DISCRETE OPTIMIZATION PROBLEM

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In the paper a mathematical model is considered that allows simultaneous optimization of a program motion and an ensemble of perturbed motions. Analytical expressions for functional variations are suggested that help constructing various directed methods of optimization. Given mathematical apparatus can be effectively used in the optimization of the dynamics of charged particles in linear accelerators.

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1. INTRODUCTION

Discrete problems of control are important in the theory and practice of optimal control, because many problems are described exactly by differential equations. In practice the information on the stage of the process comes in discrete moments of time and the control of the process also comes step by step.

Problems of control in discrete systems received attention of many researches. Two approaches to the problem can be found. The first approach is based on the Bellman principle of optimality. The second one is the variational approach which links to the apparatus principle of L.S. Pontryagin.

Conventional formulations of optimal control problems are quite known and they had been studied quite well [1]. These problems can be considered as problems of control of particular trajectories. At the same time, in works of D.A.Ovsyannikov such methods of optimal control and optimization of ensemble of trajectories or beam trajectories have been developed [2].

Let us note that the problems of control of ensemble of trajectories naturally emerge under the study of optimization of charged particle beam dynamics in accelerating and focusing structures.

2. MATHEMATICAL MODEL

Let particle dynamics be given by a difference equation system:

$$x(k+1) = f(k,x,u) = \begin{cases} f_1(k,x(k),u(k)), k = 0,1 \\ f_2(k,x(k),x(k-1),x(k-2),u(k)), k = 2,\dots,N-1 \end{cases} \quad (1)$$

$$y(k+1) = F(k,x,y,u) = \begin{cases} F_1(k,x(k),y(k),u(k)), k = 0,1 \\ F_2(k,x(k),...,x(k-2),y(k),...,y(k-2),u(k)), \\ k = 2,\dots,N-1. \end{cases} \quad (2)$$

Here $x(k)$ is the $n-$ dimensional phase vector defining the program motion, $y(k)$ is the $m-$ dimensional phase vector defining the perturbed motion, $u(k)$ is the $r-$ dimensional control vector; $f(k,x,u)$ is the $n-$ dimensional vector function defining the process dynamics at each step. For all $k \in \{0,1,\dots,N\}$ the vector function $f(k,x,u)$ is assumed to be definite and continuous on $\Omega_x \times U(k)$ in all its arguments (x,u) along with partial derivatives with respect to these variables. $F(k,x(k),y(k),u(k))$ is the $m-$ dimensional vector function, for all $k \in \{0,1,\dots,N\}$ it is assumed to be definite and continuous on $\Omega_x \times \Omega_y \times U(k)$ in all its arguments (x,y,u) along with partial derivatives with respect to these variables and second partial derivatives

$$\frac{\partial^2 F_l}{\partial y_i \partial y_j}, \frac{\partial^2 F_l}{\partial y_i \partial x_m}, \frac{\partial^2 F_l}{\partial y_i \partial u_k}, \quad l,i = 1,2,\dots,m;$$
$$m = 1,2,\dots,n; \quad k = 1,2,\dots,r.$$

Here Ω_x is the region in R^n, Ω_y is the region in R^m, $U(k)$, $k = 0,1,\dots,N-1$ is the compact set in R^r. In this case we consider the Jacobian

$$J_k = J(k,x(k),y(k),u(k)) = \left| \frac{\partial F(k,x(k),y(k),u(k))}{\partial y(k)} \right|$$

to be nonzero for all changes of k, $x(k)$, $y(k)$, $u(k)$.

The given type of system (1)-(2) is determined by the kind of discrete equations which can be described as the motion of charged particles in accelerator with drift tubes [2]. System (1)-(2) can be reduced to

$$x(k+1) = f(k,x(k),u(k)),$$
$$y(k+1) = F(k,x(k),y(k),u(k)).$$

However, this is not useful as this leads to the rise of steps of phase vectors and the loss of physical sense of beam trajectories.

