# ABOUT THE NEW QUANTUM MECHANICS

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In our previous papers it is demonstrated that in high energy region (at very small distances) there is a proper quantum theory principally distinguished from the Heisenberg-Schroedinger one (which is well adopted to the description of moderate energy region). It means that usual quantum theory is not complete (not closed) and may be completed. Here we formulate the completeness procedure for Schroedinger wave mechanics (it is to some extent a mathematical problem). Hereby naturally to use the theory of commutative functional rings in which new wave theory is already in ready form. Our approach to the closeness problem permits by transparent way to come to light non-uniqueness of the present quantum theory and to get the unique completeness procedure for Heisenberg picture.

PACS: 13.75Gx

In spite of equivalence between the Heisenberg picture of quantum mechanics (algebra of observables) and Schroedinger one (space of wave functions), see [1], procedures of their completeness are quite different. Enough to notice that if to begin from operator formalism of quantum mechanics so it will be seen that completeness procedure is rather non-unique (see for example [2]). May be therefore this approach did not still lead to any successes. Here we choose another approach to the problem proceeding from the wave variables of quantum mechanics. We show that in this approach closeness procedure is unique and leads to the unique extension of operator formalism of quantum mechanics. So, from the point of view of closeness problem Heisenberg and Schroedinger pictures are not equivalent: Heisenberg formulation is more closed than Schroedinger one and the reason of this circumstance is the possibility to eliminate wave variables from Heisenberg picture at all introducing instead of state vectors $|\psi\rangle$ corresponding projection operators $P_\psi = |\psi\rangle\langle\psi|$ [1,3] [1]. Then the main formula of quantum mechanics is written in the form $\overline{A}_\psi = SpAP_\psi$, where $A$ is an observable and $\overline{A}_\psi$ is its mean value in the state $|\psi\rangle$.

In [4] we have studied the topological structure of momentum and configuration spaces of new quantum theory well adopted for description of physical phenomena at very small distances [2]. Here we continue our investigation. We begin from the consideration of the well known class of wave functions used in the usual quantum (wave) mechanics and show that the class may be extended by natural way.

1. First of all we would like to notice that even in the case of non-relativistic quantum mechanics (where one does not take into account the connection between spin and statistics) field variables (wave functions) $\psi(X)$ are used not only as elements of any linear space $L$ but also as elements of any commutative ring (algebra) $A$. In fact at description of field interactions products of the type $\psi(X)\varphi(X)$ where $X \in M_X$ ($M_X$ is configuration space) and integrals $\int_{M_X}\psi(X)\varphi(X)dX$ ($dX$ is the Lebesgue measure on $M_X$) are used.

In momentum picture connected with configuration one by the Fourier transformation $\psi(X) = \int_{M_p}\psi(p)e^{ipX}dp$ ($p \in M_p$ is momentum space, $dp$ is the Lebesgue measure on $M_p$; one says that $M_p$ is dual to the $M_X$ in the sense of Fourier-transform) contractions $\psi * \varphi = \int_{M_p}\psi(p)\varphi(q-p)dp$ as else one multiplication law and integrals $\int_{M_p}\psi(p)\varphi(-p)dp$ appear. In connection with the latter we have to consider also usual multiplication of functions on $M_p$ $\psi(p)\varphi(p)$.

If $M_p$ is locally compact but non-compact manifold with Lebesgue measure $\mu_L(M_p) = \infty$ (just this case takes place in particle physics and usual quantum mechanics where a certain equilibrium between $M_X$

[1] As is well known projection operator $P_\psi$ impossible to decompose into a *sum* $\alpha P_1 + \beta P_2$ $(\alpha + \beta = 1)$ of others projection operators $P_1, P_2$ (von Neumann theorem [1]). Von Neumann considered this fact to be impossibility of introducing of hidden parameters in the framework of quantum mechanics. But as it turns out $P$ may be decomposed into (direct) *product* $QQ^+$ of two other operators $Q, Q^+$ behind which hidden parameters stand (see further).

