EVALUATION OF $1/\nu$ TRANSPORT AND PARALLEL CURRENT DENSITY IN STELLARATORS USING THE INTEGRATION TECHNIQUE ALONG THE MAGNETIC FIELD LINE*

Winfried Kernbichler[1], Victor V. Nemov[2], Sergei V. Kasilov[2], Martin F. Heyn[1]

[1]Institut für Theoretische Physik, Technische Universität Graz
Petersgasse 16, A–8010 Graz, Austria

[2]Institute of Plasma Physics
National Science Center “Kharkov Institute of Physics and Technology”
Ul. Akademicheskaya 1, 61108 Kharkov, Ukraine

1. INTRODUCTION

In the present paper, using an analytic solution of the linearized drift kinetic equation in the long-mean-free-path regime, formulas for neoclassical transport coefficients and for the parallel current density are obtained for stellarator configurations with realistic magnetic field geometry. As in the standard neoclassical theory, for the solution of the linearized drift kinetic equation, the deviation of the distribution function from a Maxwellian is expanded into a series with respect to the collision frequency. The leading order term in this expansion is proportional to $1/\nu$. This leading term is sufficient to obtain the particle and energy fluxes in this regime. In [1], this term is calculated taking into account all classes of trapped particles. Finally, the results are presented in a form containing a line integral along the magnetic field line and an integration over the perpendicular adiabatic invariant of trapped particles.

For the calculation of the parallel current density, also the next term in the expansion over the collision frequency is necessary. In contrast to $1/\nu$ transport, where the contribution of multiply trapped particles within many local magnetic field minima is small, they play an essential role in the formation of the parallel current density. In [2], a method to calculate the bootstrap current is proposed which utilizes Boozer coordinates and which is also based on a line integration along the magnetic field line. Here, this procedure is generalized in two ways: (i) the contribution of trapped particles is taken into account; and (ii) calculations can also be done directly in real space coordinates. As a consequence of (i) also the local current density can be calculated and can be shown to be consistent with results obtained from ideal MHD equilibrium equations. In addition, also the problem with the interpretation of the boundary condition at the trapped-passing boundary in [2] is solved.

The method is very flexible and can be used in cases when only a real space realization of the magnetic field is available as well as in cases when a Boozer representation exists. The second case is especially interesting for stellarator optimization studies because results can be obtained numerically on a very fast time scale.

2. DRIFT-KINETIC EQUATION

The starting point is the linearized drift-kinetic equation in the long-mean-free-path regime with a Lorentz collision operator which describes pitch angle scattering but does not conserve momentum,

$$\sigma \frac{\partial \tilde{f}}{\partial s} + \frac{V^\psi}{|v_\parallel|}\frac{\partial f_M}{\partial \psi} = 4\nu A \frac{\partial}{\partial J_\perp}\left(\frac{|v_\parallel| J_\perp}{B}\frac{\partial \tilde{f}}{\partial J_\perp}\right), \quad (1)$$

where σ is the sign of the parallel velocity, s is the distance measured along the magnetic field line, ψ is the magnetic surface label, $V^\psi = \mathbf{V}\cdot\nabla\psi$ is a radial component of the drift velocity, $v_\parallel^2 = v^2 - J_\perp B$, $J_\perp = v_\perp^2/B$ is the perpendicular adiabatic invariant, B is the magnetic field module, $f_M = f_M(\psi, w)$ is the Maxwellian distribution function, $\tilde{f}$ is the correction to the distribution function according to $f = f_M + \tilde{f}$, $w = mv^2/2 + e\Phi$ is the total energy, Φ is the electrostatic potential, and νA is the pitch-angle scattering frequency. As discussed in [2], for small magnetic field modulations within the magnetic surface, the momentum preserving term will change the resulting value of the average parallel current by a factor which is weakly dependent on the magnetic field geometry and, therefore, can be taken from tokamak theory.

The problem in the interpretation of the boundary condition at the trapped-passing boundary in

---

*This work has been carried out within the Association EURATOM-ÖAW.

