Just on the bases of these relations the S-matrix formalism is build [1,5].

In terms of ψ -functions the field operator Ψ has the usual form

$$\begin{aligned}
\Psi &= \sum_p (a_{pin} {}^+\psi_p + b_{pin}^\dagger {}^-\psi_p) = \\
&= \sum_p (a_{pout} {}^+\psi_p + b_{pout}^\dagger {}^-\psi_p)
\end{aligned} \tag{13}$$

Here a_{pin} is the destruction operator of a particle in the state ${}^+\psi_p$ and b_{pin}^\dagger is the creation operator of an antiparticle in the state ${}^-\psi_p$. The sum is over all p including p in the Klein region.

In paper [9] the field operator is written as

$$\begin{aligned}
\Psi &= \sum_k (a_{kin} p_{kin}(x) + b_{kin}^\dagger n_{kin}(x)) = \\
&= \sum_k (a_{kout} p_{kout}(x) + b_{kout}^\dagger n_{kout}(x)),
\end{aligned} \tag{14}$$

see (31a), (31b) in [9]. Disregarding here the normalization factors, the connections to our ψ -functions are

$$\begin{aligned}
p_{kin} &= {}^-\psi_k, n_{kin} = {}^+\psi_k, \\
p_{kout} &= -\psi_k, n_{kout} = {}^+\psi_k.
\end{aligned} \tag{15}$$

The authors of [9] name their p_{kin} -function the state of incoming particle on the grounds that there are no other waves for $x_3 \rightarrow \pm\infty$. In our nomenclature this is the outgoing negative frequency state as explained above. So the disagreement is in the sign of frequencies and in the in- and out- labeling. The authors of subsequent papers [10,11] accepted the classification of [9].

Another argument in favor of our choice (9) is as follows [1]. In the case of a constant electric field we can work either with stationary or non-stationary solutions. For the latter the classification of states is obvious. The experiment should be described in terms of wave packets and using stationary or non-stationary solutions should produce the same result. This agrees with the fact that in both cases the relations (10) and (11) have the same form with the same coefficients c_{1p} , c_{2p} . By the way it is shown in [1] why the strict S-matrix formulation is possible despite the fact that the constant field is not turned off for $t \rightarrow \pm\infty$. The reason is that for the given set of quantum numbers p the formation length for pair production is finite. Outside this length the field does not create pairs and does not prevent the S-matrix formulation as does any field that does not create pairs.

2. ORTHONORMALIZATION OF WAVE FUNCTIONS

The choice (9) assumes that ${}^+\psi$ and ${}^-\psi$ and also ${}^+\psi$ and ${}^-\psi$ are orthogonal. We shall show in this section that this is so. For definiteness we assume that (4) is satisfied. The Klein- Gordon equation has the form

$$\left[\frac{d^2}{dx_3^2} - 2eA^0 p^0 + e^2 A_0^2 + p_0^2 - m_1^2 \right] f_p(x_3) \times e^{i[p_1 x_1 + p_2 x_2 - p^0 t]} = 0 \quad (16)$$

The J^0 component of a transition current is given by the expression

$$J^0(\psi', \psi) = i[\psi'^* D_0 \psi - (D_0 \psi')^* \psi], \quad D_0 = \frac{\partial}{\partial t} - ieA_0. \quad (17)$$

For our potential A^0 and functions $f_{p'}$, f_p we have

$$J^0(f', f) = [p^0 + p'^0 - 2eA^0(x_3)] f_{p'}^* f_p = [\pi^0(x_3) + \pi'^0(x_3)] f_{p'}^* f_p. \quad (18)$$

