LOCALIZED ELECTROSTATIC WAVES IN TWO-DIMENSIONALLY NON-UNIFORM MAGNETIZED PLASMA

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The phenomenon of localized electrostatic wave has been investigated in magnetized plasma described by two-dimensional Helmholtz equation. For its description, a formalism has been developed that, like WKB method, exploits smallness of wavelength before the space scale of media non-uniformity. Using the formalism, the localized waves in radially non-uniform plasma cylinder are studied. Besides localized waves with circular trajectories, the waves with elliptic trajectories are found and, their fields are calculated. These calculations are checked by the finite difference simulation. The coincidence between the results obtained with two different methods is noticeable.

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1. ELECTROSTATIC MODE

Many wave phenomena in different areas of physics are described by two-dimensional Helmholtz equation:

$$\Delta\Phi + G(x,y)\Phi = 0\,, \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In the regions where $\mathrm{Re}\,G > 0$ its solutions correspond to propagating wave. One of these phenomena is the electrostatic wave in magnetized low-density plasma. It is described by Poisson equation:

$$\nabla \cdot \hat{\varepsilon} \cdot \nabla\varphi = 0\,. \tag{2}$$

The dielectric tensor of cold low-density plasma in the steady magnetic field directed along z -axis

$$\hat{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix} \tag{3}$$

is diagonal. The only plasma dependent component of the tensor $\varepsilon_z = 1 - \omega_{pe}^2 / \omega^2$ could be negative. If potential has a Fourier harmonic dependence in z direction $\varphi = \Phi(x,y)\exp(ik_z z)$, Eq. (2) could be reduced to Eq. (1) with

$$G = -k_z^2 \varepsilon_z\,. \tag{4}$$

Quantity G is positive when electron plasma frequency is higher than the wave frequency and the absolute value of the phase velocity component along magnetic field ω / k_z exceeds electron thermal velocity.

2. CYLINDRICAL LOCALIZED MODE

The appearance of localized modes we illustrate by a simple one-dimensional example. We investigate Eq. (1) in cylindrical geometry (r,ϕ) with $G = G(r)$ and $\Phi = \Phi_m(r)\exp(im\phi)$. In this case Eq. (1) reduces to the following equation:

$$\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\Phi_m + D\Phi_m = 0\,, \tag{5}$$

where $D = G - m^2 / r^2$. We consider the parabolic plasma density profile. For it

$$G = G_0\left(1 - r^2 / a^2\right). \tag{6}$$

Quantity D is either negative at the interval $r \in (0,a)$ or indefinite. The marginal case is when $G_0 = G_{0g} \equiv 4m^2 / a^2$. In this case D is negative in the domain exempt the point $r = a/\sqrt{2}$. In this point $D = 0$ and $\frac{dD}{dr} = 0$. If G_0 is slightly higher than G_{0g}, the narrow region with $D > 0$ around the point $r = a/\sqrt{2}$ appears. There a standing wave solution of Eq. (5) could exist. It could be found analytically using Teylor expansion of D near the point $r = a/\sqrt{2}$. Neglecting the term $\frac{1}{r}\frac{d}{dr}\Phi_m$, Eq. (5) could be reduced to the parabolic cylinder equation. The field of the first radial number eigenmode reads:

$$\Phi_{m,1} = \exp\left[-\frac{\left(r - a/\sqrt{2}\right)^2}{a^2}\sqrt{G_{0,1}a^2}\right]. \tag{7}$$

The corresponding eigenvalue of G_0 is $G_{0,1} = \frac{4|m|(|m|+1)}{a^2}$ It slightly exceeds G_{0g}. The eigenmode is strongly radially localized if

$$\sqrt{G_{0,1}a^2} >> 1 \text{ or } |m| >> 1\,. \tag{8}$$

It is necessary to mention that condition (8) and the condition of smallness of the terms neglected in Eq. (5) coincide.

The solution found corresponds to the localized in radial direction wave propagating along the cylindrical (guiding) surface $r = a/\sqrt{2}$ in z direction and rotating in azimuthal direction. The existence of localized wave of such type is the result of appearance of two close cut-off points in radial direction. If right cut-off point is the result of plasma density decrease to the periphery of plasma column, the left one could be associated with the curvature of the cylindrical surface.

3. FORMALISM FOR LOCALIZED MODE

We assume that localized wave may exist in two-dimensionally non-uniform plasma and has a guiding surface $y = f(x)$. We introduce coordinates u across the surface and v along it. Below are the implicit formulas for them.

$$x = x_s + u\frac{f'(x_s)}{\sqrt{1+f'^2(x_s)}}, \tag{9}$$

$$y = f(x_s) - u\frac{1}{\sqrt{1+f'^2(x_s)}}, \tag{10}$$

$$v = \int_0^{x_s} \sqrt{1+f'^2(x)}dx \tag{11}$$

where $f' = \frac{df}{dx}$. Coordinate u is the distance between point (x, y) and curve $y = f(x)$ with appropriate sign Coordinate v is the length of the segment of the curve between the initial point and the point (x_s, y_s) (see Fig.1). This coordinate system is orthogonal. In it the guiding surface equation is $u = 0$.