[2] where the "last" class of trapped particles cannot be identified, does not appear if instead of an irrational surface one first considers a rational surface. In this case, the number of classes of trapped particles stays finite, and the boundary conditions are clearly defined. Then, the irrational surface can be considered as a limiting case of a "true" rational surface [3] which satisfies the closure condition for the equilibrium currents (Pfirsch-Schlüter),

$$Y_{PS}(L) = 0, \qquad Y_{PS}(s) = \int\limits_0^s \mathrm{d}s' \frac{|\nabla\psi| k_G}{B^2}, \tag{2}$$

where

$$\begin{aligned} k_G &= \nabla\psi \cdot \mathbf{h} \times \nabla \ln B / |\nabla\psi| \\ &= (\mathbf{h} \times (\mathbf{h} \cdot \nabla)\mathbf{h}) \cdot \nabla\psi / |\nabla\psi|, \end{aligned} \tag{3}$$

is the geodesic curvature of the magnetic field line and $\mathbf{h} = \mathbf{B}/B$. With the help of (3) the radial drift velocity can be presented in the form

$$\begin{aligned} V^\psi \equiv \mathbf{V} \cdot \nabla\psi &= (v^2 + v_\parallel^2) \frac{|\nabla\psi| k_G}{2\omega_c} \\ &= -|v_\parallel| \frac{\partial}{\partial J_\perp} \frac{|v_\parallel|}{B} V_G, \end{aligned} \tag{4}$$

where

$$V_G = (v^2 + \frac{1}{3} v_\parallel^2) \frac{|\nabla\psi| k_G}{\omega_c}. \tag{5}$$

The correction $\tilde{f}$ to the distribution function is expanded in a series over the collision frequency

$$\tilde{f} = f_{-1} + g_0 + f_0 + g_1 + f_1 + \ldots, \tag{6}$$

with $f_k, g_k \sim \nu^k$, where f_k is constant and g_k varies along the magnetic field line. Thus, the kinetic equation is split into the series of equations

$$\sigma \frac{\partial g_0}{\partial s} = \frac{\partial}{\partial J_\perp} \left[\frac{|v_\parallel|}{B} \left(4\nu A J_\perp \frac{\partial f_{-1}}{\partial J_\perp} + V_G \frac{\partial f_M}{\partial \psi} \right) \right], \tag{7}$$

$$\sigma \frac{\partial g_{k+1}}{\partial s} = 4\nu A \frac{\partial}{\partial J_\perp} \left[\frac{|v_\parallel| J_\perp}{B} \frac{\partial}{\partial J_\perp} (g_k + f_k) \right], \tag{8}$$

where $k = 0, 1, \ldots$ The requirements for g_k are that these functions are periodic for passing particles,

$$g(s) = g(s + L), \tag{9}$$

where L is the magnetic field field period along the field line, and satisfy

$$g_k(s_0)|_{\sigma=1} = g_k(s_0)|_{\sigma=-1}, \tag{10}$$

in the reflection points $s_0 = s_{\min}$ and $s_0 = s_{\max}$ where $v_\parallel(s_0) = 0$ for trapped particles. The only requirement to f_k is that these functions are independent from σ in the trapped particle region. Conditions (9) and (10) leave freedom for the constant part of g_k along the field line. This freedom is removed by imposing the conditions

$$g_k(s_{\min}) = 0, \tag{11}$$

where $s_{\min}$ is the left reflection point for trapped particles or the point of the global magnetic field maximum on the field line for passing particles. With this, one obtains

$$g_0(s) = \sigma \frac{\partial}{\partial J_\perp} \int\limits_{s_{\min}}^s \mathrm{d}s' \left[\frac{|v_\parallel|}{B} \left(4\nu A J_\perp \frac{\partial f_{-1}}{\partial J_\perp} + V_G \frac{\partial f_M}{\partial \psi} \right) \right], \tag{12}$$

$$g_{k+1}(s) = 4\sigma\nu A \frac{\partial}{\partial J_\perp} \int\limits_{s_{\min}}^s \mathrm{d}s' \left[\frac{|v_\parallel| J_\perp}{B} \frac{\partial}{\partial J_\perp} (g_k + f_k) \right], \tag{13}$$

