The Schur ℓ_1 Theorem for Filters

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We study the classes of filters $\mathcal{F}$ on $\mathbb{N}$ such that the weak and strong $\mathcal{F}$-convergence of sequences in ℓ_1 coincide. We study also an analogue of ℓ_1 weak sequential completeness theorem for filter convergence.

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1. Preliminaries

Every theorem of Classical Analysis, Functional Analysis or of the Measure Theory that states a property of sequences leads to a class of filters for which this theorem is valid. Sometimes this class of filters is trivial (say, all filters or the filters with countable base), but in some cases this approach leads to a new class of filters, and the characterization of this class can be a rather non-trivial task. Among these non-trivial examples there are Lebesgue filters (for which the Lebesgue dominated convergence theorem is valid), Egorov filters which

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correspond to the Egorov theorem on almost uniform convergence [7], and those filters $\mathcal{F}$ for which every weakly $\mathcal{F}$ convergent sequence has a norm-bounded subsequence [6].

One of the reasons to study these problems is that they throw a new light on classical results. Say, it is known that the dominated convergence theorem can be deduced from the Egorov theorem. The question, whether the converse is true, has no sense in the classical context: if both the theorems are true, how can we see that one of them is not deducible from the other one? But, if one looks at the correspondent classes of filters, the problem makes sense and in fact there are Lebesgue filters which are not Egorov ones (in particular the statistical convergence filter).

In this paper we study the Schur theorem on the coincidence of weak and strong convergence in ℓ_1 in a general setting when the ordinary convergence of sequences is substituted by a filter convergence. We show that for some filters this theorem is valid and for some is not and give necessary and sufficient conditions (close one to another) for its validity. After that we consider the Schur theorem for ultrafilters. We also study a related problem of weak sequential completeness for filter convergence.

Recall that a *filter* $\mathcal{F}$ on a set N is a nonempty collection of subsets of N satisfying the following axioms: $\emptyset \notin \mathcal{F}$; if $A, B \in \mathcal{F}$, then $A \bigcap B \in \mathcal{F}$; and for every $A \in \mathcal{F}$ if $B \supset A$, then $B \in \mathcal{F}$. All over the paper if the contrary is not stated directly we consider filters on a countable set N. Sometimes for simplicity we put $N = \mathbb{N}$.

A sequence (x_n), $n \in N$ in a topological space X is said to be $\mathcal{F}$-*convergent* to x (and we write $x = \mathcal{F} - \lim x_n$ or $x_n \to_{\mathcal{F}} x$) if for every neighborhood U of x the set $\{n \in N : x_n \in U\}$ belongs to $\mathcal{F}$.

In particular, if one takes as $\mathcal{F}$ the filter of those sets whose complement is finite (the *Fréchet filter*), then $\mathcal{F}$-convergence coincides with the ordinary one.

The natural ordering on the set of filters on N is defined as follows: $\mathcal{F}_1 \succ \mathcal{F}_2$ if $\mathcal{F}_1 \supset \mathcal{F}_2$. If G is a centered collection of subsets (i.e., all finite intersections of the elements of G are nonempty), then there is a filter containing all the elements of G. The smallest filter containing all the elements of G is called *a filter generated by G*.

Let $\mathcal{F}$ be a filter. A collection of subsets $G \subset \mathcal{F}$ is called *the base of $\mathcal{F}$* if for every $A \in \mathcal{F}$ there is a $B \in G$ such that $B \subset A$.

A filter $\mathcal{F}$ on N is said to be *free* if it dominates the Fréchet filter. All the filters below are supposed to be free. Thus every ordinary convergent sequence will be automatically $\mathcal{F}$-convergent.

A maximal in the natural ordering filter is called an *ultrafilter*. The Zorn lemma implies that every filter is dominated by an ultrafilter. A filter $\mathcal{F}$ on N is an ultrafilter if and only if for every $A \subset N$ either A or $N \setminus A$ belongs to $\mathcal{F}$.

A subset of N is called *stationary* with respect to a filter $\mathcal{F}$ (or just $\mathcal{F}$-stationary) if it has nonempty intersection with each member of the filter. Denote the collection of all $\mathcal{F}$-stationary sets by $\mathcal{F}^*$. For an $I \in \mathcal{F}^*$ we call the collection of sets $\{A \cap I : A \in \mathcal{F}\}$ *the trace of $\mathcal{F}$ on I* (which is evidently a filter on I), and by $\mathcal{F}(I)$ we denote the filter on N generated by the trace of $\mathcal{F}$ on I. Clearly $\mathcal{F}(I)$ dominates $\mathcal{F}$. Any subset of N is either a member of $\mathcal{F}$ or the complement of a member of $\mathcal{F}$, or the set and its complement are both $\mathcal{F}$-stationary sets. $\mathcal{F}^*$ is precisely the union of all ultrafilters dominating $\mathcal{F}$. $\mathcal{F}^*$ is a filter base if and only if it is equal to $\mathcal{F}$ and $\mathcal{F}$ is an ultrafilter.

Theorem 1.1. *Let X be topological space, x_n, $x \in X$ and let $\mathcal{F}$ be a filter on N. Then the following conditions are equivalent:*

(1) *(x_n) is $\mathcal{F}$-convergent to x;*

(2) *(x_n) is $\mathcal{F}(I)$-convergent to x for every $I \in \mathcal{F}^*$;*

(3) *x is a cluster point of $(x_n)_{n \in I}$ for every $I \in \mathcal{F}^*$.*

P r o o f. Implications $(1) \Rightarrow (2)$ and $(2) \Rightarrow (3)$ are evident. Let us prove that $(3) \Rightarrow (1)$ do not $\mathcal{F}$-converge to x. Then there is such a neighborhood U of x that in each $A \in \mathcal{F}$ there is a $j \in A \coprod$ such that $x_j \notin U$. Consequently $I = \{j \in \mathbb{N} : x_j \notin U\}$ is stationary and x is not a cluster point of $(x_n)_{n \in I}$. ∎

More about filters, ultrafilters and their applications one can find in most of advanced General Topology textbooks, for example, in [10].

