FIELD STRENGTH FOR GRADED YANG–MILLS THEORY

K. Ilyenko

Institute for Radiophysics and Electronics, NASU
12 Ak. Proskura St., Kharkiv - 61085, Ukraine

The graded field strength is defined for *osp*(2/1;C) non-degenerate gauge algebra. We show that a pair of Grassman odd scalar fields find their place as a constituent part of the graded gauge potential on the equal footing with an ordinary (Grassman even) one-form taking values in the proper Lie subalgebra, *su*(2), of the graded Lie algebra. Some possibilities of constructing a meaningful variational principle are discussed.

PACS: 11.10.Ef, 11.15.-q

1. INTRODUCTION

As well-known the number of gauge bosons of a theory is given by the number of generators of the corresponding gauge Lie algebra. The requirement of being a Lie algebra stems from the reality of the action functional which follows from the properties of the Fermat principle in optics and the Feynman path integral. In particular, this leads to unitary evolution of a system with such an action functional. Nevertheless, no one is restricted from looking for not necessarily of Lie-type algebras as the gauge algebras provided they are physically meaningful.

2. GRADING OF GAUGE ALGEBRA

Such an algebra is advocated in the current contribution. This is a graded extension of *su*(2) gauge algebra by a pair of odd generators, τ_A, which anticommute with one another and commute with the three even generators, T_a, of *su*(2). We use the square brackets to denote both commutation and anticommutation operations of the generators with understanding of their proper usage. The defining relations have the following form, [1-3]:

$$[T_a, T_b] = i\varepsilon_{abc}T_c\,,\quad [T_a, \tau_A] = \tfrac{1}{2}(\gamma_a)_A{}^B\tau_B\,,$$

$$[\tau_A, \tau_B] = -\tfrac{i}{2}(\gamma_a)_{AB}T_a \qquad (1)$$

Lowercase Roman indices run from 1 to 3; uppercase Roman indices run over 1, 2; $(\gamma_a)_{AB} = (\gamma_a)_A{}^C\epsilon_{CB}$; ε_{abc} ($\varepsilon_{123} = 1$) and ϵ_{AB} ($\epsilon_{12} = 1$) are the Levi-Civita totally antisymmetric symbols in three and two dimensions; the Clifford matrices $(\gamma_a)_A{}^B$ and ϵ_{AB} are given by

$$(\gamma_a)_A{}^B = \left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\right];$$

$$\epsilon_{AB} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \qquad (2)$$

In the adjoint representation the matrices T_a and τ_A can be written as follows:

$$T_1 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -i & 0 & 0\\ 0 & i & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 1/2 & 0\\ 0 & 0 & 0 & 0 & -1/2 \end{array}\right),$$

$$T_2 = \left(\begin{array}{ccc|cc} 0 & 0 & i & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ -i & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 1/2\\ 0 & 0 & 0 & 1/2 & 0 \end{array}\right),$$

$$T_3 = \left(\begin{array}{ccc|cc} 0 & -i & 0 & 0 & 0\\ i & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & -1/2\\ 0 & 0 & 0 & 1/2 & 0 \end{array}\right). \qquad (3)$$

The non-degenerate super Killing form, $B(T_\alpha, T_\beta)$, is defined by

$$B(T_\alpha, T_\beta) = str(T_\alpha T_\beta) = \left(\begin{array}{c|c} \delta_{ab} & 0\\ \hline 0 & i\epsilon_{AB} \end{array}\right), \qquad (4)$$

where the supertrace operation is adopted from [4] and the Greek indices run over the whole set of the graded Lie algebra generators. It turns out that all of the generators are grade star Hermitian: the even ones being just Hermitian in an ordinary sense while the odd generators obey more complicated relations (cf. [5,6]). We assign a degree, $\deg T_\alpha$, 0 to the even and 1 to the odd generators.

3. THE GAUGE POTENTIAL

Given an *su*(N) Lie algebra one defines a gauge potential, which takes values in the algebra, by introducing $(N^2 - 1)\times n$ one-forms $A^a_\mu(x)dx^\mu$, n being the dimensions of space-time, and subtracting them with the algebra generators $T_a A^a_\mu(x)dx^\mu$. Note that from the standpoint of graded Lie algebra we have a composite object of the degree (0,1), the first position shows that generator T_a is an even element of the algebra and the second position corresponds to the fact that $A^a_\mu(x)dx^\mu$ is a one-form of the algebra of exterior forms on M^n. This has a suggestive generalization to the case when one has the degree 1 odd part of a gauge algebra: (s)he needs to construct the homogeneous compliment of the expression above, namely, the element of degree (1,0). It has the form $\tau_A \Phi^A(x)$ and must be added to the element of degree (0,1) to form the complete graded gauge potential

$$\mathrm{A}(x) = \tau_A \Phi^A(x) \quad + \qquad (5)$$

homogeneous = odd⊗even + even⊗odd

Here $\Phi^A(x)$ are zero-forms on M^n. Thus one obtains a proper element A(x) of the degree one of the direct product of two graded algebras (compare with [7], p. 629).

