EXACT RELATIVISTIC PLASMA DISPERSION FUNCTIONS

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On the ground of the theory of singular integrals with Cauchy kernel the exact plasma dispersion functions (PDFs) are introduced and studied. Those PDFs make more exact the weakly relativistic PDFs and generalize them on the case of arbitrary plasma temperature.

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1. INTRODUCTION

Plasma waves have a wide range of applications. Essential to each of the applications is knowledge of the dielectric properties of the plasma. Analytical treatment of those properties leads to expressions for the dielectric tensor in terms of plasma dispersion functions (PDFs).

As the non-relativistic PDF, $W(z)=\exp(-z^2)[1+(2i/\sqrt{\pi})\int_0^z \exp(t^2)dt]$, introduced into the theory of plasma and tabulated in complex region in the works [1,2], respectively, (or PDF, $Z(z)=i\sqrt{\pi}W(z)$, introduced in the work [3]), so and the weakly relativistic PDFs, introduced in the work [4], are approximate ones, since have been derived through some approximations in the parameter $1/\mu$ ($\mu=(c/V_{T0})^2$, $V_{T0}=\sqrt{T/m_0}$, m_0 is the rest mass of electron).

The primary purpose of the present work is introducing the exact relativistic PDFs, which makes more exact the non-relativistic PDF and the weakly relativistic PDFs and generalize them on the case of arbitrary plasma temperature. It is achieved on the ground of a deep connection between the theory of plasma waves and the theory of Cauchy type integrals.

2. INTRODUCING OF THE EXACT RELATIVISTIC PDFs

Starting from the 1st integral form of Trubnikov's plasma dielectric tensor, that neglects ion dynamics, one can write it in Cartesian coordinates with z-axis directed along static magnetic field in the next equivalent form [5]

$$\varepsilon_{ij}=\delta_{ij}-\left(\frac{\omega_p}{\omega}\right)^2\frac{\mu}{2K_2(\mu)}\sum_{n=-\infty}^{\infty}\frac{1}{|n|!}\left(\frac{\lambda}{2}\right)^{|n|}\sum_{k=0}^{\infty}\frac{C_{n,k}}{k!}\left(\frac{\lambda}{2}\right)^k\times$$
$$\pi_{ij}(n,k)\,\tau_n^{(|n|+k,r_{ij})}(N_{//},n\Omega_c/\omega,\mu), \quad (1)$$

where ω and ω_p are an angular wave and plasma frequencies, $K_2(\mu)$ is Macdonald function of second order, $N_{//}=k_{//}c/\omega$ is longitudinal refractive index, $\Omega_c=eB/(m_0c)$ is the electron cyclotron frequency,

$$C_{n,k}=\frac{(-1)^k|n|!\,[2(|n|+k)]!}{(2|n|+k)!\,(|n|+k)!},\quad \pi_{xx}=\frac{n^2}{\lambda},\quad \pi_{xy}=-\pi_{yx}=i\frac{n(|n|+k)}{\lambda},$$

$$\pi_{xz}=\pi_{zx}=\frac{n}{\sqrt{\lambda}},\quad \pi_{yy}=\frac{n^2}{\lambda}+\frac{2k(2|n|+k)(|n|+k-1)}{(2|n|+2k-1)\lambda},$$

$$\pi_{yz}=-\pi_{zy}=-i\frac{n(|n|+k)}{\sqrt{\lambda}},\quad \pi_{zz}=1,$$

$$\tau_n^{(|n|+k,r_{ij})}(N_{//},n\Omega_c/\omega,\mu)=\frac{\mu^{r_{ij}/2}}{(|n|+k)!}\left(\frac{\mu}{2}\right)^{|n|+k}\times$$

$$\int_{-\infty}^{\infty}d\bar p_{//}\,\bar p_{//}^{\,r_{ij}}e^{-\mu\sqrt{1+\bar p_{//}^2}}\int_0^{\infty}dx\,e^{-\mu x}\frac{x^{|n|+k}\left(x+2\sqrt{1+\bar p_{//}^2}\right)^{(|n|+k)}}{x+\beta_n}. \quad (2)$$

In the expression (2) $\bar p=p/(m_0c)$ is normalized momentum, $x=\sqrt{(1+\bar p_{//}^2+\bar p_\perp^2)}-\sqrt{(1+\bar p_{//}^2)}$, $\beta_n=\sqrt{(1+\bar p_{//}^2)}-N_{//}\bar p_{//}-n\Omega_c/\omega$ and $r_{xx}=r_{xy}=r_{yx}=r_{yy}=0$, $r_{xz}=r_{zx}=r_{yz}=r_{zy}=1$, $r_{zz}=2$.