For the standard Banach space terminology we refer to [8]. All the spaces, functionals and operators (although this does not matter) are assumed to be over the field of reals. Before we pass to the main results let us recall some notations and geometric properties of ℓ_1. Denote by e_n the n-th element of the canonical basis of ℓ_1 and by e_n^* the n-th coordinate functional on ℓ_1. In this notations for every $x \in \ell_1$ we have

$$x = \sum_{n \in \mathbb{N}} e_n^*(x) e_n.$$

Recall that e_n are separated from 0 by the functional $f(x) = \sum_{n \in \mathbb{N}} e_n^*(x)$, i.e., 0 is not a weak cluster point of (e_n). The following lemma can be easily extracted from the block-basis selection method (see [8], volume 1). We give the proof for completeness.

Lemma 1.2. *Let $y_n \in \ell_1$, $\inf_{n \in \mathbb{N}} \|y_n\| = \varepsilon_0 > \varepsilon > 0$ and let $\{m(n)\}$ be an increasing sequence of naturals. Denote $z_i = \sum_{k \in (m(i), m(i+1)]} e_k^*(y_i) e_k$. If under these notations $\sup_{n \in \mathbb{N}} \|y_n - z_n\| < \varepsilon/2$ (i.e., (y_n) is a small perturbation of the block-basis (z_n)), then (y_n) is equivalent to the sequence $(\|y_n\| e_n)$ and consequently 0 is not a weak cluster point of (y_n).*

P r o o f. We must find $c_1, c_2 > 0$ such that for every collection of scalars a_n

$$c_1 \sum_{n \in \mathbb{N}} |a_n| \|y_n\| \leq \left\| \sum_{n \in \mathbb{N}} a_n y_n \right\| \leq c_2 \sum_{n \in \mathbb{N}} |a_n| \|y_n\|.$$

The upper estimate with $c_2 = 1$ follows immediately from the triangular inequality. The lower one holds with $c_1 = 1 - \varepsilon_0 / \varepsilon$

$$\left\| \sum_{n \in \mathbb{N}} a_n y_n \right\| \geq \sum_{n \in \mathbb{N}} |a_n| \|z_n\| - \sum_{n \in \mathbb{N}} |a_n| \|y_n - z_n\|$$

$$\geq \sum_{n \in \mathbb{N}} |a_n| \|y_n\| - 2 \sum_{n \in \mathbb{N}} |a_n| \|y_n - z_n\| \geq \left(1 - \frac{\varepsilon}{\varepsilon_0}\right) \sum_{n \in \mathbb{N}} |a_n| \|y_n\|.$$

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2. Simplified Schur Property for Filters

There are several natural ways to generalize the Schur theorem for filters instead of sequences. Let us start with the one leading to a class of filters which we are able to characterize completely in combinatorial terms.

Definition 2.1. *A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be a simple Schur filter (or is said to have the simplified Schur property) if for every coordinate-wise convergent to 0 the sequence $(x_n) \subset \ell_1$ if (x_n) weakly $\mathcal{F}$-converges to 0, then $\mathcal{F} - \lim \|x_n\| = 0$.*

For an infinite set $I \subset \mathbb{N}$ let us call a *blocking* of I a disjoint partition $D = \{D_k\}_{k \in \mathbb{N}}$ of I into nonempty finite subsets.

Definition 2.2. *A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be block-respecting if for every $I \in \mathcal{F}^*$ and for every blocking D of I there is a $J \in \mathcal{F}^*$, $J \subset I$ such that $|J \cap D_k| = 1$ for all k, where the "modulus" of a set stands for the number of elements in the set.*

Remark 2.3.
$$\forall_{k \in \mathbb{N}} |J \cap D_k| \leq 1 \tag{2.1}$$
instead of $|J \cap D_k| = 1$, one will obtain an equivalent definition.

Remark 2.4. *If $\mathcal{F}$ is block-respecting, then $\mathcal{F}(J)$ for every $J \in \mathcal{F}^*$ is also block-respecting.*

Lemma 2.5. *Let $\mathcal{F}$ be a block-respecting filter and let $(x_n) \subset \ell_1$ form a coordinate-wise convergent to 0 sequence, which does not $\mathcal{F}$-converge to 0 in norm. Then there is a $J \in \mathcal{F}^*$, such that the sequence $(x_n), n \in J$ is equivalent to $(a_i e_i)$, where e_i form the canonical basis of ℓ_1, $a_i \geq 1$.*

P r o o f.   Due to Th. 1.1 there is an $I \in \mathcal{F}^*$ such that $\inf_{n \in I} \|x_n\| > \varepsilon > 0$. Fix a decreasing sequence of $\delta_k > 0$, $\sum_{k \in \mathbb{N}} \delta_k \leq \varepsilon/8$. Using the definition of ℓ_1 let us select an increasing sequence of naturals $(m(n))$ and such that for every $n \in \mathbb{N}$

$$\sum_{k \geq m(n)} |e_k^*(x_n)| < \delta_n, \tag{2.2}$$

and using the coordinate-wise convergence of x_n to 0 select an increasing sequence of integers (n_i) such that $n_0 = 0$, $D_i := (n_{i-1}, n_i] \cap I \neq \emptyset$ and for every $i \in \mathbb{N}$ and $j \geq n_{i+1}$

$$\sum_{k \leq m(n_i)} |e_k^*(x_j)| < \delta_i. \tag{2.3}$$

Taking in account the respect which $\mathcal{F}$ has to the blocks D_i let us select a $J = \{j_1, j_2, \ldots\} \in \mathcal{F}^*$, $J \subset I$ such that $j_i \in (n_{i-1}, n_i]$ for all $i \in \mathbb{N}$. Since $J \in \mathcal{F}^*$, either $J_1 = \{j_1, j_3, j_5 \ldots\}$ or $J_2 = \{j_2, j_4, j_6 \ldots\}$ is an $\mathcal{F}$-stationary set as well. Let, say, $J_2 \in \mathcal{F}^*$. Let us show that in fact vectors $y_i = x_{j_{2i}}$ are small perturbations of the block-basis $z_i = \sum_{k \in (m(n_{2(i-1)}), m(n_{2i})]} e_k^*(y_i) e_k$, which due to Lem. 1.2 completes the proof. So,

$$\|y_i - z_i\| = \sum_{k \leq m(n_{2i-2})} |e_k^*(x_{j_{2i}})| + \sum_{k > m(n_{2i})} |e_k^*(x_{j_{2i}})|.$$

