STRUCTURES OF SOLUTIONS TO THE PROBLEMS FOR MULTI-LAYER MEDIA AND THEIR APPLICATIONS TO THE CALCULATION OF FUEL–ELEMENT THERMAL FIELDS

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The technique of constructing approximate solutions to heat conduction problems of media is presented. The calculation is based on the structures of solutions constructed with the help of the left, normalized-to-the-first-order equations of region boundaries and their characteristic parts. The structures of solutions contain uncertain components, and whatever their choice may be, the boundary conditions and the conditions of medium conjugation are fulfilled exactly. The arbitrary choice of uncertain components is used to satisfy the basic differential equation.

PACS: 61.20.-Ja.

The thermal power of the reactor core is limited, from the thermal viewpoint, by the maximal temperature of fuel and fuel element cladding. If the fuel elements have a finned, intricately-shaped surface, a gas gap between the fuel and the cladding, or a multi-layer structure, the existing methods of solving boundary-value problems give no way of taking into account these limitations [1]. This paper describes the procedure of solving boundary-value problems with mixed boundary conditions and a complicated geometrical shape of fuel elements.

Mathematically, the problem is presented in terms of a boundary-value problem:

$$\lambda_s\left(\frac{\partial^2 T_s}{\partial x^2}+\frac{\partial^2 T_s}{\partial y^2}\right)= \begin{cases} -F(x,y), \ \forall (x,y)\in \Omega_0 \\ 0, \ \forall (x,y)\in \bar{\Omega}_0, \end{cases} \quad s=0,\ 1,\ 2, \tag{1}$$

$$T_0\big|_{\Gamma_0}=T_1\big|_{\Gamma_1}, \quad -\lambda_0\frac{\partial T_0}{\partial \nu_0}\bigg|_{\Gamma_0}=-\lambda_1\frac{\partial T_1}{\partial \nu_0}\bigg|_{\Gamma_0}, \tag{2}$$

$$T_1\big|_{\Gamma_1}=T_2\big|_{\Gamma_1}, \quad -\lambda_1\frac{\partial T_1}{\partial \nu_1}\bigg|_{\Gamma_1}=-\lambda_2\frac{\partial T_2}{\partial \nu_1}\bigg|_{\Gamma_1}, \tag{3}$$

$$-\lambda_2\frac{\partial T_2}{\partial \nu_c}\bigg|_{\Gamma_c}=\alpha_2(T_2-T_c), \tag{4}$$

where $F(x,y)=Q/V_0$ - specific power of heat evolution (wt/m^3, Q – heat evolution power), $V_0=\pi R_0 L=\pi d^2 L/4$ – volume of fuel element, s – number of areas, T_c – ambient temperature.

The proposed approach of constructing approximate solutions of boundary-value problems combines the possibilities of the *R*–function method [2] to take into account the complicated character of boundary conditions (2) – (4) for intricately-shaped fuel elements (Figs. 1, 2) and the properties of exact solutions to allow for the effect of concentrated fuel elements.