where $k = 0, 1, \ldots$ One can see that the functions f_k and g_k are symmetric with respect to σ for odd k and antisymmetric for even k. Therefore, in the trapped particle region $f_k = 0$ for even k. Equations for f_k (constraint conditions for g_{k+1}) are obtained from the conditions (9) and (10) for passing particles and for trapped particles, respectively,

$$\frac{\partial}{\partial J_\perp} J_\perp F_k = 0, \tag{14}$$

$$F_{-1} = I \frac{\partial f_{-1}}{\partial J_\perp} + \frac{H}{12\nu A J_\perp} \frac{\partial f_M}{\partial \psi} \tag{15}$$

$$F_k \equiv I \frac{\partial f_k}{\partial J_\perp} + \oint \mathrm{d}s \frac{v_\parallel}{B} \frac{\partial g_k}{\partial J_\perp}, \tag{16}$$

where $k = 0, 1, \ldots$ and

$$I = \oint \frac{\mathrm{d}s}{B} v_\parallel, \qquad H = 3 \oint \mathrm{d}s \frac{v_\parallel}{B} V_G. \tag{17}$$

For passing particles the integrals (17) are defined as

$$\oint \mathrm{d}s v_\parallel A = \int\limits_{s_{\min}}^{s_{\max}} \mathrm{d}s |v_\parallel| A, \tag{18}$$

with $s_{\max} = s_{\min} + L$, whereas for trapped particles they are defined as

$$\oint \mathrm{d}s v_\parallel A = \sum_{\sigma=\pm1} \int\limits_{s_{min}}^{s_{max}} \mathrm{d}s |v_\parallel| A. \tag{19}$$

The boundary conditions for f_k at the boundary between different classes of trapped particles or at the trapped-passing boundary ($J_\perp = J_{b1}$ and $J_\perp = J_{b2}$ in Fig. 1, respectively) are obtained from the integration of (7) and (8) over the area of the boundary layer and subsequent summation over σ.

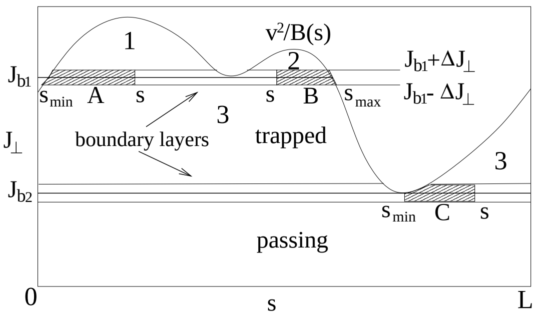

Figure 1: Boundaries between classes, J_{b1} and J_{b2}, and regions of integration in the boundary layer. Regions 1, 2 and 3 correspond to different classes of trapped particles.

Using the condition (10) at the reflection points, one obtains at these boundaries

$$\left[F_k^{(j)}\right]_+ = \left[F_k^{(j)} + F_k^{(j+1)}\right]_- , \qquad (20)$$

$$\sum_{\sigma=\pm 1} \left[F_k^{(1)}\right]_+ = \left[F_k^{(1)}\right]_- , \qquad (21)$$

respectively. Here the superscript (j) enumerates the particle population. Particles have index j if they can pass through the magnetic field minimum j but never reach minimum $j-1$. The minima are numbered starting from the global maximum of the magnetic field on the field line. Subscripts "+" and "−" denote the regions with particle passing through the local magnetic field maximum (region 3 in Fig. 1) and particles reflected in its neighborhood (regions 1 and 2 in Fig. 1), respectively. There exist two further conditions related to the distribution functions f_k, namely that f_k stays finite at the bottom of each magnetic well, $J_\perp = v^2/B_{min}$, where B_{min} is a local minimum of B, and that f_k stays finite at the boundary into the passing region $J_\perp = 0$. This leads at those boundaries to

$$F_k^{(j)} = 0. \qquad (22)$$

Conditions (20), (21) and (22) can be satisfied simultaneously, using (22) in the whole phase space. Thus, one obtains

$$\frac{\partial f_{-1}}{\partial J_\perp} = -\frac{H}{12\nu A J_\perp I}\frac{\partial f_M}{\partial \psi}, \qquad (23)$$

$$g_0(s) = \sigma \frac{\partial f_M}{\partial \psi}\frac{\partial}{\partial J_\perp}\int\limits_{s_{\min}}^{s} \mathrm{d}s' \frac{|v_\parallel|}{B}\left(\frac{H}{3I} + V_G\right). \qquad (24)$$