Taking into account inequalities (2.2), (2.3) and that $j_{2i} \in (n_{2i-1}, n_{2i}]$, we get $\|y_i - z_i\| \leq 2\delta_{j_{2i}}$ which implies the condition of Lem. 1.2.   ∎

Theorem 2.6. *A filter $\mathcal{F}$ on $\mathbb{N}$ has the simplified Schur property if and only if $\mathcal{F}$ is block-respecting.*

P r o o f.   The "if" part of the theorem follows immediately from Lem. 2.5. So let us turn to the "only if" part. Assume that $\mathcal{F}$ is not block-respecting, i.e., there is an $I \in \mathcal{F}^*$ and there is a blocking D of I such that every $J \subset I$ satisfying (2.1) is not $\mathcal{F}$-stationary. In other words $\mathbb{N} \setminus J \in \mathcal{F}$ for every $J \subset I$ satisfying (2.1). Since the finite intersection of the filter elements again belongs to $\mathcal{F}$, we can reformulate the fact that $\mathcal{F}$ is not block-respecting as follows: there is an $I \in \mathcal{F}^*$ and such a blocking $D = \{D_k\}_{k \in \mathbb{N}}$ of I that $\mathbb{N} \setminus J \in \mathcal{F}$ for every $J \subset I$ satisfying

$$\sup_{k \in \mathbb{N}} |J \cap D_k| < \infty. \tag{2.4}$$

Now, using Dvoretzky's almost Euclidean section theorem, let us select an increasing sequence of integers $0 = m_0 < m_1 < m_2 < \ldots$ and a sequence of vectors

$x_n \in \ell_1$ such that $x_n = 0$ when $n \notin I$; $x_n \in \mathrm{Lin}\{e_k\}_{k \in (m_{i-1}, m_i]}$ when $n \in D_i$, and for every collection of scalars a_n

$$\left(\sum_{n \in D_i} |a_n|^2 \right)^{1/2} \leq \left\| \sum_{n \in D_i} a_n x_n \right\| \leq \left(2 \sum_{n \in D_i} |a_n|^2 \right)^{1/2}. \qquad (2.5)$$

This sequence converges coordinate-wisely to 0 and is not $\mathcal{F}$-convergent to 0 in norm, because $\|x_n\| \geq 1$ for every $n \in I$. Let us prove x_n's weak $\mathcal{F}$-convergence to 0, which will show that $\mathcal{F}$ does not have the simplified Schur property. Well, take an $f \in \ell_1^*$ with $\|f\| = 1$, fix an $\varepsilon > 0$ and consider the set of indexes $A = \{n : f(x_n) < \varepsilon\}$. We must prove that $A \in \mathcal{F}$. Since the complement of A lies in I, it is sufficient to show that $J = \mathbb{N} \setminus A = \{n : f(x_n) \geq \varepsilon\}$ satisfies (2.4). In other words, we must estimate $d_k = |J \cap D_k|$ from above uniformly in k. Let us do this. Consider $y_k = \sum_{n \in J \cap D_k} x_n$. Then $f(y_k) \geq \varepsilon d_k$ and due to (2.5) $\|y_k\|^2 \leq 2d_k$. Hence

$$\varepsilon d_k \leq f(y_k) \leq \sqrt{2d_k}$$

and $d_k \leq 2/\varepsilon^2$. ∎

Remark 2.7. *One can see that in the "only if" part of the Th. 2.6 proof the sequence (x_n) is bounded by $\sqrt{2}$. So, if one restricts the Def. 2.1 to the bounded sequences, the class of filters does not change. In fact this is a little bit surprising because a weakly $\mathcal{F}$ convergent sequence can converge to infinity in norm [6]. If one analyzes the characterization [6] of those "good" filters $\mathcal{F}$ for which every weakly $\mathcal{F}$ convergent sequence has a norm-bounded subsequence, one can see that every simple Schur filter is "good". The only obstacle to see this without refereeing to [6] is the coordinate-wise convergence which appears in Def. 2.1. To see that this obstacle is not fatal one really needs to go to the proofs of [6].*

3. Schur Filters

Let us pass now to the study of the most natural Schur theorem generalization, which is easier to formulate, but is much more complicated to characterize in combinatorial terms.

Definition 3.1. *A filter $\mathcal{F}$ on N is said to be a Schur filter (or is said to have a Schur property) if for every weakly $\mathcal{F}$-convergent to 0 sequence $(x_n) \subset \ell_1$, $n \in N$ the $\mathcal{F} - \lim \|x_n\|$ equals 0.*

Evidently, every Schur filter has the simplified Schur property. By now we do not know if the converse holds true as well.

To simplify the exposition we mostly consider $N = \mathbb{N}$, but the general case cannot differ from this particular one.

Definition 3.2. *$\mathcal{F}$ is said to be a diagonal filter if for every decreasing sequence $(A_n) \subset \mathcal{F}$ of the filter elements and for every $I \in \mathcal{F}^*$ there is a $J \in \mathcal{F}^*$, $J \subset I$ such that $|J \setminus A_n| < \infty$ for all $n \in \mathbb{N}$.*

Lemma 3.3. *If a filter $\mathcal{F}$ on $\mathbb{N}$ is diagonal, then for every $I \in \mathcal{F}^*$ and for every coordinate-wise $\mathcal{F}$-convergent to 0 sequence $(x_n) \subset \ell_1$ there is a $J \in \mathcal{F}^*$, $J \subset I$ such that x_n coordinate-wisely converge to 0 along J.*

P r o o f. Fix a decreasing sequence of subsets U_n, forming a base of neighborhoods of 0 in the topology of coordinate-wise convergence. Define $A_n = \{k \in \mathbb{N} : x_k \in U_n\}$. Since $\mathcal{F}$ is diagonal there is a $J \in \mathcal{F}^*$, $J \subset I$ such that $|J \setminus A_n| < \infty$ for all $n \in \mathbb{N}$. This is the J we desire. ∎

Remark 3.4. *As one can see from the proof, the only property of the coordinate-wise convergence topology we used is the countable base of 0 neighborhoods existence. Also one can easily prove the inverse to the Lem. 3.3 result: if $\mathcal{F}$ is not diagonal, then there is a $I \in \mathcal{F}^*$ and a coordinate-wise $\mathcal{F}$-convergent to 0 sequence $(x_n) \subset \ell_1$ such that for every $J \in \mathcal{F}^*$, $J \subset I$ the sequence (x_n) does not converge coordinate-wisely to 0 along J.*