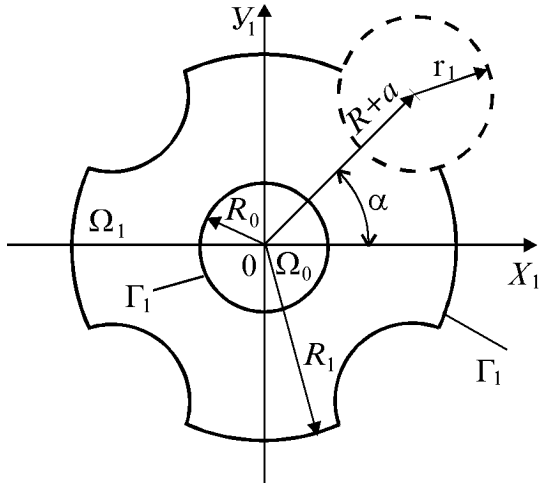

Fig. 1. *Model of a fin-shaped source of an elliptic type*

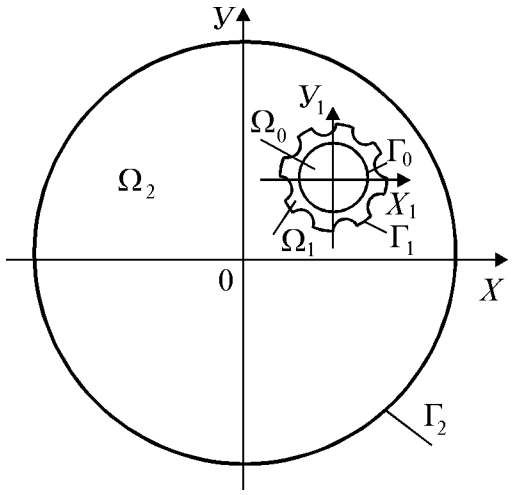

Fig. 2. *Model of the technological cartridge with a fin-shaped source of an elliptic type*

According to the variation principle [3], the boundary-value problem (1) – (4) is equivalent to the variation problem of finding the function $\theta_s(x,y)$ that leads to the minimum of the following functional on the set $D(A)$:

$$I(\theta_s) = \iint_{\Omega_0} \left[\lambda_0 \nabla \theta_0)^2 - 2\theta_0 F \right] d\Omega_0 + \iint_{\Omega_s} \left[\lambda_s \nabla \theta_s)^2 \right] d\Omega_s + \alpha_c \int_{\Gamma_c} \theta_2^2 d\Gamma_c, \quad (5)$$

where $\theta_s = T_s - f_0$ is a new unknown function; f_0 is a certain function satisfying the boundary conditions (2) – (4); s is the region of the i – the boundary.

The functions $\theta_s(x,y)$ (s=0,1,2), minimizing the functional (5), can be represented by structural formulas [4] as

$$\theta_0(x,y) = \Phi_0, \quad (6)$$

$$\theta_1(x,y) = \Phi_0 + \frac{(\omega_1\omega_2)^2\omega_0}{(\omega_1\omega_2)^2 + \omega_0^2}\left(\frac{\lambda_0}{\lambda_1} - 1\right) D_0\Phi_0, \quad (7)$$

$$\begin{aligned}\theta_2(x,y) = {} & \left[1 - \frac{\alpha_2}{\lambda_2}\frac{(\omega_0\omega_1)^2\omega_2}{(\omega_0\omega_1)^2 + \omega_2^2}\right]\Phi_0 + \\ & + \frac{(\omega_0\omega_2)^2\omega_1}{(\omega_0\omega_2)^2 + \omega_1^2}\left[\left(\frac{\lambda_1}{\lambda_2} - 1\right) D_1\Phi_0\right] - \\ & - \frac{\lambda_2}{\alpha_2}\frac{(\omega_0\omega_1)^2}{(\omega_0\omega_1)^2 + \omega_2^2} D_2\Phi_0 + \\ & + \frac{\lambda_2}{\alpha_2}\frac{(\omega_0\omega_1)^2\omega_2}{(\omega_0\omega_1)^2 + \omega_2^2} D_2\left[\frac{(\omega_0\omega_1)^2}{(\omega_0\omega_1)^2 + \omega_2^2} D_2\Phi_0\right].\end{aligned} \quad (8)$$

The equations $\omega_s(x,y)=0$ are the normalized equations of boundaries Γ_s and satisfy the following conditions [2]:

1) $\omega_s(x,y) \in C^2(\Omega_s)$,
2) $\omega_s(x,y) > 0, \ \forall (x,y) \in \Omega_s$,
3) $\omega_s(x,y) = 0, \ \forall (x,y) \in \Gamma_s$,
4) $\left.\frac{\partial \omega}{\partial \nu_s}\right|_{\Gamma_s} = 1$,

where ν_s is the inward normal of the contour Γ_s, bounding the region Ω_s.