We assume, that $x(0) = x_0$ is fixed and the initial state of the system (2) is described by the set M_0 - a compact set of nonzero measure in R^m. Let us call the sequence of vectors $\{u(0),u(1),\dots,u(N-1)\}$ as the control of the system (1)-(2) and denote it by u for brevity. Its associated sequence of vectors $\{x(0),x(1),\dots,x(N)\}$ is called the trajectory of program motion and is denoted by $x = x(x_0,u)$. Denote by $x(k) = x(k,x_0,u)$ the phase state of the program particle at k th step. Similarly, let us call the sequence of vectors $\{y(0),y(1),\dots,y(N)\}$ as the trajectory of perturbed motion and denote it by $y = y(x,y_0,u)$. Denote by $y(k) = y(k,x,y_0,u)$ the phase state of the particle at the k -th step.

The set of trajectories $y(x,y_0,u)$ corresponding to the initial state x_0, the control u and different states $y_0 \in M_0$ will be referred to as a bundle of trajectories or simply the bundle. The phase state of beam at the k

th step is also called as the cross section of the bundle of trajectories and is denoted by $M_{k,u}$, i.e.

$$M_{k,u} = \{ y(k) : y(k) = y(k, y_0, x, u), y_0 \in M_0 \},$$

the controls satisfying conditions $u(k) \in U(k), \quad k = 0,1,\dots,N-1$ are admissible.

We introduce the following functionals:

$$I_1(u) = \sum_{k=1}^{N-1} c_k g_k(x_k) + c_N g_N(x_N), \qquad (3)$$

$$I_2(u) = \sum_{k=1}^{N-1} \int_{M_{k,u}} \varphi_k(x_k, y_k, u_k) dy_k + \int_{M_{N,u}} g(y_N) dy_N, \qquad (4)$$

$$I(u) = I_1(u) + I_2(u) \qquad (5)$$

characterizing the phase state of the bundle of trajectories and the control parameters. Here $x_k = x(k)$, $y_k = y(k) \in M_{k,u}$, the function φ_k is defined and continuous on $\Omega_x \times \Omega_y \times U(k)$ for all k in all its arguments together with partial derivatives with respect to x, y and u; g_k is continuously differentiable function defined in Ω_x for all k; g is a continuously differentiable function defined in Ω_y, c_k is constant for all k.

Functional (3) characterizes the dynamics of the program motion, functional (4) characterizes the dynamics of the ensemble of perturbed motions, while functional (5) allows simultaneous estimation of the program and perturbed motions as well as their simultaneous optimization. We will consider the functional minimization problem for all admissible controls.

3. VARIATION OF FUNCTIONAL

Let us consider the admissible control u and $\tilde{u}$. Their associated trajectories are denoted by $x(x_0, u)$ and $\tilde{x}(x_0, \tilde{u})$, and associated trajectories of perturbed motions are denoted by

$$y(x, y_0, u) \text{ и } \tilde{y}(\tilde{x}, y_0, \tilde{u}). \qquad (6)$$

The difference $\Delta u(k) = \tilde{u}(k) - u(k)$ is called the variational of the control u at the kth step, difference $\Delta x(k) = \Delta x(k, x_k) = \tilde{x}(k, x_0, \tilde{u}) - x(k, x_0, u)$ is called the trajectory increment $x(x_0, u)$ at the kth step, and the difference

$\Delta y(k) = \Delta y(k, y_k) = \tilde{y}(\tilde{x}, y_0, \tilde{u}) - y(x, y_0, u)$ is called the trajectory increment of perturbed motion at the kth step. With Δu and $\Delta x, \Delta y$ referred to as the variation of control u and the increments of trajectories of $x(x_0, u)$ and $y(x, y_0, u)$ respectively. It is evident that, by the common properties of continuity $\|\Delta x\| \to 0$ as $\|\Delta u\| \to 0$, and $\|\Delta y\| \to 0$ as $\|\Delta u\| \to 0$ uniformly in $y_0 \in M_0$, $\|\Delta x\| = \max_{k=0,1,\dots,N-1} \|\Delta x(k)\|$, here $\|\Delta x(k)\| = \sqrt{(\Delta x(k), \Delta x(k))}$. The norm Δy and norm Δu are defined in a similar manner.

Let us denote variations of trajectories of the system (1) – (2) as $\delta x(k), \delta y(k)$ at admissible variation of control Δu and given u.