We consider also the J_3 component of the transition current

$$J_3(f', f) = -if_{p'}^* \frac{\partial}{\partial x_3} f_p \equiv -i \left[f_{p'}^* \frac{\partial}{\partial x_3} f_p - f_p \frac{\partial}{\partial x_3} f_{p'}^* \right] \quad (19)$$

and calculate its derivatives over x_3 using (16)

$$\frac{d}{dx_3} J_3 = i(p^0 - p'^0) J^0. \quad (20)$$

From here we have

$$\int_{-L_d}^{L_u} dx_3 J^0(f', f) = \frac{i}{p'^0 - p^0} [J_3|_{x_3=L_u} - J_3|_{x_3=-L_d}]. \quad (21)$$

For $f' = {}_+ f_{p'}$, $f = {}_+ f_p$ we obtain (for details see [12])

$$\int dx_3 J^0({}_+ f_{p'}, {}_+ f_p) = -|c'_{2p}|^2 2\pi \delta(p'^0 - p^0). \quad (22)$$

So for ${}_+ f_p$ normalized as in (5) we have (22). One can verify that ${}_+ \psi$ and ${}_+ \psi$ are orthogonal.

3. SOLVABLE POTENTIAL

For the potential

$$A^0(x_3) = -\tanh kx_3 \quad (23)$$

the solutions of the Klein-Gordon equation are known [1]

$$\begin{aligned} {}_- f_p(x_3) &= \frac{1}{\sqrt{2|\pi_3(-)}} (-z)^{-i\mu} (1-z)^\lambda \times \\ &\times F(-i\mu - iv + \lambda, -i\mu + iv + \lambda; 2i\mu + 1; z) \\ -z &= e^{2kx_3}, \pi_3(-) = 2k\mu, \pi_3(+)= 2kv, \\ \pi^0(\pm) &= p^0 + ea, \\ \lambda &= 1/2 + \tilde{\lambda}, \tilde{\lambda} = \sqrt{\frac{1}{4} - \left(\frac{ea}{k}\right)^2}. \end{aligned} \quad (24)$$

Here $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function. $\pi_3(\pm)$ are real in the Klein region. Three other solutions with

quantum numbers p can be obtained from (24) by employing the discrete symmetry of the Klein-Gordon equation [13]. Thus ${}_+ f_p(x_3)$ can be obtained from ${}_+ f_p(x_3)$ by substitution $\mu \rightarrow -\mu$,

$${}_+ f_p(x_3) = \frac{1}{\sqrt{2|\pi_3(-)}} (-z)^{i\mu} (1-z)^\lambda \times F(i\mu - iv + \lambda, i\mu + iv + \lambda; 2i\mu + 1; z) \quad (25)$$

${}_+ f_p(x_3)$ (${}_+ f_p(x_3)$) can be obtained from ${}_+ f_p(x_3)$ (${}_+ f_p(x_3)$) by substitutions (not changing the Klein-Gordon equation (16)) $x_3 \rightarrow -x_3, a \rightarrow -a, \mu \rightarrow \nu, \pi^0(-) \leftrightarrow \pi^0(+)$:

$${}_+ f_p(x_3) = \frac{1}{\sqrt{2|\pi_3(+)}} (-z)^{i\mu} (1-z^{-1})^\lambda \times F(-i\mu - iv + \lambda, i\mu - iv + \lambda; -2i\mu + 1; z^{-1}) \quad (26)$$

$${}_+ f_p(x_3) = \frac{1}{\sqrt{2|\pi_3(+)}} (-z)^{-i\mu} (1-z^{-1})^\lambda \times F(-i\mu + iv + \lambda, i\mu + iv + \lambda; 2i\mu + 1; z^{-1}) \quad (27)$$

The coefficients c'_{1p} , c'_{2p} in (6) and (8) have the form

$$\begin{aligned} c'_{1p} &= \frac{\sqrt{\pi_3(+)} \Gamma(2i\mu + 1) \Gamma(2iv)}{\sqrt{\pi_3(-)} \Gamma(i\mu + iv + \lambda) \Gamma(i\mu + iv + 1 - \lambda)}, \\ c'_{1p} &= \frac{\sqrt{\pi_3(+)} \Gamma(2i\mu + 1) \Gamma(-2iv)}{\sqrt{\pi_3(-)} \Gamma(i\mu - iv + \lambda) \Gamma(i\mu - iv + 1 - \lambda)}. \end{aligned} \quad (28)$$

Two special cases are: the step potential ($k \rightarrow \infty$ in (23)) and constant electric field ($k \rightarrow 0, a \rightarrow \infty, ak = E = \text{Const}$).