Bortnatici and Ruffina shown that for the case $n\ge 0$ anti-Hermitian parts of the functions, $F^0_{q+3/2}(N_{//},n\Omega_c/\omega,\mu)=e^{\mu}\sqrt{\mu/2\pi}\,\tau_n^{(q,0)}(N_{//},n\Omega_c/\omega,\mu)$, can be exactly expressed in terms of modified spherical Bessel functions [6]. Obviously, the same result takes place and for functions, $Z^0_{q+3/2}(N_{//},n\Omega_c/\omega,\mu)=\tau_n^{(q,0)}(N_{//},n\Omega_c/\omega)/(2K_2(\mu))$, that appear in the expression (1) since they differ from functions, $F^0_{q+3/2}(N_{//},n\Omega_c/\omega,\mu)$, by the factor not depending of two first arguments. In first, let us generalize these results on the case of arbitrary harmonic number, n, using some facts from the theory of Cauchy type integrals [7,8]. Then on the same way we'll give the exact analytical expressions for the Hermitian parts of the functions, $Z^0_{q+3/2}(N_{//},n\Omega_c/\omega,\mu)$ and define these whole functions as the exact relativistic PDFs.

Using in the inside integral of the expression (2) the first of the Sokhotskii-Plemelj formulas,

$$F^+(t)=\frac{\varphi(t)}{2}+\frac{1}{2\pi i}P\int_{-\infty}^{+\infty}\frac{\varphi(\tau)d\tau}{\tau-t},$$
$$F^-(t)=-\frac{\varphi(t)}{2}+\frac{1}{2\pi i}P\int_{-\infty}^{+\infty}\frac{\varphi(\tau)d\tau}{\tau-t}, \quad (3)$$

corresponding to the Landau rule of passing the pole, and passing to the arguments used by Robinson [9-11]: $x=\bar p_{//}\sqrt{\mu/2}$, $z=\mu(1-n\Omega_c/\omega)$, $a=\mu N_{//}^2/2$ one can, at once, receive

$$\operatorname{Im}Z^{r_{ij}}_{q+3/2}(a,z,\mu)=\frac{\pi e^{z-\mu}(-1)^{q+1}2^{r_{ij}/2}}{\sqrt{2\mu}K_2(\mu)q!}\times$$
$$\int_{x^-}^{x^+}e^{-2a^{1/2}x}\left[z-2a^{1/2}x+x^2-\left(z-2a^{1/2}x\right)^2/(2\mu)\right]^q dx, \quad (4)$$

where limits of integration, $x^\pm$, can be obtained from the condition of appearing the pole in the inside integral of the expression (2): $\beta_n\le 0$. Then for $0\le N_{//}<1$ the limits of integration equal $x^\pm=\beta[a^{1/2}(1-z/\mu)\pm\sqrt{a-z+z^2/(2\mu)}]$ and for $N_{//}>1$ ones equal $x^+=+\infty$, $x^-=\beta[a^{1/2}(1-z/\mu)-\sqrt{a-z+z^2/(2\mu)}]$; here for shortness was used denoting $\beta=\mu/(\mu-2a)$. The direct integration in the expression (4) gives anti-Hermitian parts of functions $Z^{r_{ij}}_{q+3/2}(a,z,\mu)$. So, for the case $r_{ij}=0$ and $0\le N_{//}<1$

$$\operatorname{Im} Z^{0}_{q+3/2}(a,z,\mu) = -\frac{\pi^{3/2}\beta^{1/2}}{\sqrt{2\mu}K_2(\mu)}\left(\sqrt{\frac{a-z+z^2/(2\mu)}{a}}\right)^{q+1/2}\times$$
$$\exp[\beta(z-2a)-\mu]\, I_{q+1/2}\left(\beta 2a^{1/2}\sqrt{a-z+z^2/(2\mu)}\right), \quad (z<a^*), \tag{5}$$

$$\operatorname{Im} Z^{0}_{q+3/2}(a,z,\mu) \equiv 0, \ (z\ge a^*), \tag{6}$$

here $I_{q+1/2}(x)$ is modified Bessel function of half-integer order and $a^*=\mu\left(1-1/\sqrt{\beta}\right)$. By the similar way for $r_{ij}=0$ and $N_{//}>1$

$$\operatorname{Im} Z^{0}_{q+3/2}(a,z,\mu) = -\frac{(-\pi\beta)^{1/2}}{\sqrt{2\mu}K_2(\mu)}\left(\sqrt{\frac{a-z+z^2/(2\mu)}{a}}\right)^{q+1/2}\times$$
$$\exp[\beta(z-2a)-\mu]K_{q+1/2}\left(-\beta 2a^{1/2}\sqrt{a-z+z^2/(2\mu)}\right), \tag{7}$$

for arbitrary z; here $K_{q+1/2}(x)$ is Mac Donald function of half-integer order.