Now we will write down for system (1) – (2) according equations in variations:

$$\delta x(k+1) = \frac{\partial f(k)}{\partial x(k)} \delta x(k) + \frac{\partial f(k)}{\partial u(k)} \Delta u(k), \qquad k = 0,1$$

$$\delta x(k+1) = \frac{\partial f(k)}{\partial x(k)} \delta x(k) + \frac{\partial f(k)}{\partial x(k-1)} \delta x(k-1) +$$

$$\frac{\partial f(k)}{\partial x(k-2)} \delta x(k-2) + \frac{\partial f(k)}{\partial u(k)} \Delta u(k),$$

$$k = 2,\dots,N-1, \qquad (7)$$

$$\delta y(k+1) = \frac{\partial F(k)}{\partial x(k)} \delta x(k) + \frac{\partial F(k)}{\partial y(k)} \delta y(k) + \frac{\partial F(k)}{\partial u(k)} \Delta u(k),$$

$$k = 0,1,$$

$$\delta y(k+1) = \frac{\partial F(k)}{\partial x(k)} \delta x(k) + \frac{\partial F(k)}{\partial x(k-1)} \delta x(k-1) + \frac{\partial F(k)}{\partial x(k-2)} \delta x(k-2)$$

$$+ \frac{\partial F(k)}{\partial y(k)} \delta y(k) + \frac{\partial F(k)}{\partial y(k-1)} \delta y(k-1) + \frac{\partial F(k)}{\partial y(k-2)} \delta y(k-2) +$$

$$\frac{\partial F(k)}{\partial u(k)} \Delta u(k), \qquad k = 2,\dots,N-1. \qquad (8)$$

In this case, $\|\Delta x - \delta x\|$ and $\|\Delta y - \delta y\|$ are infinitesimals of higher order than $\|\Delta u\|$.

Let us consider the mapping of the set $M_{k,u}$ into the set $M_{k,\tilde{u}}$ that is defined by the trajectories (6) emanating from the same points of the set M_0. Denote it by

$$\tilde{y}_k = \tilde{y}(y_k). \qquad (9)$$

Let us write down the Jacobian of this transformation [2]:

$$\det\left(\frac{\partial \tilde{y}_k}{\partial y_k} \right) = 1 + div_y \Delta y(k, y_k) + o(\|\Delta y(k, y_k)\|),$$

where $div_y \Delta y(k, y_k)) = \sum_{i=1}^{m} \frac{\partial \Delta y_i(k, y(k))}{\partial y_i(k)}$.

Furthermore, by the above assumptions, $|div_y \Delta y(k, y_k) - div_y \delta y(k, y_k)| = o(\|\Delta u\|)$ is uniform with respect to $k = 1,2,\dots,N$ and $y_0 \in M_0$.

Similarly [2], it is easy to show, that there take place following relationships:

$$div_y \delta y(k+1) = div_y \delta y(k) +$$

$$J_k^{-1} \left(\frac{\partial J_k}{\partial y(k)} \delta y(k) + \frac{\partial J_k}{\partial x(k)} \delta x(k) + \frac{\partial J_k}{\partial u(k)} \Delta u(k) \right),$$

$$k = 0,1,$$

$$div_y \delta y(k+1) = div_y \delta y(k) + J_k^{-1} \left(\frac{\partial J_k}{\partial y(k)} \delta y(k) + \right.$$

$$\frac{\partial J_k}{\partial y(k-1)} \delta y(k-1) + \frac{\partial J_k}{\partial y(k-2)} \delta y(k-2) + \frac{\partial J_k}{\partial x(k)} \delta x(k) +$$

$$\left. + \frac{\partial J_k}{\partial x(k-1)} \delta x(k-1) + \frac{\partial J_k}{\partial x(k-2)} \delta x(k-2) + \frac{\partial J_k}{\partial u(k)} \Delta u(k) \right),$$

$$k = 2,\dots,N-1. \qquad (10)$$

Taking into account equations (7), (8), (10) and initial values of variations $\delta x(0) = 0, \delta y(0) = 0,$

$div_y \delta y(0) = 0$, and using the methods of investigation of functionals of types (3)-(5), presented at the works [1-2, 4] variations of functionals (3)-(5) (at admissible variation of control Δu) can be represented in the following form:

$$\delta I_1 = \sum_{k=0}^{N-1} \xi^T(k+1) \frac{\partial f(k, x(k), u(k))}{\partial u(k)} \Delta u(k), \quad (11)$$

where $\xi(k)$ is the following auxiliary function:

$$\xi^T(k) = \sum_{i=k}^{m} \xi^T(i+1) \frac{\partial f(i)}{\partial x(k)} + c_k \frac{\partial g_k(x_k)}{\partial x(k)},$$

here $m = \begin{cases} N-1, при & k = N-1, N-2 \\ k+2, при & k = 1,2,\ldots,N-3 \end{cases}$