4. CONSTANT ELECTRIC FIELD

In this Section we obtain the propagator for the scalar particle in a constant electric field and show that in terms of our in- and out-states it has the form dictated by the general theory. The vector-potential (23) reduces to

$$A_\mu = \delta_{\mu 0} E x_3, \quad A^0 = -A_0. \quad (29)$$

With this potential the Klein-Gordon equation (16) takes on the form

$$\left[\frac{d^2}{dZ^2} + \frac{Z^2}{4} - \frac{\lambda}{2} \right] f(x_3) = 0 \quad (30)$$

$$Z = \sqrt{2|eE|} \left(x_3 + \frac{p^0}{eE} \right) = -\sqrt{\frac{2}{|eE|}} \pi^0(x_3), \quad \lambda = \frac{m_1^2}{|eE|}. \quad (31)$$

The solutions, normalized on unity current, are, see Eq. (8.2.5) in [14],

$$\begin{aligned} {}_- \psi_{p^+} &= {}_+ f_p = c_p D_\nu (-e^{i\pi/4} Z), \\ {}_- \psi_{p^-} &= {}_- f_p = c_p D_\nu^* (-e^{-i\pi/4} Z), \end{aligned}$$

$$\begin{aligned}
+\psi_{p^-} = f_p = c_p D_v (e^{i\pi/4} Z), \\
+\psi_{p^-} = f_p = c_p D_v^* (e^{-i\pi/4} Z), \\
v = \frac{i\lambda}{2} - \frac{1}{2}, c = (2|eE|)^{-1/4} \exp\left(\frac{\pi\lambda}{8}\right), p = p_1, p_2, p^0. \quad (32)
\end{aligned}$$

Now in the relations (6) and (8)

$$c'_{1p} = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1-i\lambda}{2}\right)} \exp\left[\frac{\pi}{4}(\lambda-i)\right], c'_{2p} = i \exp\left[\frac{\pi}{2}\lambda\right], \quad (34)$$

and in relations (10) and (11)

$$\begin{aligned}
c_{1p} = -\frac{c'_{1p}}{c'_{2p}} = -\frac{\sqrt{2\pi}}{\Gamma\left(\frac{1-i\lambda}{2}\right)} \exp\left[-\frac{\pi}{4}(\lambda-i)\right], \\
c_{2p} = \frac{1}{c'_{2p}} = -i \exp\left[-\frac{\pi}{2}\lambda\right]. \quad (35)
\end{aligned}$$

We note that $-\psi$ and $-\tilde{\psi}$ in (9) coincide in this case. According to (22) to normalize ψ_p on $\pm 2\pi\delta(p^0 - p'^0)$ we have to replace c_p in (32) and (33) by

$$\frac{c_p}{|c'_{2p}|} = (2|eE|)^{-1/4} \exp\left[-\frac{3\pi\lambda}{8}\right]. \quad (36)$$

Thus we may assume that ψ -functions in (10) and (11) are normalized in this way. The same relations hold for non-stationary states. The important thing is that the relations (10) and (11) constitute all the necessary ingredients for S -matrix theory [1,5].

We note now that the relation (18) takes on the form

$$\begin{aligned}
J^0(f_{p'}, f_p) = -\sqrt{\frac{|eE|}{2}} (Z + Z') f_{p'}^* f_p, \\
Z' = \sqrt{2|eE|} \left(x_3 + \frac{p^0}{eE}\right), \quad e = -|e|. \quad (37)
\end{aligned}$$

On the other hand, using (30) we find

$$\frac{d}{dZ} \left[f_{p'}^* \frac{d}{dZ} f_p \right] = -\frac{1}{\sqrt{8|eE|}} (p'^0 - p^0) (Z + Z') f_{p'}^* f_p. \quad (38)$$

Now it easy to verify that relations (19), (21) and (22) remain valid as well as the orthogonality of $+\psi$ and $-\psi$ and also of $+\tilde{\psi}$ and $-\tilde{\psi}$.