It is known [7, 8] that for the density of Cauchy integral, $\varphi(t)$, satisfied to the Holder condition and integrated along the real axis in the Cauchy sense real and imagine parts of the boundary function, $F^+(t)=u(t)+iv(t)$, are mutually single significantly connected by the Hilbert transforms

$$iv(t)=\frac{1}{\pi i}P\int_{-\infty}^{+\infty}\frac{u(\tau)d\tau}{\tau-t}, \qquad u(t)=\frac{1}{\pi i}P\int_{-\infty}^{+\infty}\frac{iv(\tau)d\tau}{\tau-t}. \tag{8}$$

Here and everywhere letter P before integrals shows that integrals are understood in the sense of principle value. Consequently, implying $v(t)=\operatorname{Im} Z^{0}_{q+3/2}(a,z,\mu)$ one can use the second transform to express real parts of the boundary functions, $Z^{0}_{q+3/2}(a,z,\mu)$, through their imagine parts. Then for the case $0\le N_{//}<1$ one can receive

$$\operatorname{Re} Z^{0}_{q+3/2}(a,z,\mu) = \frac{\pi^{1/2}\beta^{1/2}e^{-\beta(a^*-2a)-\mu}}{\sqrt{2\mu}K_2(\mu)\left(\sqrt{a}\right)^{q+1/2}}P$$
$$\int_0^{+\infty}\frac{\left(\sqrt{a+\tau+\tau^2/(2\mu)}\right)^{q+1/2} I_{q+1/2}\left(\beta 2a^{1/2}\sqrt{a+\tau+\tau^2/(2\mu)}\right)e^{-\beta u}du}{u-a^*+z}. \tag{9}$$

In correspondence with the Sokhotskii-Plemely formula, giving the right sign (as in the weakly relativistic case) of anti-Hermitian parts of the functions, $Z^{0}_{q+3/2}(a,z,\mu)$, and corresponding to passing of the contour of integration above the pole, one can receive the exact integral form for the whole those functions

$$Z^{0}_{q+3/2}(a,z,\mu) = \frac{\sqrt{\pi\beta}e^{-\mu\sqrt{\beta}}}{\sqrt{2\mu}K_2(\mu)\left(\sqrt{a}\right)^{q+1/2}}\times$$
$$\int_0^{+\infty}\frac{\left(\sqrt{u(u/(2\mu)+1/\sqrt{\beta}}\right)^{q+1/2} I_{q+1/2}\left(\beta 2a^{1/2}\sqrt{u(u/(2\mu)+1/\sqrt{\beta}}\right)e^{-\beta u}du}{u-\mu(1-\sqrt{\beta})+z}. \tag{10}$$

For the case $N_{//}>1$ using the expression (7) for anti-Hermitian parts of the functions, $Z^{0}_{q+3/2}(a,z,\mu)$, by the similar way one can obtain

$$Z^{0}_{q+3/2}(a,z,\mu) = -\sqrt{-\beta}\,\frac{\exp[-2\beta a-\mu]}{\sqrt{2\pi\mu}K_2(\mu)\left(\sqrt{a}\right)^{q+1/2}}\times$$
$$\int_{-\infty}^{\infty}\frac{\left(\sqrt{a-\tau+\tau^2/(2\mu)}\right)^{q+1/2} K_{q+1/2}\left(-\beta 2a^{1/2}\sqrt{a-\tau+\tau^2/(2\mu)}\right)\exp(\beta\tau)d\tau}{\tau-z}, \tag{11}$$

where the contour of integration is passing below the pole.

Integral forms (10), (11) define analytically the exact relativistic PDFs for the cases $0\le N_{//}<1$ and $N_{//}>1$, respectively.