[2] Necessity in this investigation is conditioned by the goal to construct the mathematically consequent quantum theory of particle interaction in high energy region free from ultraviolet divergences. It turned out that the simple combination of special relativity (theory of high velocities) with principles of Heisenberg-Schroedinger quantum mechanics (theory of operators in *separable* Hilbert space) is insufficient to solve this problem. The main obstacle is an inconsistency between differential topology (a priori induced in the region of small distances) and the nature of small distances, see [4].

and $M_p$ is established by Fourier-transform with Lebesgue measure) ring $A$ may be topologized by several way. Usually one limits oneself by consideration of class of measurable absolutely integrable functions $\psi(p)$ satisfying the condition $\lim_{|p|\to\infty}\psi(p)=0$. It forms normed (Banach) ring $L_1$, endowed by the norm $\|\psi\|_1=\int|\psi(p)|dp$ obeying obviously the condition $\|\psi*\varphi\|_1\le\|\psi\|_1\|\varphi\|_1$. This norm is connected with measure on $M_p$ that is very important for physics. So the ring $A$ (in $X$-picture) with usual multiplication and the ring $L_1$ (in $p$-picture) with contraction are considered to be *isomorphic* [5].

It is remarkable that $L_1(\sim A)$ is the ring without unity $e$: it does not contain unity function $\psi(p)=1$ [3] and therefore may be (in spite of its topological closedness) extended with conserving multiplication law $*$ and usual multiplication (but not measure on $M_p$ and $M_X$, see further). Wider ring we denote $A'$.

2. If the set of maximal regular ideals of ring $A$ (under usual multiplication) to denote $\mathfrak{M}$ (maximal ideal $I_p\subset A$ is collection of functions having zero in the point $p\in M_p$) so it is clear that $\mathfrak{M}$ is isomorphic to the manifold $M_p$ [5].

Ring $A$ may be enclosed into ring $A'$ in the form of maximal ideal $I_\infty=A\subset A'$. All set of maximal ideals of $A'$ is denoted $\mathfrak{M}$'.

According to the abstract theory of commutative rings [5] manifold $\mathfrak{M}$ may be obtained from $\mathfrak{M}$' by means of pricking out the only one ideal $I_\infty=A$ consisting of functions obeying the condition $\psi(\infty)=0$. In its turn manifold $\mathfrak{M}$' may be obtained from $\mathfrak{M}\sim M_p$ by means of closeness or compactification of the latter, i.e. $\mathfrak{M}'\sim\overline{M}_p$. Unlike $\mathfrak{M}$ manifold $\mathfrak{M}$' is closed and hence compact [5]. So, $A$ is connected with $\mathfrak{M}$, and $A'$ is connected with $\mathfrak{M}$'.

Importance of $A'$ is in the following: as usually it contains additional (hidden from point of view of $A$) elements (parameters) which correspond to the more fundamental (in definite sense) objects than elementary particles standing behind the ring $A$. *Ring extension $A\subset A'$ is the base of our approach to the completeness problem in wave mechanics.*

3. Ring $A'$ may be made topological one. Usually it is identified with normed (Banach) ring of limited functions $L_\infty$ topology of which is given rise by the metric $\rho(\psi,\chi)=\|\psi-\chi\|_\infty$ where $\|\psi\|_\infty=\max_{p\in M_p}|\psi(p)|$ is the norm obeying the condition $\|\psi\varphi\|_\infty\le\|\psi\|_\infty\|\varphi\|_\infty$. With such a topology $L_\infty$ is *strongly non-separable* ring [5].

It is remarkable that norm $\|\bullet\|_\infty$ generally speaking by no means is connected with measure on $M_p$ (although in [6] this norm is considered to be somehow depending on a measure on $M_p$). It is the reason why ring $L_\infty$ is not used in quantum mechanics where measurement process on $M_p$ plays very important role. In quantum mechanics $L_\infty(\sim A')$ is considered to be a dual (conjugate) space to the space $L_1$ of measurable (under the Lebesgue measure on $M_p$) functions, i.e. as a space of linear functionals over space $L_1$. Hereby $L_\infty$ is endowed by weak topology in which it is *weakly separable* ring. So that in context of quantum mechanics $L_\infty$ (as functionals) is not identical with $A'$. Of course further we will have to remove this *discrepancy*.