Solutions (32), in which the factor depending on x_1, x_2, t is dropped for brevity, are characterized by the quantum number p^0 (and also p_1 and p_2). If instead of vector-potential (29) we use in the Klein-Gordon equation the vector-potential

$$\begin{aligned}
A'_\mu = -\delta_{\mu 3} E x^- = A_\mu - \frac{\partial}{\partial x^\mu} \eta, \quad x^\pm = t \pm x_3, \\
\eta = E(t x_3 - x_3^2 / 2), \quad (39)
\end{aligned}$$

we obtain the solutions ψ'_{p^-} characterized by the quantum number p^- see [15,16]. Of course, we can go back to the potential (29) and obtain

$$\psi'_{p^-}(x|A) \equiv \psi'_{p^-} = e^{ie\eta} \psi'_{p^-}, \quad \psi'_{p^-} = \psi'_{p^-}(x|A'). \quad (40)$$

Making modifications due to the present assumption $e=-|e|$, we have from the results in [15,16]

$$\begin{aligned}
+\psi'_{p^-}(x) = \frac{1}{(4|eE|)^{1/4}} \exp\left[-\frac{i}{2} p^- x^+ + \frac{i}{4} eE(x^-)^2 + v^* \ln z\right] \\
z = \frac{\pi^-}{\sqrt{|eE|}}, \quad \pi^- = p^- - eE x^-. \quad (41)
\end{aligned}$$

The factor depending on x_1, x_2 is dropped for brevity. So this is the positive-frequency out-state. To corroborate this we may add to the arguments in [15,16] the following physical justification. The classical particle with the negative charge starts from the region with large negative x_3 , is slowed down and is reflected back to where it comes. Its kinetic momentum π_3 is negative and grows in magnitude for $t \rightarrow \infty$. Hence

$$\pi^- = (\pi^0 - \pi_3)|_{t \rightarrow \infty} \rightarrow \infty. \quad (42)$$

Going back to the quantum state (41) we note that the particles (antiparticles) are in the region where $\pi^- > 0$ ($\pi^- < 0$). Large π^- indicates that we are far from the region where pairs are created ($\pi^- \approx 0$). In the region where $\pi^- < 0$ the wave function (41) must be small for small probability of pair production. Thus in this region

$$z = e^{-i\pi}(-z), \quad z < 0. \quad (43)$$

Similarly,

$$\begin{aligned}
-\psi'_{p^-}(x) = \frac{1}{(4|eE|)^{1/4}} \times \\
\times \exp\left[-\frac{i}{2} p^- x^+ + \frac{i}{4} eE(x^-)^2 + v^* \ln(-z)\right] \quad (44)
\end{aligned}$$

For positive z in (44) we have

$$-z = e^{-i\pi} z, \quad z > 0. \quad (45)$$

The complete set of ψ'_{p^-} -solutions must satisfy the relations (10) and (11). This gives

$$\begin{aligned}
+\psi'_{p^-} = \theta(\pi^-) c_{1p} + \psi'_{p^-}, \quad -\psi'_{p^-} = \theta(-\pi^-) c_{1p} - \psi'_{p^-}, \\
\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases} \quad (46)
\end{aligned}$$

The proper time representation of the scalar particle propagator for the vector-potential (39) is [17,1]

$$\begin{aligned}
G(x', x|A') = \frac{eE}{(4\pi)^2} \exp\left[-i\frac{eE}{2} y_3(x'^+ + x^-)\right] \int_0^\infty \frac{ds}{s \sinh(eEs)} \\
\times \exp\left[-ism^2 + i\frac{y_1^2 + y_2^2}{4s} - \frac{i}{4}(y_0^2 - y_3^2)eE \coth(eEs)\right], \\
y = x' - x. \quad (47)
\end{aligned}$$