Let us demonstrate this inverse theorem. By the negation of the diagonality definition a decreasing sequence of $A_n \in \mathcal{F}$ and an $I \in \mathcal{F}^*$ exist such that if $S \subset I$ satisfies the condition $|S \setminus A_n| < \infty$ for all $n \in \mathbb{N}$, then $\mathbb{N} \setminus S \in \mathcal{F}$. Without loss of generality, one may assume that all the $D_n := A_{n+1} \setminus A_n$ are infinite and $\bigcup_n D_n = I$. Then every $J \in \mathcal{F}^*$, $J \subset I$ must satisfy the condition

$$|\{n \in \mathbb{N} : |J \cap D_n| = \infty\}| = \infty. \qquad (3.1)$$

For every $n \in I$ denote by $f(n)$ such index that $n \in D_{f(n)}$. Consider the following sequence (x_n): for $n \in \mathbb{N} \setminus I$ put $x_n = 0$, and for $n \in I$ put $x_n = e_n + e_{f(n)}$. This sequence is the one we need.

Theorem 3.5. *If a filter $\mathcal{F}$ on $\mathbb{N}$ is diagonal and block-respecting, then $\mathcal{F}$ has the Schur property.*

P r o o f. Let $(x_n) \subset \ell_1$ be weakly $\mathcal{F}$-convergent to 0. Arguing "ad absurdum" suppose that there is an $I \in \mathcal{F}^*$ such that

$$\inf_{n \in I} \|x_n\| > \varepsilon > 0. \qquad (3.2)$$

Due to Lem. 3.3 there is a $J \in \mathcal{F}^*$, $J \subset I$ such that x_n coordinate-wisely converge to 0 along J. Since $\mathcal{F}(J)$ is block-respecting (Remark 2.4), the condition (3.2) contradicts Th. 2.6. ∎

It was shown in Th. 2.6 that the block-respect of $\mathcal{F}$ was a necessary condition in order to be a Schur filter. Our next goal is to show that the diagonality of $\mathcal{F}$ is not a necessary condition. To do this define a special filter on N. Let $D = \{D_n\}_{n \in \mathbb{N}}$ be a disjoint partition of N into infinite subsets. For every sequence $C = \{C_n\}_{n \in \mathbb{N}}$ of finite subsets $C_n \subset D_n$ and every $m \in \mathbb{N}$ introduce the set $B_{m,C} = \bigcup_{n=m}^{\infty}(D_n \setminus C_n)$. The sets $B_{m,C}$ form a filter base. Denote the corresponding filter by $\mathcal{F}_D$. One can easily see that $\mathcal{F}_D$ is an example of a non- diagonal block-respecting filter. In fact this filter "almost" appeared in Remark 3.4. To make the picture clearer, we may represent N as an infinite matrix $\mathbb{N} \times \mathbb{N}$, with $D_n = \{(k,n) : k \in \mathbb{N}\}$ being its columns.

Definition 3.6. *A filter $\mathcal{F}$ on N is said to be self-reproducing if for every $I \in \mathcal{F}^*$ there is a $J \in \mathcal{F}^*$, $J \subset I$ such that the structure of the trace of $\mathcal{F}$ on J is the same as of the original filter $\mathcal{F}$, i.e., there is a bijection $s : N \to J$ that maps $\mathcal{F}$ onto its trace on J: $A \in \mathcal{F} \iff s(A) \in \mathcal{F}(J)$.*

Theorem 3.7. *$\mathcal{F}_D$ is a Schur filter, i.e., diagonality is not a necessary condition for the filter's Schur property.*

P r o o f.   First, remark that a subset $J \subset N$ is $\mathcal{F}_D$-stationary if and only if condition (3.1) is met. In particular, for every infinite subset $M \subset \mathbb{N}$ and for every selection of infinite subsets $A_n \subset D_n$, $n \in M$ the set $\bigcup_{n \in M} A_n$ is an $\mathcal{F}_D$-stationary set. Let us call such sets of the form $\bigcup_{n \in M} A_n$ "standard sets". Every $\mathcal{F}_D$-stationary set contains a standard subset. Remark also that the structure of the trace of $\mathcal{F}_D$ on a standard subset J is exactly the same as of the original filter $\mathcal{F}_D$, i.e., $\mathcal{F}_D$ is self-reproducing.

To prove the theorem assume contrary that there is a sequence $(x_n) \subset \ell_1$, $n \in N$ that $\mathcal{F}_D$-weakly converges to zero but the norms do not $\mathcal{F}_D$-converge to zero. So there is an $\varepsilon > 0$ and such a standard set $J \subset N$, that $\|x_n\| \geq \varepsilon$ for all $n \in J$. According to the previous remark, we may assume without loss of generality that $J = N$, i.e $\|x_n\| \geq \varepsilon$ for all $n \in N$. Passing from x_n to $x_n / \|x_n\|$, we may suppose that $\|x_n\| = 1$ for all n. For every fixed $m \in \mathbb{N}$ select a subsequence of $D'_m \subset D_m$ such that $x_n, n \in D'_m$ coordinate-wisely converge to an element $y_m \in \ell_1$. Passing to a new standard set of indexes $\bigcup_{m \in \mathbb{N}} D'_m$, we reduce the situation to the case when $x_n, n \in D_m$ converge coordinate-wisely to y_m for every $m \in \mathbb{N}$.

Notice that due to the weak $\mathcal{F}_D$-convergence to zero of the whole sequence $(x_n), n \in N$, y_m converge coordinate-wisely to zero. In fact, for arbitrary coordinate functional e_k^* and for every $\varepsilon > 0$ there is a set of the form $B_{m,C}$ such that $|e_k^*(x_j)| < \varepsilon$ for all $j \in B_{m,C}$. This means that for $i \in \mathbb{N}$, $i > m$ we have

$$|e_k^*(y_i)| = \lim_{j \in D_i} |e_k^*(x_j)| \leq \varepsilon.$$

This means in its turn the desired coordinate-wise convergence to zero of (y_m).