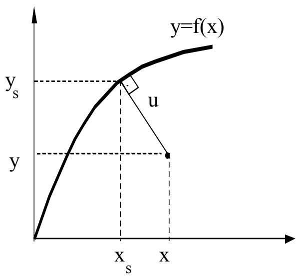

Fig.1. Cross-section of the guiding surface (bold line) by the surface $z = 0$ (trajectory) and coordinates u and x_s

Let us introduce the characteristic length scale a and consider the short-wavelength case $Ga^2 >> 1$. The solution of Eq. (1) for the radial mode number n we write in the following form:

$$\Phi = S_{n-1}\frac{\exp(-gu^2 + i\psi)}{\sqrt{k}}. \tag{12}$$

Here $\vec{k} = \nabla\psi$, g and ψ are functions on v which vary in space with the characteristic space scale a, S_{n-1} is the polynomial in u order of $n-1$ with coefficients dependent on v excluding that one before the lowest degree. It should be even if n is odd and vice versa. As in the above example, we consider the case $n = 1$ in that $S_0 = 1$. After substitution of the expression (12) to Eq. (1) terms of different orders appear. The largest one, order of G or k^2, is $D\Phi$, where

$$D = \mathrm{Re}G - k^2. \tag{13}$$

This term should be nullified at the guiding surface:

$$D\big|_{u=0} = 0. \tag{14}$$

Since the wave is localized in the direction of u coordinate we use Teylor expansion for D. With account of Eq. (14) it reads:

$$D(u,v) = u\frac{\partial D}{\partial u}\bigg|_{u=0} + \frac{u^2}{2}\frac{\partial^2 D}{\partial u^2}\bigg|_{u=0} + ... \tag{15}$$

The first term in (15) has order of $k^{3/2}$ and, being the only term of such order, should be nullified too:

$$\frac{\partial D}{\partial u}\bigg|_{u=0} = 0. \tag{16}$$

Eqs. (14, 16) are enough to determine the guiding surface and value of k on it $k_0 = k\big|_{u=0}$. They used for this purpose in paper [1] directly. Using them and the fact that the wave vector $\vec{k}_0$ is tangent to the trajectory, the cut of the guiding surface by the plane $z = 0$, we obtain the following differential equations for the trajectory:

$$\dot{\vec{r}} \equiv \frac{d\vec{r}}{d\tau} = \vec{k}_0, \tag{17}$$

$$\dot{\vec{k}}_0 = -\frac{k_0\nabla D}{\frac{\partial D}{\partial k}} = \frac{1}{2}\nabla D. \tag{18}$$

The equations for g and ψ could be found when the remainder terms of Eq. (1) order of k are taken into account:

$$\dot{g} = -2ig^2 - \frac{i}{4}\frac{\partial^2 D}{\partial u^2}\bigg|_{u=0}, \tag{19}$$

$$\dot{\psi} = k_0^2 - g + \frac{i}{2}\mathrm{Im}G. \tag{20}$$

The localized wave exists if the term $\frac{\partial^2 D}{\partial u^2}\Big|_{u=0}$ in Eq. (19) is negative. If this term does not depend on τ there is solution of this equation with $\dot{g} = 0$ and $\mathrm{Im}\, g = 0$. As it follows from Eq. (20), purely real g provides corrections to the phase, but not to the amplitude. If $\frac{\partial^2 D}{\partial u^2}\Big|_{u=0}$ vary with τ, real part of g and the wave channel width change. The variation of $\mathrm{Re}\, g$ causes the appearance of the imaginary part of g. Its presence in Eq. (20) provides subsequent change of the wave amplitude.

The remainder terms in Eq. (1) are order of $k^{1/2}$ and k^0. They are disregarded. Therefore, the error in the solution is proportional to $k^{-1/2}$. This error has higher order than for WKB solutions for those the error is proportional to k^{-1}.

4. TWO-DIMENSIONAL LOCALIZED MODES

We use the formalism presented above for studying the electrostatic localized waves in azimuthally symmetric plasma cylinder with G given by expression (6). In this simple case the equations for trajectory (17) and (18) are solvable analytically. The expression for particular set of trajectories is the following:

$$\vec{r} = \vec{e}_x\sqrt{a^2 - r_0^2}\sin\left(\frac{\sqrt{G_0}\tau}{a}\right) - \vec{e}_y r_0\cos\left(\frac{\sqrt{G_0}\tau}{a}\right) \tag{21}$$

Other trajectories could be produced by rotation of these ones around z-axis. The trajectories are elliptic. In the case $r_0 = a/\sqrt{2}$ the trajectory is round. In this case the results of second section could be fully reproduced within the approach developed.