$k = 1,2,\ldots,N-1,$

with the terminal condition $\xi^T(N) = c_N \dfrac{\partial g_N(x_N)}{\partial x(N)}$,

$\xi(k)$ — n-vector;

$$\delta I_2 = \sum_{k=0}^{N-1} \int_{M_{k,u}} \left(J_j p^T(k+1) \frac{\partial F(k)}{\partial u(k)} + J_k \gamma^T(k+1) \frac{\partial f(k)}{\partial u(k)} + q(k+1) \frac{\partial J_k}{\partial u(k)} + \frac{\partial \varphi_k(x(k), y(k), u(k))}{\partial u(k)} \right) dy_k \Delta u(k), \quad (12)$$

here $q(k), p(k), \gamma(k)$ are the following auxiliary functions:

$$q(k) = J_k q(k+1) + \varphi_k,$$

$$p^T(k) = \sum_{i=k}^{m} \prod_{j=k}^{i-1} J_j q(i+1) \frac{\partial J_i}{\partial y(k)} + \sum_{i=k}^{m} \prod_{j=k}^{i} J_j p^T(i+1) \frac{\partial F(i)}{\partial y(k)} + \frac{\partial \varphi_k(x(k), y(k), u(k))}{\partial y(k)},$$

$$\gamma^T(k) = \sum_{i=k}^{m} \left(\prod_{j=k}^{i-1} J_j q(i+1) \frac{\partial J_i}{\partial x(k)} + \prod_{j=k}^{i} J_j p^T(i+1) \frac{\partial F(i)}{\partial x(k)} + \prod_{j=k}^{i} J_j \gamma^T(i+1) \frac{\partial f(i)}{\partial x(k)} \right) + \frac{\partial \varphi_k(x(k), y(k), u(k))}{\partial y(k)},$$

here $m = \begin{cases} k+2, & k = 1,\ldots,N-3 \\ N-1, & k = N-2, N-1 \end{cases}$,

if $k > i-1$, then let be $\prod_{j=k}^{i-1} J_j = 1$;

with the terminal conditions:

$$p^T(N) = \frac{\partial g(y_N)}{\partial y(N)}, \quad q(N) = g(y_N), \quad \gamma(N) = 0,$$

here $\gamma(k)$ is the n-vector, $p(k)$ is the m-vector, $q(k)$ is the scalar.

The variation of functional (5) has the following form:

$$\delta I = \delta I_1 + \delta I_2. \quad (13)$$

The relationships (11), (12) can be written in the following form

$$\delta I_1 = gradI_1 \Delta u, \; \delta I_2 = gradI_2 \Delta u,$$

where $gradI_1 = \left(\xi^T(1) \dfrac{\partial f(0)}{\partial u(1)}, \ldots, \xi^T(N) \dfrac{\partial f(N-1)}{\partial u(N-1)} \right)$,

and components of $gradI_2$ will be integrals on $M_{k,u}$ of corresponding expressions in brackets in formula (12). Obviously, the sum of these gradients will be the gradient of functional (5).

The representation of functional variations (11)-(13) allows constructing a variety of directional methods of optimization for functionals (3)-(5). In particular, we can use the gradient methods of optimization.

4. CONCLUSION

In this paper we considered the mutual optimization of a particular trajectory and the ensemble of trajectories. Analytical representations for variations of examined functionals were found. Note that we often find necessity of joint optimization in solving various optimization problems, in particular, in the problems of charged particle beam dynamic optimization in linear accelerators [3].

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ОПТИМИЗАЦИЯ В ДИСКРЕТНЫХ СИСТЕМАХ ПРИ МИНИМАКСНОМ КРИТЕРИИ

Е.Д. Котина

Предлагается математическая модель оптимизации программного движения (движения синхронной частицы) и ансамбля возмущенных движений. Рассматриваются минимаксные функционалы, позволяющие оценивать динамику частиц по “наихудшим” частицам.

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Є.Д. Котина

Запропонована математична модель оптимізації програмного руху (руху синхронної частки) і ансамблю обурених рухів. Розглядаються мінімаксні функціонали, що дозволяють оцінювати динаміку часток по “найгіршим” частках.