Introduce some notation: for $n \in N$ denote by $f(n)$ such index that $n \in D_{f(n)}$. Put $z_n = x_n - y_{f(n)}$. Consider two cases. The first one: $\|z_n\| \to_{\mathcal{F}_D} 0$. In this case $\|y_m\| \to 1$ as $m \to \infty$, but on the other hand the condition $y_{f(n)} = x_n - z_n \xrightarrow{w}_{\mathcal{F}_D} 0$ implies ordinary weak convergence of (y_m) to 0, which is impossible according to the Schur theorem. In the remaining case, there is a standard set on which $\|z_n\|$ are bounded from below, so we may again without loss of generality assume that $\|z_n\| > \varepsilon > 0$ for all $n \in N$.

Claim. There is such a standard set $J \subset N$ that the sequence $(z_n)_{n \in J}$ is equivalent to the canonical basis of ℓ_1.

P r o o f o f t h e c l a i m. Fix a decreasing sequence of $\delta_k > 0$, $k \in N$, $\sum_{k \in N} \delta_k \le \varepsilon/8$. Using the definition of ℓ_1 let us select naturals $m(n)$ such that for every $n \in N$ the condition

$$\sum_{k \ge m(n)} |e_k^*(z_n)| < \delta_n$$

holds true. Take an arbitrary $n_1 \in D_1$. Now using consequently the coordinate-wise convergence to 0 of sequences (z_n), $n \in D_m$ for values of $m = 1, 2,\ \ 1, 2, 3,$ $1, 2, 3, 4, \ldots$ select a sequence $(n_i) \subset N$ in such a way that $n_2 \in D_1$, $n_3 \in D_2$, $n_4 \in D_1$, $n_5 \in D_2$, $n_6 \in D_3$, etc. (like triangle enumeration of a matrix) and for every $i \in \mathbb{N}$

$$\sum_{k \le s(i)} |e_k^*(z_{n_{i+1}})| < \frac{\varepsilon}{2^{i+3}},$$

where $s(i)$ denotes $\max_{k \le i} m(n_k)$. Under this construction $J = (n_i)_{i \in \mathbb{N}}$ is a standard set, and z_{n_i} is just a small perturbation of the block-basis

$$w_i = \sum_{k \in (s(i-1), m(n_i)]} e_k^*(z_{n_i}) e_k,$$

which due to Lem. 1.2 means that **the claim is proved**.

Now the last step. Once more without loss of generality assume that $J \subset N$ from the Claim in fact equals N, i.e., $(z_n)_{n \in N}$ is equivalent to the canonical basis of ℓ_1. Then for every bounded sequence of scalars $(a_n)_{n \in N}$ there is a functional $x^* \in \ell_1^*$ such that $x^*(z_n) = a_n$ for all $n \in N$. Select these $a_n = \pm 1$ in such a way that for every $i \in \mathbb{N}$

$$|\{n \in D_i : a_n = 1\}| = |\{n \in D_i : a_n = -1\}| = \infty.$$

Then for the corresponding functional x^* we have for every $i \in \mathbb{N}$

$$\limsup_{n \in D_i} x^*(x_n) - \liminf_{n \in D_i} x^*(x_n) = \limsup_{n \in D_i} x^*(z_n) - \liminf_{n \in D_i} x^*(z_n) = 2,$$

which contradicts the weak $\mathcal{F}_D$-convergence of x_n. ∎

4. Category Respecting and Strongly Diagonal Filters and Ultrafilters

Let us introduce one more class of filters, which are block-respecting and diagonal at the same time.

Definition 4.1. $\mathcal{F}$ *is said to be strongly diagonal if for every decreasing sequence* $(A_n) \subset \mathcal{F}$ *of the filter elements and for every* $I \in \mathcal{F}^*$ *there is a* $J \in \mathcal{F}^*$, $J \subset I$ *such that*

$$|(J \cap A_n) \setminus A_{n+1}| \leq 1 \text{ for all } n \in \mathbb{N}. \tag{4.1}$$

According to Th. 3.5 all strongly diagonal filters have the Schur property.

Definition 4.2. *A filter* $\mathcal{F}$ *on* $\mathbb{N}$ *is said to be category respecting if for every compact metric space* K *and for every family of closed subsets* $(F_A)_{A \in \mathcal{F}}$ *of* K, *if*

$$F_A \subset F_B, \text{ whenever } B \subset A \text{ in } \mathcal{F},$$

and $K = \bigcup_{A \in \mathcal{F}} F_A$, *then* $\text{int}(F_B) \neq \emptyset$ *for some* $B \in \mathcal{F}$.

The obvious examples of category respecting filters are those of countable base. Moreover, every filter with a base of cardinality $k < \mathfrak{m}$ is category respecting (see [5, p. 3–4] for the definition of $\mathfrak{m}$ and Th. 13A, p. 16 for the corresponding result). But the Martin axiom means that $\mathfrak{m}$ equals the cardinality of continuum so, if we accept the Martin axiom together with the negation of the continuum hypothesis, we can go to some filters with the uncountable base.

The proof of the Schur property for ℓ_1 by using the Baire theorem, as presented in [3, Prop. 5.2], gives a hint that the category respecting filters are related to the Schur property. The next theorem shows that in fact to be category respecting is a stronger restriction than to have the Schur property.

Theorem 4.3. *If* $\mathcal{F}$ *is a category respecting filter on* $\mathbb{N}$, *then* $\mathcal{F}$ *is strongly diagonal.*

P r o o f. Assume contrary that $\mathcal{F}$ is not strongly diagonal, i.e., there is a decreasing sequence $(A_n) \subset \mathcal{F}$ of the filter elements and there is an $I \in \mathcal{F}^*$ such that for all $J \in \mathcal{F}^*$, $J \subset I$ the condition (4.1) is not met. Without loss of generality, we may assume that the filter is defined only on I (pass to the trace of $\mathcal{F}$ on I), that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ (this intersection is not stationary, so we may just erase this intersection from I) and that all $D_n := A_n \setminus A_{n+1}$ are not empty. If one picks up a sequence of finite subsets

$$C_n \subset D_n, \ \sup_{n \in \mathbb{N}} |C_n| < \infty, \ \text{ then } \ \mathbb{N} \setminus \bigcup_{n \in \mathbb{N}} C_n \in \mathcal{F}. \tag{4.2}$$