Now we multiply (47) by

$$\exp\left\{-i\left[p_1y_1 + p_2y_2 - \frac{1}{2}p^-y^+\right]\right\}$$

and integrate over y_1, y_2 and y^+ , see (54). For the integral over y^+ we have

$$\int_{-\infty}^{\infty} dy^+ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{i}{2} p^- y^+ - \frac{i}{4} eE y^+ (x^- + x'^-) - \frac{i}{4} y^- y^+ eE \right] \delta\left[\frac{p^-}{2} - \frac{eE}{4} (x^- + x'^-) - \frac{y^-}{4} eE \right] dy^+ = 2\pi\delta\left[\frac{p^-}{2} - \frac{eE}{4} (x^- + x'^-) - \frac{y^-}{4} eE \right]. \quad (48)$$

In the expressions (47) and (48) the charge e can have any sign. For $e=-|e|$ we obtain for the right hand side of (48)

$$\frac{8\pi}{|eE y^-|} \delta(\coth\tau - R) = \frac{8\pi}{|eE y^-|} \sinh^2\tau \delta(\tau - \tau_0),$$

$$R = \frac{\pi'^- + \pi^-}{\pi'^- - \pi^-}, \quad (49)$$

$$\tau = |eE|s, \tau_0 = \frac{1}{2} \ln \frac{R+1}{R-1} = \frac{1}{2} \ln \frac{\pi'^-}{\pi^-},$$

$$\pi^- = p^- + |eE|x^-, \pi'^- = p^- + |eE|x'^-,$$

$$y^- = x'^- - x^-. \quad (50)$$

We note that the reversal of sign of π^- and π' does not change R . In order to have the nonzero δ -function argument, R must lie in the interval $1 < R < \infty$ because $s > 0$ in (47). This is possible only in two cases

$$0 < \pi^- < \pi', \text{ i.e. } x^- < x'^-; \quad (51)$$

$$0 > \pi^- > \pi', \text{ i.e. } x^- > x'^-. \quad (52)$$

Taking into account that

$$\int_{-\infty}^{\infty} dy_1 \exp\left[\frac{i}{4s} y_1^2 - ip_1 y_1\right] = 2\sqrt{\pi s} \exp\left[\frac{i\pi}{4} - isp_1^2\right] \quad (53)$$

and similarly for the integral over y_2 , we get for the case (51)

$$\int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} dy^+ G(x', x|A) e^{-\left[p_1 y_1 + p_2 y_2 - \frac{1}{2} p^- y^+\right]} = i(\pi^- \pi'^-)^{\frac{1}{2}} \exp\left\{-\frac{i\lambda}{2} \ln \frac{\pi'^-}{\pi^-} + \frac{i}{4} eE \left[(x'^-)^2 - (x^-)^2\right]\right\}. \quad (54)$$

The relation $e^{\tau_0} = \sqrt{\frac{\pi'^-}{\pi^-}}$ is used here, see (50). We

note also that according to (41) and (44) both ${}^+\psi'_{p^-}(x') {}^+\psi'_{p^-}(x)$ under condition (51) and ${}^-\psi'_{p^-}(x') {}^-\psi'_{p^-}(x)$ under condition (52) can be written as

$$\frac{1}{2\sqrt{\pi} \pi'^-} \exp\left\{-\frac{ip^-}{2} (x'^- - x^-) + \frac{i}{4} eE [(x'^-)^2 - (x^-)^2] - \frac{i\lambda}{2} \ln \frac{\pi'^-}{\pi^-}\right\}. \quad (55)$$