For the problem under consideration (Fig. 1) the equations $\omega_s(x,y)$ (s=0,1,2) have the following form

$$\omega_s(x,y) = \frac{R_s^2 - (x - x_s)^2 + (y - y_s)^2}{2R_s},$$

where R_s is the radius of the region Ω_s, and x_s is its center.

For the finned region Ω_1 the equation is written as follows:

$$\omega_1(x,y) = f_1 + f_2 - \sqrt{f_1^2 + f_2^2},$$

where

$$f_1 = \frac{R_1^2 - (x - x_1)^2 - (y - y_1)^2}{2R_1},$$

$$f_2 = \frac{\varphi}{\sqrt{\varphi^2 + \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2}}$$

$$\varphi = \prod_{i=1}^{n} \frac{\varphi_1 + \varphi_2 - r^2}{2r},$$

where

$$\varphi_1 = \left[x - (R + a)\cos\left(\frac{2\pi(i-1) + n\alpha}{n}\right)\right]^2,$$

$$\varphi_2 = \left[y - (R + \alpha)\sin\left(\frac{2\pi(i-1) - n\alpha}{n}\right)\right]^2,$$

n is the number of fins.

At a=0, the center of the circle forming the fin, will lie on the axis $0X$. If a=0, then the centers of circles will be on the circle of radius R_1, i.e., on the boundary Γ_1. The radius r can be used to control the size of fins and consequently their number.

D_s is the differential operator of the form

$$D_s = \frac{\partial \omega_s}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \omega_s}{\partial y}\frac{\partial}{\partial y},$$

possessing the properties

$$D_s\theta_s\big|_{\Gamma_s} = \left.\frac{\partial \theta_s}{\partial \nu_s}\right|_{\Gamma_s},$$

which follow from conditions 1), 4) of equations $\omega_s(x,y)$.

We now show that the structural formulas (6) – (8) exactly satisfy the boundary conditions (2) – (4):

$$\theta_0(x,y) = \Phi_0, \qquad \theta_1(x,y)\big|_{\Gamma_0} = \Phi_0,$$

$$-\lambda_0 \left.\frac{\partial \theta_0}{\partial \nu_0}\right|_{\Gamma_0} = -\lambda_0 D_0\Phi_0,$$

$$-\lambda_1 \left.\frac{\partial \theta_1}{\partial \nu_0}\right|_{\Gamma_0} = -\lambda_0 D_0\Phi_0,$$

$$\theta_1(x,y)\big|_{\Gamma_1} = \Phi_0,$$

$$\theta_2(x,y)\big|_{\Gamma_1} = \Phi_0,$$

$$-\lambda_1 \left.\frac{\partial \theta_1}{\partial \nu_1}\right|_{\Gamma_1} = -\lambda_1 D_1\Phi_0,$$

$$-\lambda_2 \left.\frac{\partial \theta_2}{\partial \nu_1}\right|_{\Gamma_1} = -\lambda_1 D_1\Phi_0,$$

$$-\lambda_2 \left.\frac{\partial \theta_2}{\partial \nu_2}\right|_{\Gamma_2} = \alpha_2 \Phi_0 - \lambda_2 D_2 \Phi_0,$$

$$\alpha_2 T_2|_{\Gamma_2} = \alpha_2 \Phi_0 - \lambda_2 D_2 \Phi_0.$$

The uncertain element $\Phi_0(x,y)$ of structural formulas (6) – (8) can be presented by the expansion [2], [4]

$$\Phi_0(x,y) = \sum_{i=1}^{n} c_i \varphi_i(x,y), \qquad (9)$$

where $\varphi_i(x,y)$ stands for the functions of a linearly–independent system (Chebyshev, Legendre, Hermite polynomials, etc. [5]), c_i denotes the unknown constants.