4. GRADED FIELD STRENGTH

The graded field strength, F(x), is defined by means of exterior derivative of the graded gauge potential, A(x), and adding its wedge product with itself, A$(x) \wedge$ A(x):

$$\mathrm{F}(x) = d\mathrm{A}(x) + (ig/2)\mathrm{A}(x) \wedge \mathrm{A}(x). \qquad (6)$$

Here we must clarify the meaning of both operations.

As to the first term on the right-hand side of (6) we obtain

$$d\mathrm{A}(x) = \tau_A d\Phi^A(x) + \tfrac{1}{2!} T_a F^a_{\mu\nu}(x) dx^\mu \wedge dx^\nu . \qquad (7)$$

Here $F^a = \frac{1}{2!}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\varepsilon^{abc} A^b_\mu A^c_\nu)\ dx^\mu \wedge dx^\nu$ is just the Yang-Mills field strength.

The second term on the right-hand side of (6) needs more explanations. Firstly, as in the case with Yang-Mills theory, one defines the wedge product for algebra valued forms A(x). Secondly, we have to deal with the odd part of gauge algebra and zero-forms involved in the graded gauge potential. Thus we define, [7, p. 632],

$$\mathrm{A} \wedge \mathrm{A} = (T_\alpha \otimes A^\alpha) \wedge (T_\beta \otimes A^\beta)$$

$$\equiv (-1)^{\deg A^\alpha \cdot \deg T_\beta} [T_\alpha, T_\beta] \otimes A^\alpha \wedge A^\beta . \qquad (8)$$

Here A^α comprises the whole set of fields $\{\Phi^A, A^a\}$ and the deg A^α means the degree in the exterior algebra of differential forms on M^n. One can easily see that this straight generalization is bilinear, goes into usual Yang-Mills result for ordinary Lie algebras and makes possible the graded Jacobi identity for the direct product of two graded algebras: the exterior and matrix ones.

Finally, we unite (7) and (8) in one expression:

$$\mathrm{F} = \tau_A \varphi^A + T_a [F^a + (1/2!)(\gamma^a)_{AB} \Phi^A \Phi^B], \qquad (9)$$

where F^a is defined after formula (7), and

$$\varphi^A(x) = d\Phi^A(x) + \tfrac{1}{2!}(\gamma_a)^A{}_B A^a(x) \Phi^B(x) . \qquad (10)$$

We intend to preserve the graded property of gauge potential in the expression for the field strength. The first summand in (9) is of the degree (1,1), the expression $T_a F^a$ is of the degree (0,2), while the expression $T_a(\gamma^a)_{AB}\Phi^A\Phi^B$ falls out of this pattern. Allowing for the symmetric property of the Clifford matrices $(\gamma^a)_{AB} = (\gamma^a)_{BA}$, we are forced to assume the functions $\Phi^A(x)$ as being Grassman-valued odd variables, which obey the property $\Phi^A \Phi^B = -\Phi^B \Phi^A$. Then, equation (9) takes the form

$$\mathrm{F}(x) = \tau_A \varphi^A(x) + T_a F^a(x) . \qquad (11)$$

Thus, we have built a homogeneous expression of degree two representing the field strength.

5. DISCUSSION AND OUTLOOK

The very possibility to define the graded four-potential and field strength opens up a number of further questions.

Firstly, a problem of graded gauge invariance arises. We hope that existing in the literature (see [8]) introduction of Grassman-odd parameters of transformations for odd generators of the algebra could provide a sensible solution to this problem.

Secondly, as mentioned at the beginning of the current contribution, one is interested in a definition of a *real*-valued Lagrangian density. This would lead to the physically meaningful Euler-Lagrange equations. In particular, we are going to consider an invariant with respect to the gauge algebra automorphisms expression:

$$S = \int str(\mathrm{F} \wedge {*\mathrm{F}}) \qquad (12)$$

Thus we intend to undertake an extensive research on the opportunities, which follow from the given development.

ACKNOWLEDGMENTS

The author would like to thank Yu.P. Stepanovsky and V. Pidstrigach for many helpful discussions.

REFERENCES

1. V. Kac. *Representations of classical Lie superalgebras.* Differentiat Geometrical Methods in Mathematical Physics II, edited by K. Bleuler, H.R. Petry and A. Reete. *Lecture Notes in Mathematics.* 1977, 676 p.
2. J.W. Hughes. Representations of osp(2,1) and the metaplectic representation // *J. Math. Phys.* 1981, v. 22, p. 245-250.
3. R. Brooks, A. Lue. The monopole equations in topological Yang-Mills // *J. Math. Phys.* 1996, v. 37, p. 1100-1105.
4. J.F. Cornwell. *Group Theory in Physics*, part 3, L., 1989, 628 p.
5. M. Scheunert, W. Nahm, Y. Rittenberg. Graded Lie algebras:generalization of Hermitian representations // *J. Math. Phys*. 1977, v. 18, p. 146-154.
6. M. Scheunert, W. Nahm, Y. Rittenberg. Irreducible representations of the osp(2,1) and spl(2,1) graded Lie algebras // *J. Math. Phys.* 1977, v. 18, p. 155-162.
7. S. Lang. *Algebra.* 3-nd edition, R., 1985, 829 p.
8. V.I. Ogievetskii, L. Mizinchesku. Symmetries between boson and fermion superfields // *Uspekhi Fiz. Nauk.* 1975, v. 117, p. 637-683 (in Russian).