Let us introduce the following compact topological spaces $\widetilde{D}_n$: if D_n is finite, then $\widetilde{D}_n = D_n$ with discrete topology; if D_n is infinite, then $\widetilde{D}_n = D_n \bigcup \{\infty_n\}$ – the one-point compactification of D_n. Recall that $K = \prod_{n \in \mathbb{N}} \widetilde{D}_n$ is compact in coordinate-wise convergence topology and metrizable. Define a family of closed sets $(F_A)_{A \in \mathcal{F}}$ in K as follows:

$$F_A = \{x \in K : \pi_n(x) \in \widetilde{D}_n \setminus A \text{ for all } n \in \mathbb{N}\},$$

where $\pi_n : K \to \widetilde{D}_n$ stands for the n-th coordinate projection. These sets are closed and have empty interior (the interior can be non-empty only if for a sufficiently large m $D_n \cap A = \emptyset$ for all $n \geq m$, which is not the case because $\bigcup_{k > m} D_k = A_m \in \mathcal{F}$). For every $x \in K$ the set $A(x) = \mathbb{N} \setminus \bigcup_{n \in \mathbb{N}} \{\pi_n(x)\}$ is a filter element (due to (4.2)) and $x \in F_A$. So the union of all $(F_A)_{A \in \mathcal{F}}$ equals K. Contradiction. ∎

Corollary 4.4. *If $\mathcal{F}$ is a category respecting filter on $\mathbb{N}$, then $\mathcal{F}$ is a Schur filter.*

Corollary 4.5. *Every filter with a countable base is strongly diagonal.*

Theorem 4.6. *Under the assumption of continuum hypothesis there is a strongly diagonal ultrafilter.*

P r o o f. Denote by Ω the set of all countable ordinals. Let us enumerate as $(I(\alpha), A(\alpha))$, $\alpha \in \Omega$ all the pairs (I, A), where I is an infinite subset of $\mathbb{N}$, and A is a decreasing sequence of infinite subsets of $\mathbb{N}$: $A(\alpha) = (A_n(\alpha))_{n \in \mathbb{N}}$, $\mathbb{N} \supset A_1(\alpha) \supset A_2(\alpha) \dots$. We construct recurrently an increasing family $\mathcal{F}_\alpha, \alpha < \omega_1$ of filters with countable base and an increasing family of sets $\Omega_\alpha \subset \Omega$ as follows: $\mathcal{F}_1$ is the Frechét filter, $\Omega_1 = \emptyset$. If we have an ordinal of the form $\alpha + 1$, we proceed as follows: we find the smallest $\beta \in \Omega \setminus \Omega_\alpha$ such that $I(\beta) \in \mathcal{F}_\alpha^*$ and such that $A(\beta)$ consists of $\mathcal{F}_\alpha$ elements. Applying Cor. 4.5, we find a $J \in \mathcal{F}_\alpha^*$, $J \subset I = I(\beta)$ such that (4.1) holds true for $A_n = A_n(\beta)$. Define $\mathcal{F}_{\alpha+1}$ as the filter generated by $\mathcal{F}_\alpha$ and J, and put $\Omega_{\alpha+1} = \Omega_\alpha \cup \{\beta\}$.

If α is a limiting ordinal, put $\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$ and $\Omega_\alpha = \bigcup_{\beta < \alpha} \Omega_\beta$.

Define the filter $\mathcal{F}$ we need as $\mathcal{F} = \bigcup_{\beta < \omega_1} \mathcal{F}_\beta$. Let us demonstrate that $\mathcal{F}$ is an ultrafilter. To do this we must prove that $\mathcal{F}^* \subset \mathcal{F}$. Let $B \in \mathcal{F}^*$. Then $B \in \mathcal{F}_\alpha^*$ for all α. Let $\beta \in \Omega$ be the smallest ordinal, for which $I(\beta) = B$ and $A(\beta)$ consists of the filter $\mathcal{F}$ elements. Then there is an α, for which all $A_n(\beta)$ belong to $\mathcal{F}_\alpha$. If $\beta \in \Omega_\alpha$ this means that the pair $(I(\beta), A(\beta))$ has appeared in our recurrent construction, and a subset J of B (and hence B itself) was added to the filter. If not, then not later than on the step $\alpha + 1 + \beta$ this pair $(I(\beta), A(\beta))$ has appeared in our recurrent construction and a subset J of B was added to the filter. By the same argument $\mathcal{F}$ is strongly diagonal. ∎

Notice that the diagonality of an ultrafilter $\mathcal{F}$ is equivalent to the following well-known property: $\mathcal{F}$ is a "P-point of $\beta\mathbb{N}$". The consistency of P-points non-existence is a celebrated result of Shelah [11]. So, since every strongly diagonal filter is diagonal some set theoretic assumption is needed for the last theorem. By the way in the setting of ultrafilters a property equivalent to "block-respect", called "Q-point of $\beta\mathbb{N}$" was also studied and the non-existence of Q-points is also known to be consistent [9].

To conclude this section let us present an example of a strongly diagonal filter which is not category respecting. This example resembles strongly the proof of Th. 4.3. Let $D = \{D_n\}_{n\in\mathbb{N}}$ be a disjoint partition of N into infinite subsets. For every sequence $C = \{C_n\}_{n\in\mathbb{N}}$ of finite subsets $C_n \subset D_n$ introduce the set $B_C = \bigcup_{n\in\mathbb{N}}(D_n \setminus C_n)$. The sets B_C form a filter base. Denote the corresponding filter by $\mathcal{F}_d$. A set $J \subset \mathbb{N}$ is $\mathcal{F}_d$-stationary if and only if there is an $n \in \mathbb{N}$ such that $|J \cap D_n| = \infty$. One can easily see that $\mathcal{F}_d$ is strongly diagonal. To show that it is not category respecting consider the same system of subsets $(F_A)_{A\in\mathcal{F}}$ of the same compact K as in the proof of Th. 4.3. The only difference is that now in the definition of K we do not need to consider the case of finite D_n. These sets F_A are closed, they have empty interior, but their union contains all the K, which is impossible if $\mathcal{F}_d$ is category respecting.

5.   Weak Sequential Completeness Theorem for Filters

Definition 5.1. *A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be weak ℓ_1 complete (or in abbreviated form WC1-filter) if for every $\mathcal{F}$-convergent in the topology $\sigma(\ell_1^{**}, \ell_1^*)$ bounded sequence $(x_n) \subset \ell_1$ its weak* limit $x \in \ell_1^{**}$ in fact belongs to ℓ_1.*

It is known that every Banach space with the Schur property is weakly sequentially complete. The next theorem together with Th. 4.6 shows that the picture for filters is more colorful.