Making the inverse Fourier transform of (54) i.e. multiplying it by

$$2^{-1} (2\pi)^{-3} \exp\left[i(p_1 y_1 + p_2 y_2 - \frac{1}{2} p^- y^+)\right]$$

and integrating over p_1, p_2 and p^- we get

$$G(x', x|A) = i \int \frac{dp_1 dp_2 dp^-}{(2\pi)^3} \begin{cases} \theta(\pi^-) \theta(\pi'^-) {}^+\psi'_{p^-}(x') {}^+\psi'_{p^-}(x), & x'^- > x^-, \\ \theta(-\pi^-) \theta(-\pi'^-) {}^-\psi'_{p^-}(x') {}^-\psi'_{p^-}(x), & x'^- < x^- \end{cases} \quad (56)$$

Here θ -functions take care of the conditions (51) and (52). Besides we have

$$\theta(\pi^-) \theta(\pi'^-) = \theta(\pi^-), \quad x'^- > x^-,$$

$$\theta(-\pi^-) \theta(-\pi'^-) = \theta(-\pi^-), \quad x'^- < x^-. \quad (57)$$

Taking into account (46) we obtain from (56) and (57) the sought for representation, which for the potential (29) has the form

$$G(x', x|A) = i \int \frac{dp_1 dp_2 dp^-}{(2\pi)^3 c_{1p}^*} \begin{cases} {}^+\psi_{p^-}(x') {}^+\psi_{p^-}(x), & x'^- > x^-, \\ {}^-\psi_{p^-}(x') {}^-\psi_{p^-}(x), & x'^- < x^- \end{cases} \quad (58)$$

One can verify [16] that the functions defined by the upper and lower lines on the right hand side of (56) (and (58)) coincide outside the light cones

$$y^+ y^- = y_0^2 - y_3^2 < 0. \quad (59)$$

According to (40) and (41) we have

$${}^+\psi_{p^-}(x) = \frac{1}{(4|eE|)^{1/4}} \times \exp\left[-\frac{i}{2} p^- x^+ + ieE\left(\frac{t^2}{2} - \frac{(x^-)^2}{4}\right) + v^* \ln z\right], \quad (60)$$

${}^-\psi_{p^-}$ is obtained from (60) by substitution $z \rightarrow -z$ under the logarithm sign.

Now we note that the classification of ψ_{p^-} - functions was obtained in [15,16] from the obvious classification of the non-stationary solutions ψ_{p_3} by an integral transformation

$$\psi_{p^-}(x|A) = \int_{-\infty}^{\infty} dp_3 M^*(p_3, p^-) \varphi_{p_3}(x|A), \quad (61)$$

$$M^*(p_3, p^-) = (2\pi |eE|)^{-1/2} \exp\left\{i \frac{(p^-)^2 + 4p^- p_3 + 2p_3^2}{4eE}\right\}. \quad (62)$$

Inserting ${}^+\varphi_{p_3}(x|A)$ in the expression on the right hand side of (61), we get ${}^+\psi_{p^-}(x|A)$ on the left hand side and so on. The reversed relations of (61) can be considered as the definitions of φ_{p_3} . For the vector potential $\vec{A}_\mu = -\delta_{\mu 3} Et$, considered in [15,16], we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} dp^- M(p_3, p^-)^+ \psi_{p^-}(x|\tilde{A}) = {}^+ \varphi_{p_3}(x|\tilde{A}) = \\
& = B_p e^{ip_3 x_3} D_{V^*} (e^{i\pi/4} T) \\
& \int_{-\infty}^{\infty} dp^- M(p_3, p^-)_- \psi_{p^-}(x|\tilde{A}) = {}_- \varphi_{p_3}(x|\tilde{A}) = \\
& = B_p e^{ip_3 x_3} D_{V^*} (-e^{i\pi/4} T) \\
& B_p = (2|eE|)^{-1/4} \exp\left[-\frac{\pi\lambda}{8} - \frac{i\lambda}{4} \ln 2 - \frac{i3\pi}{4}\right], \\
& T = \sqrt{2|eE|} \left(t - \frac{p_3}{|eE|}\right). \tag{63}
\end{aligned}$$