To check the analytical solutions we use finite difference modeling of Eq. (1). For it a boundary problem should be formulated. We choose $x > 0$ half-space as the domain. At $x = 0$ and $y < 0$ we specify the value of Φ that is given by our analytical solution. At $x = 0$ and $y > 0$ at the expected wave field location we use the boundary condition for traveling wave $\left.\left(\frac{\partial \Phi}{\partial y} + ik\Phi\right)\right|_{x=0} = 0$, where the values of k are also taken from analytical solution.

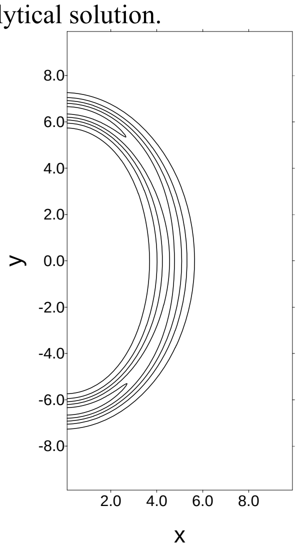

Fig.2. Contours of the module of wave field $|\Phi|$ *for* $a = 8\,cm$, $r_0 = 6.5\,cm$ and $G_0 = 6 \cdot 10^2\ cm^{-2}$

We close boundary of the domain far from the expected wave location so that the character of boundary conditions there influences small enough on the numerical solution. The result of calculation for $a = 8\,cm$, $r_0 = 6.5\,cm$ and $G_0 = 6 \cdot 10^2\ cm^{-2}$ at the mesh containing 200000 points is given in Fig.2. The figure showing the wave amplitude relates both to analytical and numerical calculations because the difference between them is not visible. Only small difference is observed also for the phase of the wave.

CONCLUSIONS

We have investigated the interesting phenomenon of localized wave in the media described by two-dimensional Helmholtz equation. One of such media could be magnetized low-density plasma.

We have developed a formalism that, like WKB method, exploits smallness of wavelength before the space scale of media non-uniformity. This formalism allows one to calculate the distribution of the wave field in media with arbitrary two-dimensional continuous non-uniformity. The accuracy of it is less than that one of WKB approximation. It is proportional to $k^{1/2}$ while the accuracy of WKB is proportional to k.

Using the formalism, the localized waves in radially non-uniform plasma cylinder are studied. Besides localized waves with circular trajectories, the waves with elliptic trajectories are found and, their fields are calculated. These calculations are checked by the finite difference simulation. The coincidence between the calculation results is noticable.

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REFERENCE

1. V.E.Moiseenko, E.Tennfors Off-Axis Global Alfvén Eigenmodes in a Tokamak// *Proc. 1996 Int.Conf. on Plasma Physics (ICPP 96), Nagoya 9-13 Sept., 1996*/ NIFS, 1996, Vol.1, pp.138-141.

ЛОКАЛИЗОВАННЫЕ ЭЛЕКТРОСТАТИЧЕСКИЕ ВОЛНЫ В ДВУМЕРНО НЕОДНОРОДНОЙ ЗАМАГНИЧЕННОЙ ПЛАЗМЕ

В.Е. Моисеенко

Исследована локализованная электростатическая волна в замагниченной плазме, описываемая двумерным уравнением Гельмгольца. Для ее описания разработан формализм, использующий, как и приближение ВКБ, малость длины волны по сравнению с пространственным масштабом неоднородности среды. Разработанный формализм использован для изучения локализованных волн в радиально неоднородном плазменном цилиндре. Кроме волн с круговыми траекториями, найдены волны с траекториями в виде эллипсов и рассчитаны их поля. Эти расчеты верифицированы с помощью конечно-разностного моделирования.

ЛОКАЛІЗОВАНІ ЕЛЕКТРОСТАТИЧНІ ХВИЛІ В ДВОВИМІРНО НЕОДНОРІДНІЙ ЗАМАГНІЧЕНІЙ ПЛАЗМІ

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Досліджено локалізовану електростатична хвилю в замагніченій плазмі, що описується двовимірним рівнянням Гельмгольца. Щодо її описання розроблено формалізм, який використовує, як і метод ВКБ, малість довжини хвилі в порівнянні з просторовим масштабом неоднорідності середовища. Формалізм застосовано до вивчення локалізованих хвиль в радіально неоднорідному плазмовому циліндрі. Окрім хвиль з кільцевими траєкторіями, знайдено хвилі з траєкторіями у вигляді еліпсів та розраховані їхні поля. Ці розрахунки верифіковані за допомогою сіткового моделювання.