Substituting (9) into (6) – (8), we obtain

$$\theta_s(x,y) = \sum_{i=1}^{n} c_i \Psi_i^{(s)}(x,y), \quad \forall (x,y) \in \Omega_s, \qquad (10)$$

where n is the number of coordinate functions defined by the formula $n = (\kappa+1)\times(\kappa+2)/2$ (κ being the power of polynomial), $\Psi_i^{(s)}(x,y)$ are the elements of basis systems of functions of types:

$$\Psi_i^{(0)}(x,y) = \varphi_i(x,y),$$

$$\Psi_i^{(1)}(x,y) = \varphi_i + b(h_3 - 1) D_0 \varphi_i,$$

$$\Psi_i^{(2)}(x,y) = (1 - h_4 b_1)\varphi_i + b_2(h_5 - 1) D_1 \varphi_i -$$

$$- h_6 b_3 D_2 \varphi_i + h_6 b_1 D_2 (b_3 D_2 \varphi_i),$$

$$b = \frac{(\omega_1 \omega_2)^2 \omega_0}{(\omega_1 \omega_2)^2 + \omega_0},$$

$$b_1 = \frac{(\omega_0 \omega_1)^2 \omega_2}{(\omega_0 \omega_1)^2 + \omega_2},$$

$$b_2 = \frac{(\omega_0 \omega_2)^2 \omega_1}{(\omega_0 \omega_2)^2 + \omega_1},$$

$$b_3 = \frac{(\omega_0 \omega_1)^2}{(\omega_0 \omega_1)^2 + \omega_2},$$

$$h_3 = \frac{\lambda_0}{\lambda_1},$$

$$h_4 = \frac{\alpha_2}{\lambda_2},$$

$$h_5 = \frac{\lambda_1}{\lambda_2},$$

$$h_6 = \frac{\lambda_2}{\alpha_2}.$$

The unknown constants $c_i (i=1,\ldots,n)$ are calculated from the minimum condition of functional (5)

$\partial I(\theta_s)/\partial c_j = 0$, $i \le j \le n$.

This requirement is equivalent to a set of linear algebraic equations

$$\sum_{i=1}^{n} A_{ij} c_i = \sum_{i=1}^{n} B_j, \quad j = 1, 2, \ldots, n,$$

where A_{ij} and B_j look like

$$A_{ij} = \sum_{s=0}^{2} \iint_{\Omega_s} \left[\frac{\partial \Psi_i^{(s)}}{\partial x} \frac{\partial \Psi_j^{(s)}}{\partial x} + \frac{\partial \Psi_i^{(s)}}{\partial y} \frac{\partial \Psi_j^{(s)}}{\partial y} \right] d\Omega_s +$$

$$+ \alpha_2 \int_{\Gamma_2} \Psi_i^{(2} \Psi_j^{(2)} \, d\Gamma_2, \quad s = 0, 1, 2,$$

$$B_j = -F \iint_{\Omega_0} \Psi_i^{(0)} d\Omega_0.$$

The integration over the regions Ω_s and the boundary Γ_c was performed using the Gauss quadrature formulas [6].

The approximate analytical solution of the boundary-value problem (1) – (4), obtained with the help of the proposed procedure, is written down as

$$T_s(x,y) = \varphi_0^{(s)}(x,y) + \theta_s(x,y), \qquad (11)$$

where $\varphi_0^{(s)}$ are the temperature fields defined by the boundary conditions.

Thus, the method is proposed for solving the boundary-value problem with complicated boundary conditions for conjugation of nonuniform media, and the conditions of heat exchange between the surface of the system and the environment. Structures from (6) to (8), developed for each constituent area, exactly satisfy the conjugation conditions (2) – (4) at the boundaries Γ_s, Γ_{s+1}. They have the properties of the passage to the limit with decreasing distances between the boundaries of contacting media. This is of importance, because with fuel swelling Ω_0 in nuclear fuel elements the clearance between the fuel and cladding is reduced, and the boundaries Γ_0 and Γ_1 are coming together almost in line [7].

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