Theorem 5.2. *An ultrafilter cannot be weak ℓ_1 complete.*

P r o o f.   Let $\mathcal{F}$ be a (free as always) ultrafilter on $\mathbb{N}$. Consider an arbitrary $f = (f_1, f_2, \ldots) \in \ell_\infty = \ell_1^*$. Then for the canonical basis (e_n) of ℓ_1 we have

$$\lim_{\mathcal{F}} f(e_n) = \lim_{\mathcal{F}} f_n,$$

which shows that the sequence (e_n) weakly* converges to the functional $\lim_{\mathcal{F}}$ on ℓ_∞, which evidently does not belong to ℓ_1.   ∎

To show that a WC1-filter may have no Schur property (and even to be without the simplified Schur property), let us recall some elements of statistical convergence theory [4, 2].

A sequence (x_k) in a topological space X is *statistically convergent to x* if for every neighborhood U of x the set $\{k : x_k \notin U\}$ has natural density 0, where the natural density of a subset $A \subset \mathbb{N}$ is defined to be $\delta(A) := \lim_n n^{-1}|\{k \leq n : k \in A\}|$.

Denote $\mathcal{F}_s = \{I \subset \mathbb{N} : \delta(\mathbb{N} \setminus I) = 0\}$ and remark that $\mathcal{F}_s$ is a filter. As it is easy to see, an $\mathcal{F}_s$-convergence and statistical convergence coincide, and a set J is $\mathcal{F}_s$-stationary provided $\delta(J) \neq 0$.

Recall that a scalar sequence (x_k) is said to be *strongly Cesaro-summable* if there is a scalar x such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |x - x_j| = 0.$$

It is known that a bounded scalar sequence is statistically convergent if and only if it is strongly Cesaro-summable (for a general version of this criterion see [1, Th. 8]). Let us apply this fact.

Theorem 5.3. *$\mathcal{F}_s$ is a WC1-filter but does not have the simplified Schur property.*

P r o o f. Consider the blocking of $\mathbb{N}$ into $D_n = (2^n - 1, 2^{n+1}]$. Every set $J \subset \mathbb{N}$ intersecting each of D_n by no more than one element, has zero natural density and consequently cannot be $\mathcal{F}_s$-stationary. Hence $\mathcal{F}_s$ is not block-respecting and by Th. 2.6 $\mathcal{F}_s$ does not have the simplified Schur property.

Let us show now the weak ℓ_1 completeness of $\mathcal{F}_s$. Let $(x_n) \subset \ell_1$ be a bounded sequence and let weak* $\mathcal{F}_s$-limit of (x_n) be equal to an $x^{**} \in \ell_1^{**}$. This means that for every $f \in \ell_1^*$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |f(x^{**} - x_j)| = 0.$$

Hence the vectors $\frac{1}{n} \sum_{j=1}^{n} x_j$ weakly*-converge to x^{**} as $n \to \infty$. By the ordinary weak sequential completeness of ℓ_1 this means that $x^{**} \in \ell_1$. ∎

Our next goal is to show that if one avoids ultrafilters in a reasonable sense, then the same sufficient condition, which we have for the Schur property, works for the WC1 as well.

Definition 5.4. *A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be a paper filter (p-filter) if all traces of $\mathcal{F}$ on $\mathcal{F}$-stationary subsets are not ultrafilters.*

Theorem 5.5. *If a p-filter $\mathcal{F}$ on $\mathbb{N}$ is diagonal and block-respecting, then $\mathcal{F}$ is a WC1-filter.*

P r o o f.   Let $(x_n) \subset \ell_1$ be a bounded sequence and let $\mathcal{F}$-limit of (x_n) in the topology $\sigma(\ell_1^{**}, \ell_1^*)$ be equal to an $x^{**} \in \ell_1^{**} \setminus \ell_1$. Consider the standard projection $P : \ell_1^{**} \to \ell_1$, which maps every element of ℓ_1^{**} (i.e., a linear functional on ℓ_∞) into its restriction on c_0. Denote $x = Px^{**}$. Without loss of generality, we may assume that $x = 0$: otherwise consider $x_n - x$ instead of x_n. This assumption means that x_n coordinate-wisely converge to 0 with respect to the filter $\mathcal{F}$. Due to Lem. 3.3 there is an $I \in \mathcal{F}^*$, such that x_n coordinate-wisely converge to 0 along I. Since $\mathcal{F}(I)$ is block-respecting (Remark 2.4), we may apply Lem. 2.5 to get such a $J \in \mathcal{F}^* J \subset I$, that the sequence (x_n), $n \in J$ is equivalent to the canonical basis of ℓ_1 (here we also use the boundedness of the sequence). Since $\mathcal{F}(J)$ is not an ultrafilter we can decompose J into two disjoint $\mathcal{F}$-stationary subsets J_1 and J_2. Consider a functional $x^* \in \ell_1^*$ which takes value 1 on all x_n, $n \in J_1$ and is equal to -1 on every x_n, $n \in J_2$. Then

$$1 = \lim_{\mathcal{F}(J_1)} x^*(x_n) = x^*(x^{**}) = \lim_{\mathcal{F}(J_2)} x^*(x_n) = -1.$$

This contradiction completes the proof. ∎

To proceed further let us introduce the sum and the product of filters.