3. ANALYTICAL PROPERTIES OF THE EXACT PDFs

In first, let us study the integral form (10) defining the exact relativistic PDFs for the case $0\le N_{//}<1$ that is interesting from the point of view of the EC waves description in magnetized plasma; for shortness of writing from now on we are missing arguments of these functions. Using the recurrent formula for modified Bessel functions, $I_{\nu+1}(z)=I_{\nu-1}(z)-(2\nu/z)I_\nu(z)$, one can receive

$$\beta a Z^0_{q+5/2}+(q+1/2)Z^0_{q+3/2}-\beta[a-z+z^2/(2\mu)]Z^0_{q+1/2}=$$
$$\frac{\beta}{2\mu}\frac{\sqrt{\pi\beta}e^{-\mu\sqrt{\beta}}}{\sqrt{2\mu}K_2(\mu)\left(\sqrt{a}\right)^{q-1/2}}\times \tag{12}$$
$$\int_0^{\infty}(a^*-u)\left(\sqrt{u(u/(2\mu)+1/\sqrt{\beta}}\right)^{q-1/2} I_{q-1/2}\left(\beta 2a^{1/2}\sqrt{u(u/(2\mu)+1/\sqrt{\beta}}\right)e^{-\beta u}du.$$

Both integrals in the expression (12), using the known integral [12]

$$\int_0^{+\infty}\left(\sqrt{x^2+xz}\right)^{\nu} I_\nu\left(c\sqrt{x^2+xz}\right)e^{-px}dx=\frac{c^\nu}{2^\nu\sqrt{\pi}}\left(\frac{z}{\sqrt{p^2-c^2}}\right)^{\nu+1/2}\times$$
$$e^{pz/2}K_{\nu+1/2}\left(\frac{z}{2}\sqrt{p^2-c^2}\right), \ (\operatorname{Re}p>|\operatorname{Re}c|;\ \operatorname{Re}\nu>-1,\ |\arg z|<\pi),$$

can be integrated

$$\frac{\sqrt{\pi\beta}e^{-\mu\sqrt{\beta}}}{\sqrt{2\mu}K_2(\mu)\left(\sqrt{a}\right)^{q-1/2}}\int_0^{\infty}(s)^{q-1/2}I_{q-1/2}\left(\beta 2a^{1/2}s\right)e^{-\beta u}du=\frac{K_q(\mu)}{K_2(\mu)}, \tag{13}$$

$$\frac{\sqrt{\pi\beta}e^{-\mu\sqrt{\beta}}}{\sqrt{2\mu}K_2(\mu)\left(\sqrt{a}\right)^{q-1/2}}\int_0^{\infty}(a^*-u)(s)^{q-1/2}I_{q-1/2}\left(\beta 2a^{1/2}s\right)e^{-\beta u}du=$$
$$-\mu\frac{K_{q+1}(\mu)-K_q(\mu)}{K_2(\mu)}, \tag{14}$$

here, for shortness, noting $s=\sqrt{u(u/(2\mu)+1/\sqrt{\beta}}$ was used. Thus, from (12) we receive a recurrent relation for the exact PDFs

$$\beta a Z^0_{q+5/2}+(q+1/2)Z^0_{q+3/2}-\beta[a-z+z^2/(2\mu)]Z^0_{q+1/2}=$$
$$\frac{\beta}{2}[(1-z/\mu)K_q(\mu)+K_{q+1}(\mu)]/K_2(\mu). \tag{15}$$

For the case $N_{//}>1$, that is interesting for the description of ICR waves in relativistic plasma, by the similar way one can receive the same recursive relation (15). Only in this case it is necessary to use the recurrent formula for Mac Donald functions, $K_{\nu+1}(z)=K_{\nu-1}(z)+(2\nu/z)K_\nu(z)$, and the integral

$$\int_{-\infty}^{\infty}\left(\sqrt{x^2+z^2}\right)^{\nu}K_{\nu}\left(p\sqrt{x^2+z^2}\right)e^{bx}dx=\sqrt{2\pi}p^{\nu}\left(\frac{z}{\sqrt{p^2-b^2}}\right)^{\nu+1/2}\times$$

$$K_{\nu+1/2}\left(z\sqrt{p^2-b^2}\right),\quad (\mathrm{Re}\,\nu>-1)$$

that follows by integrating in the parameter p from the known integral [13]

$$\int_{0}^{\infty}e^{-p\sqrt{x^2+z^2}}chbxdx=\frac{pz}{\sqrt{p^2-b^2}}K_1\left(z\sqrt{p^2-b^2}\right),\quad (\mathrm{Re}\,p>|\mathrm{Re}\,b|;$$

$\mathrm{Re}\,z>0$).