By analogy we expect that

$$\begin{aligned}
\varphi_{p^0} &= \int_{-\infty}^{\infty} dp^- K(p^0, p^-) \psi_{p^-} \tag{64} \\
K(p^0, p^-) &= \\
&= (2\pi |eE|)^{-1/2} \exp\left[i \frac{(p^-)^2 + 4p^- p^0 + 2p_0^2}{4eE}\right] \tag{65}
\end{aligned}$$

where φ_{p^0} differs from $\psi_{p^0} = \psi_{p^0}$ in (32) only by an inessential phase factor, see (69). To see that this is true we use first the relation

$$\begin{aligned}
& i \frac{(p^-)^2 - 4p^- p^0 + 2p_0^2}{4eE} + i \left[-\frac{1}{2} p^- x^+ + eE \frac{t^2}{2} - eE \frac{(x^-)^2}{4} \right] = \\
& -i \frac{z^2}{4} + iz\zeta - i \frac{\zeta^2}{2} - ip^0 t \tag{66}
\end{aligned}$$

where

$$\zeta = -\sqrt{|eE|} \left(x_3 + \frac{p^0}{eE}\right), \quad z = \frac{\pi^-}{\sqrt{|eE|}}, \quad e = -|e|, \tag{67}$$

then formula 3.462(1) in [18], the prescriptions (43) and (45) and the relations between the parabolic cylinder functions, see 8.2(6)-(8) in [14]. Then we find

$$\begin{aligned}
{}^+ \varphi_{p^0} &= \int_{-\infty}^{\infty} dp^- K(p^0, p^-)^+ \psi_{p^-} = \\
& (2|eE|)^{-1/4} \exp\left[\frac{i\pi}{8} - \frac{i\lambda}{4} \ln 2 - \frac{3\pi\lambda}{8} - ip^0 t\right] D_{V^*} (e^{-i\pi/4} Z) \tag{68}
\end{aligned}$$

$$\begin{aligned}
{}_- \varphi_{p^0} &= \int_{-\infty}^{\infty} dp^- K(p^0, p^-)_- \psi_{p^-} = \\
& = (2|eE|)^{-1/4} \exp\left[\frac{i\pi}{8} - \frac{i\lambda}{4} \ln 2 - \frac{3\pi\lambda}{8} - ip^0 t\right] D_{V^*} (-e^{-i\pi/4} Z) \tag{69}
\end{aligned}$$

and similar expressions for ${}^+ \varphi_{p^0}$ and ${}_- \varphi_{p^0}$. Here Z is the same as in (31).

Comparing these functions with the ones in (32) together with (36), we see that φ_{p^0} coincide with ψ_{p^0}

up to an inessential phase factor. We could get rid of this factor by modifying $K(p^0, p^-)$.

We note here that

$$\int_{-\infty}^{\infty} dp^- K^*(p^0, p^-) K(p^0, p^-) = \delta(p^0 - p'^0) \tag{70}$$

and

$$\int_{-\infty}^{\infty} dp^0 K^*(p^0, p'^-) K(p^0, p^-) = \delta(p^- - p'^-), \tag{71}$$

So the relation (64) can be reversed

$$\psi_{p^-} = \int_{-\infty}^{\infty} dp^0 K^*(p^0, p^-) \varphi_{p^0} \tag{72}$$

Using this formula we rewrite the integrals over p^- in (58) as follows

$$\int_{-\infty}^{\infty} dp^- {}^+ \psi_{p^-}(x') {}^+ \psi_{p^-}^*(x) = \int_{-\infty}^{\infty} dp^0 {}^+ \psi_{p^0}(x') {}^+ \psi_{p^0}^*(x), \quad t' > t \tag{73}$$

$$\int_{-\infty}^{\infty} dp^- {}_- \psi_{p^-}(x') {}_- \psi_{p^-}^*(x) = \int_{-\infty}^{\infty} dp^0 {}_- \psi_{p^0}(x') {}_- \psi_{p^0}^*(x), \quad t' < t.$$