Definition 5.6. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be filters on N_1 and N_2, respectively. Define $\mathcal{F}_1 + \mathcal{F}_2$ as the filter on $N_1 \cup N_2$ consisting of those elements $A \subset N_1 \cup N_2$ that $A \cap N_1 \in \mathcal{F}_1$ and $A \cap N_2 \in \mathcal{F}_2$. The filter $\mathcal{F}_1 \times \mathcal{F}_2$ is defined on $N_1 \times N_2$ with base formed by the sets $A_1 \times A_2$, $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$.*

Definition 5.7. *A filter $\mathcal{F}$ on N is said to have the double Schur property if $\mathcal{F} \times \mathcal{F}$ is a Schur filter.*

Theorem 5.8. *Every filter $\mathcal{F}$ with the double Schur property is a WC1-filter and a Schur filter at the same time.*

P r o o f.   Consider such a bounded sequence $(x_n) \subset \ell_1$ that $\mathcal{F}$-limit of (x_n) in topology $\sigma(\ell_1^{**}, \ell_1^*)$ is equal to $x^{**} \in \ell_1^{**}$. Then the double sequence $(x_n - x_m)$ is weakly $\mathcal{F} \times \mathcal{F}$ convergent to 0. According to the double Schur property of $\mathcal{F}$ this implies that $\|x_n - x_m\| \to_{\mathcal{F} \times \mathcal{F}} 0$, i.e., (due to the completeness of ℓ_1) there is an element $x \in \ell_1$ such that $\|x_n - x\| \to_{\mathcal{F}} 0$. Evidently $x^{**} = x \in \ell_1$. ∎

6.   Domination by Schur and WC1 Filters. Open Problems

Definition 6.1. *A property $\mathcal{P}$ of filters (or a corresponding class of filters) is said to be quasi-increasing if for every $\mathcal{F} \in \mathcal{P}$ all the filters of the form $\mathcal{F}(J)$ for every $J \in \mathcal{F}^*$ have the property $\mathcal{P}$ as well.*

Remark 6.2. *$\mathcal{F}(J)$-convergence to 0 (in arbitrary fixed topology) of a sequence (x_n) is equivalent to $\mathcal{F}$-convergence to 0 in the same topology of the sequence $(x_n \chi_J(n))$. Consequently the properties defined only through convergence to 0 (like the Schur or double Schur properties) are quasi-increasing.*

Definition 6.3. *A property $\mathcal{P}$ of filters is said to be decreasing if for every $\mathcal{F} \in \mathcal{P}$ all the filters dominated by $\mathcal{F}$ have the property $\mathcal{P}$ as well.*

Evidently WC1 filters form a decreasing class. So one can improve Th. 5.8 as follows: every filter dominated by a filter with the double Schur property is a WC1-filter. This is an improvement, because of the following proposition:

Theorem 6.4. *The Schur property, the double Schur property and moreover every nontrivial quasi-increasing property $\mathcal{P}$ of filters are not decreasing.*

P r o o f. Let $\mathcal{F}_1 \in \mathcal{P}$, $\mathcal{F}_2 \notin \mathcal{P}$ be filters on N_1 and N_2, respectively. Then $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$ is a filter on $N_1 \cup N_2$ which cannot have the property $\mathcal{P}$, because $\mathcal{F}(N_2) \notin \mathcal{P}$. On the other hand $\mathcal{F}(N_1) \in \mathcal{P}$ but $\mathcal{F}(N_1)$ dominates $\mathcal{F}$. ■

One can introduce a bit weaker but still reasonable version of the Schur property, which is decreasing:

Definition 6.5. *A filter $\mathcal{F}$ on N is said to be an almost Schur filter (or is said to have the almost Schur property) if for every weakly $\mathcal{F}$-convergent to 0 sequence $(x_n) \subset \ell_1$, $n \in N$ the norms of x_n are not separated from 0 (or in other words 0 is a cluster point for $\|x_n\|$, $n \in N$).*

Theorem 1.1 easily implies that a filter $\mathcal{F}$ on N has the Schur property if and only if all the filters $\mathcal{F}(J)$ for every $J \in \mathcal{F}^*$ are almost Schur filters.

One can also introduce increasing properties:

Definition 6.6. *A property $\mathcal{P}$ of filters is said to be increasing if for every $\mathcal{F} \in \mathcal{P}$ all the filters that dominate $\mathcal{F}$ have the property $\mathcal{P}$ as well.*

Evidently the negation of an increasing property is a decreasing one and contra versa.

Definition 6.7. *Let $\mathcal{P}$ be an increasing (decreasing) class of filters. A class of filters $\mathcal{P}_1 \subset \mathcal{P}$ is said to be a basis for $\mathcal{P}$ if $\mathcal{P}$ is the smallest increasing (decreasing) class, containing $\mathcal{P}_1$.*

The problem, which looks interesting, is to construct explicitly a class of almost Schur filters, which forms a base for the class of all almost Schur filters. The same question makes sense for the negation of property to be almost Schur filter. An analogous study was done in [6] for the class of those filters $\mathcal{F}$ for which the weak $\mathcal{F}$-convergence of a sequence implies the existence of a bounded subsequence.

References

[1] *J. Connor*, Two Valued Measures and Summability. — *Analysis* **10** (1990), 373–385.

[2] *J. Connor*, A Topological and Functional Analytic Approach to Statistical Convergence. — *Analysis of Divergence*, (Orono, ME, 1997); Appl. Numer. Harmon. Anal., Birkhauser, Boston, MA, 1999, 403–413.

[3] *J.B. Conway*, A Course in Functional Analysis. — *Graduate Texts Math.*, **96**. Springer–Verlag, New York, 1985. MR **86h**:46001.

[4] *H. Fast*, Sur la Convergence Statistique. — *Colloq. Math.* **2** (1951), 241–244.

[5] *D.H. Fremlin*, Consequences of Martin's Axiom. — Cambridge Tracts Math., **84**. Cambridge Univ. Press, Cambridge, 1984. MR780933 (86i:03001).

[6] *M. Ganichev and V. Kadets*, Filter Convergence in Banach Spaces and Generalized Bases. General Topology in Banach Spaces (Taras Banakh, Ed.). NOVA Sci. Publ., Huntington, New York, 2001, 61–69.

[7] *V. Kadets and A. Leonov*, Dominated Convergence and Egorov Theorems for Filter Convergence. — *J. Math. Phys., Anal., Geom.* **3** (2007), 196–212.

[8] *J. Lindenstrauss and L. Tzafriri*, Classical Banach Spaces I and II. Springer–Verlag, Berlin, 1996.

[9] *A. W. Miller*, There are no Q-points in Laver's Model for the Borel Conjecture. — *Proc. Amer. Math. Soc.* **78** (1980), No. 1, 103–106.

[10] *S. Todorcevic*, Topics in Topology. — *Lecture Notes Math.* **1652**. Springer–Verlag, Berlin, 1997.

[11] *E.L. Wimmers*, The Shelah *P*-point Independence Theorem. — *Israel J. Math.* **43** (1982), No. 1, 28–48.