Then instead integrals (13), (14) one can obtain the integrals

$$\sqrt{-\beta}\,\frac{\exp[-2\beta a-\mu]}{\sqrt{2\pi\mu}K_2(\mu)\left(\sqrt{a}\right)^{q-1/2}}\int_{-\infty}^{\infty}(s)^{q-1/2}K_{q-1/2}\left(-\beta 2a^{1/2}s\right)\exp(\beta\tau)d\tau=$$

$$\frac{K_q(\mu)}{K_2(\mu)},\qquad (16)$$

$$\sqrt{-\beta}\,\frac{\exp[-2\beta a-\mu]}{\sqrt{2\pi\mu}K_2(\mu)\left(\sqrt{a}\right)^{q-1/2}}\int_{-\infty}^{\infty}(s)^{q-1/2}K_{q-1/2}\left(-\beta 2a^{1/2}s\right)\exp(\beta\tau)d\tau=$$

$$-\mu\frac{K_{q+1}(\mu)-K_q(\mu)}{K_2(\mu)}.\qquad (17)$$

Here, for shortness, noting $s=\sqrt{a-\tau+\tau^2/(2\mu)}$ was used.

The integrals (13), (14) and (16), (17) are moments on the density in the respective Cauchy integrals defining the exact PDFs, $Z^0_{q+1/2}$, and have the next physical sense: the integrals (13), (16) are the densities of resonance electrons in the respective Dopler spectral line of absorption, the integrals (14), (17) define z-coordinate of this spectral line.

So, from all those integrals it follows the next theorem: *z-coordinates of Dopler absorption spectral lines and densities of resonance electrons in these lines don't depend from the direction of waves propagation and are defined by the moments (13), (14), (16), (17) of anti-Hermitian parts of the exact PDFs.*

Those integrals have also and the mathematical sense: (13), (16) are first and (14), (17) second coefficients in asymptotic expansion of the PDF, $Z^0_{q+1/2}$,

$$Z^0_{q+1/2}=\frac{K_q(\mu)}{K_2(\mu)z}-\mu\frac{K_{q+1}(\mu)-K_q(\mu)}{K_2(\mu)z^2}+...=$$

$$\frac{1}{z}\left(A^q_0+\frac{A^q_1}{z}+\frac{A^q_{21}}{z^2}+...\right).\qquad (18)$$

Substituting the expression (18) into the recursive relation (15) one can receive recurrent formulas for calculating an arbitrary coefficient in this expansion through first two coefficients

$$A^q_n=\beta a A^{q+2}_{n-2}+(q+1/2)A^{q+1}_{n-2}-\beta a A^q_{n-2}+\beta A^q_{n-1}.\qquad (19)$$

One can verify the formulas for the 1st and 2nd derivatives of the exact PDFs in the parameter z

$$(Z^0_q)'=\beta[Z^0_q+(z/\mu-1)Z^0_{q-1}],$$

$$(Z^0_q)''=\beta^2\left\{Z^0_q+[2(z/\mu-1)+1/(\beta\mu)]Z^0_{q-1}+(z/\mu-1)^2Z^0_{q-2}\right\}.\qquad (20)$$

Excepting PDFs, Z^0_{q-1} and Z^0_{q-1} from the expressions (15), (20) one can obtain the usual linear deferential equation in the parameter z for the exact PDFs

$$(z-\mu)(a-z+z^2/(2\mu))(Z^0_q)''-$$

$$[2\beta(z-\mu)(a-z+z^2/(2\mu))+a+(q-2)\mu+(2q-3)(-z+z^2/(2\mu))](Z^0_q)'+$$

$$\beta[(z-\mu)(-z+z^2/(2\mu))+a+(q-2)\mu+(2q-3)(-z+z^2/(2\mu))]Z^0_q+$$

$$\beta^2(z-\mu)^3[\mu K_{q-3/2}-(z-\mu)K_{q-5/2}]/(2\mu^3K_2)=0,\qquad (21)$$

that also may be useful for their study.

At last, from the formulas (4) by differentiation in the parameter a it follows the identity

$$dZ^0_q/da=(Z^0_{q+1})'',$$

connecting derivatives of these PDFs in the parameter a and z that coincides with the respective identity for the weakly relativistic PDFs.

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ТОЧНЫЕ РЕЛЯТИВИСТСКИЕ ПЛАЗМЕННЫЕ ДИСПЕРСИОННЫЕ ФУНКЦИИ

С.С. Павлов и Ф. Кастехон

На основе теории сингулярных интегралов с ядром Коши вводятся и исследуются плазменные дисперсионные функции (ПДФ), уточняющие слабо релятивистские ПДФ и обобщающие их на случай произвольной температуры.

ТОЧНІ РЕЛЯТИВІСТСЬКІ ПЛАЗМОВІ ДИСПЕРСІЙНІ ФУНКЦІЇ

С.С. Павлов і Ф. Кастехон

На основі теорії сингулярних інтегралів з ядром Коші вводяться і досліджуються точні плазмові дисперсійні функції (ПДФ), уточнюючі слабо релятивістські ПДФ і узагальнюючі їх на випадок довільної температури.