4. Further for simplicity only one-dimensional case is considered when $M_p\sim R_p=]-\infty,\infty[$ is the real number axis. In this case there are three different ways of compactification of $R$ (see [5,6]). They are:

i) extended number axis $\overline{R}=[-\infty,\infty]$,

ii) Alexandrov compactification $\tilde{R}=R\cup\infty\sim S^1$,

iii) H. Bohr compactification $bR$ (in this case the infinite far point of $R$ is not added to the $bR$).

As is known in the case ii) completely (in i) partially) there lose linear structure and linear ordering of momentum space $R_p$. Dual space to the $\tilde{R}_p\sim S^1$ is the lattice $Z$ [7]. We do not consider this case because configuration space $Z$ does not concern the particle physics [4].

As is known usual Fourier-transform (with Lebesgue measure on $M_p$) is not feeling to the distinction between $R_p$ and $\overline{R}_p$. Dual space to the $R_p$ and $\overline{R}_p$ is one and the same $R_X$. Classes of functions on $R_p$ and $\overline{R}_p$ are the same too. Hereby we may consider highly wide spectrum of normed space $L_q$ $(1\le q\le\infty)$ considered as a space of linear functionals over the normed space $L_p$ with $p=\dfrac{q}{q-1}$ and also locally convex (in particular countable normed) topological vector spaces.

Quantum theory connected with $\overline{R}_p$ is quite equivalent (from pure physical point of view) to the Heisenberg-Schroedinger quantum mechanics on $R_p$ which uses usual Lebesgue measure on $R_p$ and separable space of functions on it. However on pure

[3] At first sight it seems that under the contraction the Dirac's $\delta$ -function might be unity in $L_1$. But being a function set on manifold of zero Lebesgue measure it has norm $\|\delta\|_1=\int\delta(p)dp=0\ne1=\|e\|_1$ that was emphasized by von Neumann [1].

[4] In fact increasing radius of circle $S^1$ (i.e. decreasing step of lattice $Z$) we will never come to the non-countable set of points (continuum) which consists in real configuration space $R_X$ but will always get only some countable (may be dense in $R_X$) set $D$ of points, see [4].

physical reasoning (exclusive importance of scalar product) quantum theory is usually connected with Hilbert space $L_2$ [1]. Further generalization connected with rigged Hilbert spaces including distributions of the type Dirac's $\delta$ -function as a matter of fact did nothing change but only broke the ring structure not leading to the solution of ultraviolet problem. It has to pay attention (this von Neumann emphasized [1]) that such a generalization is not desirable and not quite correct from point of view of theory of functions [5].

Especially we have to pay attention to the following circumstance: although $\overline{R}_p$ is closed (compact) its Lebesgue measure $\mu_L(\overline{R}_p) = \mu_L(R_p) = \infty$ . Therefore for unity function $\psi(p) = 1 \in A'$ contraction is $\int_{\overline{R}_p} dp = \infty \ (= 2\pi \int \delta^2(X)dX = 2\pi\delta(0))$. Namely with this infinity $\delta(0)$ ultraviolet catastrophe in usual field theory is connected. The point is in the infinity of Lebesgue measure of $\overline{R}_p$. That compact manifold $\overline{R}_p$ has infinite measure we consider to be a discrepancy or conflict between topology and measure [6]. We recall that one and the same reserve of functions on $R$ may be topologied by different ways [1].

5. Thus the main contradiction in approach i) consists in the following: compact manifold $\overline{R}_p$ has infinite (Lebesgue) measure. To decline this contradiction we can by means of changing the measure on $R_p$. It turns out the compactification of $R_p$ by means of iii) is unique remedy to solve completely this problem.