The conditions on t and t' are written on account of the remark after (58). Thus

$$\begin{aligned}
G(x', x | A) &= i \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \int_{-\infty}^{\infty} dp^0 \frac{1}{(2\pi)^3 c_{1p}^*} \times \\
& \times \begin{cases} {}^+ \psi_{p^0}(x') {}^+ \psi_{p^0}^*(x), & t' > t, \\ {}_- \psi_{p^0}(x') {}_- \psi_{p^0}^*(x), & t' < t \end{cases} \tag{74}
\end{aligned}$$

By the way it is clear from (74) and (40) that

$$G(x', x | A') = e^{ie(\eta(x) - \eta(x'))} G(x', x | A). \tag{75}$$

The expression (74) has the form dictated by the general theory [1,19].

Finally we note that it follows from (61) and (64) that

$$\varphi_{p^0} = \int_{-\infty}^{\infty} dp_3 N(p^0, p_3) \varphi_{p_3} \tag{76}$$

where

$$\begin{aligned}
N(p^0, p_3) &= \int_{-\infty}^{\infty} dp^- K(p^0, p^-) M^*(p_3, p^-) = \\
& = (2\pi |eE|)^{-1/2} \exp\left[-\frac{i\pi}{4} - \frac{ip^0 p_3}{|eE|}\right] \tag{77}
\end{aligned}$$

The relation (76) can be checked with the help of formula 2.11.4(7) in [20] which can be adjusted as follows

$$\int_{-\infty}^{\infty} dx \exp\left[\frac{1}{2}c^2xy\right] D_{\kappa}(cx) = \frac{2\sqrt{\pi}}{c} (-1)^{\kappa/2} D_{\kappa}\left(\frac{cy}{\sqrt{-1}}\right),$$

$$|\text{phase}|c < \frac{3\pi}{4}. \quad (78)$$

Here $\sqrt{-1} = i$ ($\sqrt{-1} = -i$) for $\kappa = \nu$ ($\kappa = \nu^*$). So we insert ${}^{\dagger}\varphi_{p_3}(x|A)$ into the right hand side of (76) and take into account that

$${}^{\dagger}\varphi_{p_3}(x|A) = e^{-i|eE|tx_3} {}^{\dagger}\varphi_{p_3}(x|\tilde{A}),$$

$$-i \frac{p_3^0}{|eE|} + p_3 x_3 = -\frac{i}{2} ZT - ip^0 t + i|eE|tx_3, \quad (79)$$

Then we get

$${}^{\dagger}\varphi_{p^0}(x|A) = \int_{-\infty}^{\infty} dp_3 N(p^0, p_3) {}^{\dagger}\varphi_{p_3}(x|A) =$$

$$= e^{\left[\frac{i\pi}{8} - \frac{\pi\lambda}{8} - \frac{i\lambda}{4} \ln 2 - ip^0 t\right]} \int_{-\infty}^{\infty} dT \exp\left[-\frac{i}{2} ZT\right] D_{\nu^*}(e^{i\pi/4} T). \quad (80)$$

Now using (78) we obtain the right hand side of (68). For $\varphi_{p_3}(x|A)$ we proceed similarly, but use the substitution $T = -x$ instead of $T = x$, when employing (78).

It is shown in [15,16] that for example ${}^{\dagger}\psi_{p^-}$ can be obtained from ${}^{\dagger}\psi_{p_3}$ by changing continuously the gauge of the electric field potential. The same is true for ${}^{\dagger}\psi_{p^-}$ and ${}^{\dagger}\psi_{p^0}$. For this reason these functions are indistinguishable and only the wave packets are observable.

It is clear that $N(p^0, p_3)$ and $M(p_3, p^-)$ have the orthogonality properties of $K(p^0, p^-)$, see (70) and (71).

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