Function g_0 (24) formally has a δ-singularity at the boundaries between classes and at the trapped passing boundary because H/I has a jump there. One can check that the integration of g_0 across the boundary gives the correct result if one assumes a fast but continuous change of H/I in the boundary layer. Integrating (1) over the boundary layer regions A, B or C (see Fig. 1), depending on the boundary and position of the point s, one obtains in the lowest order the correct jump condition for g_0. Finally one obtains for f_0 in the passing region

$$\frac{\partial f_0}{\partial J_\perp} = -\frac{\sigma}{I}\frac{\partial f_M}{\partial \psi}\int\limits_{s_{\min}}^{s_{\min}+L} \mathrm{d}s \frac{|v_\parallel|}{B}\frac{\partial^2}{\partial J_\perp^2} \times \int\limits_{s_{\min}}^{s} \mathrm{d}s' \frac{|v_\parallel|}{B}\left(\frac{H}{3I} + V_G\right). \qquad (25)$$

2. PARTICLE AND ENERGY FLUXES

The expression (23) for f_{-1} yields particle and energy fluxes in $1/\nu$ regime [1]. The definition of the averaged particle flux density F_n is given as

$$F_n = \frac{\pi}{mS}\oint\limits_\psi \frac{\mathrm{d}S}{|\nabla\psi|} B \sum_{\sigma=\pm1}\int \mathrm{d}w \int \mathrm{d}J_\perp f_{-1}\frac{\mathbf{V}\cdot\nabla\psi}{|v_\parallel|} . \qquad (26)$$

The equivalence of surface and field line averaging

$$\frac{\langle P\rangle}{\langle Q\rangle} = \oint\limits_\psi \frac{\mathrm{d}S}{|\nabla\psi|}P\left(\oint\limits_\psi \frac{\mathrm{d}S}{|\nabla\psi|}Q\right)^{-1} = \lim_{L\to\infty}\int\limits_0^L \frac{\mathrm{d}s}{B}P\left(\int\limits_0^L \frac{\mathrm{d}s}{B}Q\right)^{-1} , \qquad (27)$$

and integration by parts over $J_\perp$ changing the order of integration so that the integral over s becomes the innermost one, integration over energy and reordering of terms, yields

$$F_n = -\frac{\sqrt{8}}{9\pi^{3/2}}\frac{v_T^2\rho_L^2}{\nu R^2}\epsilon_{\text{eff}}^{3/2}\int\limits_0^\infty \frac{\mathrm{d}z\, e^{-z}z^{5/2}}{A(z)}\frac{n}{f_M}\frac{\partial f_M}{\partial r} , \qquad (28)$$

where $w = zT + e\Phi$ and the effective ripple modulation ϵ_{eff} is given by

$$\epsilon_{\text{eff}}^{3/2} = \frac{\pi R^2}{8\sqrt{2}}\lim_{L_s\to\infty}\left(\int\limits_0^{L_s}\frac{\mathrm{d}s}{B}\right)\left(\int\limits_0^{L_s}\frac{\mathrm{d}s}{B}|\nabla\psi|\right)^{-2} \times \int\limits_{B_{min}^{(abs)}/B_0}^{B_{max}^{(abs)}/B_0} \mathrm{d}b' \sum_{j=1}^{j_{max}}\frac{\hat{H}_j^2}{\hat{I}_j} , \qquad (29)$$

with the additional integrals over s

$$\hat{H}_j = \frac{1}{b'}\int\limits_{s_j^{(min)}}^{s_j^{(max)}} \frac{\mathrm{d}s}{B}\sqrt{b' - \frac{B}{B_0}}\left(4\frac{B_0}{B} - \frac{1}{b'}\right)|\nabla\psi| k_G ,$$

$$\hat{I}_j = \int\limits_{s_j^{(min)}}^{s_j^{(max)}} \frac{\mathrm{d}s}{B}\sqrt{1 - \frac{B}{B_0 b'}} . \qquad (30)$$