In [8] H. Bohr demonstrated in fact that the ring $L_\infty$ with norm $\|\bullet\|_\infty$ may be enclosed into new topological ring connected with a new measure on $R_p$ that is very important for physical theory. In Bohr's theory $A'$ is identified with non-separable Hilbert space $L_2'$ of almost periodical functions on $R_p$ endowed by the scalar product

$$(\psi,\varphi)' = \lim_{P\to\infty} \frac{1}{2P} \int_{-P}^{P} \overline{\psi}(p)\varphi(p)dp$$

and norm $\|\psi\|' = \sqrt{(\psi,\psi)'}$. Hereby norm $\|\bullet\|_\infty$ majorates the Bohr-norm $\|\bullet\|'$. In fact $\|\psi\|' = \sqrt{\lim_{P\to\infty} \frac{1}{2P} \int_{-P}^{P} |\psi(p)|^2 dp} \le \sqrt{\max_{p\in R_p} |\psi(p)|^2 \lim_{P\to\infty} \frac{1}{2P} \int_{-P}^{P} dp}$ $= \max_{p\in R_p} |\psi(p)| = \|\psi\|_\infty$ .

Hence $L_\infty \subset L_2'$. In Bohr's theory contraction is defined by the formula $\psi * \varphi = \lim_{P\to\infty} \frac{1}{2P} \int_{-P}^{P} \psi(p)\varphi(q-p)dp$. $R_p$, endowed by Bohr-measure we label $bR_p$. Obviously $\mu_B(R_p) = 1$. Hereby the ring $A'$ stays non-separable because $bR_p$ is non-Hausdorf space [4].

Factorizing $A'$ over ideal $A$ (carriers of functions from $A$ are subsets in $R_p$ of zero Bohr measure) we get the ring of Bohr's almost periodical functions on $R_p$ denoted $L_2'$: $A'/A = L_2'$. Properties of this ring are studied in details by H. Bohr [8].

Concretely we can say now that importance of our ring $A'$ is in the following: *extended wave mechanics is connected with it.* Not only particles (subring $A$ with Lebesgue measure) but particle constituents - granules (factor ring $L_2'$ with the Bohr measure) - are described by $A'$ , see [4]. We emphasize that non-separable ring $A'$ is not isomorphic to the weak separable ring $L_\infty$ used in the usual quantum theory (although they have one and the same reserve of functions, see above).

Dual space to the $bR_p$ (in Pontrjagin sense) is completely non-connected configuration space, i.e. discontinuum $R_X'$ ($R_X'$ like $R_X$ consists of non-countable set of points). Topology of $R_X'$ induced by class of almost periodical functions on $R_p$ is the strongest or discrete one.

So new wave mechanics (field theory on discontinuum) is the theory of a new level of physical reality - particle constituents. But it is not all yet.

6. Further building of the theory is situated in the channel of works [4] where first of all it is demonstrated that the second quantized version of wave mechanics on discontinuum is trivial. Thereat it is shown that spinor granules are described by bispinors $\psi_\alpha$ $(\alpha = 1,2,3,4)$ obeying to the permutation relations

$$\{\psi_\alpha, \overline{\psi}_\beta\} = 0$$

which determine anti-commutative or Grassmann ring $G_8^{(*)}$ (it is important to emphasize that spinors exist in the configuration space with dimension $n \ge 3$). Grassmann's numbers $\psi_\alpha, \overline{\psi}_\beta$ are only symbols: they have not got any realization. In particular $\psi_\alpha, \overline{\psi}_\beta$ as operators dense defined on separable Hilbert space are zeros

$$\psi_\alpha = \overline{\psi}_\beta = 0,$$

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[5] It seems the famous argument between Dirac and von Neumann will be decided on the latter's favour.

[6] It is important to keep in mind that topology of manifold and measure on it is generally speaking not connected one with other. To set topology of manifold means to set any system of its subsets possessing definite properties, see [6]. To set the measure means to set any function on these subsets satisfying definite requirements [5]. At one and the same topology there exists some non-equivalent measures. So the Lebesgue measure $\mu_L$ is not equivalent to the Bohr measure $\mu_B$: $\mu_L(R) = \infty$ , $\mu_B(R) = 1$. But topology on a class of functions is essentially connected with measure. And we may introduce even the notion of such a topology which is connected with measure on $R$. Ring $A'$ is formed by one and the same reserve of functions independently on what space $\overline{R}$ or *bR* is considered. But as topological rings weak separable $L_\infty$ ($\overline{R}$ ) and non separable $A'$ (*bR*) are principally distinct, see further and also [4].

see [4], that is insufficient of course for building of particle theory.

But then it turns out this Grassmann ring $G_8^{(*)}$ has complicated dynamical structure described by the Heisenberg algebra $h_8^{(*)}$, see [4]. Generators of this algebra, denoted $\Phi_\alpha, \overline{\Phi}_\beta$ $(\alpha, \beta = 1,2,3,4)$, obey the commutation relations

$$[\Phi_\alpha, \overline{\Phi}_\beta] = \delta_{\alpha\beta}$$

(the rest relations are zero). They play role of canonical variables of a new dynamical system called bi-Hamiltonian. In [4] we have considered the mapping

$$\psi_\alpha \to \Phi_\alpha, \overline{\psi}_\alpha \to \overline{\Phi}_\alpha$$

(which is not homomorphism of Grassmann algebra of course) called quantization of the Dirac-Grassmann fiber. Hereby Dirac's particle fields (spin *1/2*) $\psi_\alpha, \overline{\psi}_\beta$ may be represented in the form of matrix elements

$$\psi_\alpha = \left\langle \dot{f}, \Phi_\alpha f \right\rangle, \quad \overline{\psi}_\alpha = \left\langle \dot{f}, \overline{\Phi}_\alpha f \right\rangle,$$

where $f, \dot{f}$ are states of bi-Hamiltonian dynamical system belonging to the dual pair of spaces $(F, \dot{F})$, in which there is a non-standard (non-Fock, non-self-adjoin or non-unitary) representation of the Heisenberg algebra $h_8^{(*)}$. Field $\psi$ of particle with arbitrary spin is represented by the formula

$$\psi = \left\langle \dot{f}, O^{(\psi)} f \right\rangle, \tag{1}$$

where $O^{(\psi)} \in U[L]$ ($L$ is Lagrangian plane in $h_8^{(*)}$) represents particle field $\psi$ at a new level of physical reality described by new quantum mechanics. Here $f, \dot{f}$ play the role of hidden variables in a new wave mechanics. They describe a new physical substance called ether.

As a result we come to definite extension of Heisenberg matrix (operator) mechanics. In fact using expression (1) for state $|\psi\rangle$ we can write the following formula for projection operator $P$ of usual quantum mechanics

$$P = Q(\varphi) Q^+(\varphi'), \tag{2}$$

where $Q(\varphi), Q^+(\varphi')$ are other projection operators

$$Q(\varphi) = f(\varphi) >< \dot{f}(\varphi), Q^+(\varphi') = \dot{f}(\varphi') >< f(\varphi') \tag{3}$$

(here $Q$ is projection onto state $f \in F$ along the state $\dot{f} \in \dot{F}$) acting on the dual pair of topological vector spaces $(F, \dot{F})$ of «ideal» states $f$, $\dot{f}$. Concrete operator $P_\psi$ is represented in the form

$$P_\psi = SpO^{(\psi)} Q \, SpQ^+ \overline{O}^{(\psi)}, \tag{4}$$

where

$$\begin{aligned} SpO^{(\psi)}Q &= < \dot{f}, O^{(\psi)} f > = \\ &\int \bar{\dot{f}}(\varphi) O^{(\psi)}(\varphi) f(\varphi) d\mu(\varphi) = |\psi\rangle \end{aligned} \tag{5}$$

and $d\mu(\varphi)$ is the measure on the space of hidden parameters $\varphi_\alpha, \overline{\varphi}_\beta \in L$ $(\alpha, \beta = 1,2)$ [4], in which canonical variables $\Phi, \overline{\Phi}$ are written in the form

$$\Phi = \begin{pmatrix} \varphi_\alpha \\ \partial / \overline{\varphi}_\alpha \end{pmatrix}, \overline{\Phi} = \begin{pmatrix} -\partial / \varphi_\beta \\ \overline{\varphi}_\beta \end{pmatrix}. \tag{6}$$

Particles described by bilocal fields (1) are not point-like, they are smearing objects. Interactions of such objects are free from ultraviolet divergences.

So in the suggested theory ether plays decided role at the construction of mathematically consequent theory of elementary particles and their interactions: from pure technical point of view it plays the role of regulating medium [4].

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