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Cosmological scalar fields that mimic the ΛCDM cosmological model

We look for cosmologies with a scalar field (dark energy without cosmological constant), which mimic the standard ΛCDM cosmological model yielding exactly the same large-scale geometry described by the evolution of the Hubble parameter (i.e. photometric distance and angular diameter distance as functions on z). Asymptotic behavior of the field solutions is studied in the case of spatially flat Universe with pressureless matter and separable scalar field Lagrangians; the cases of power-law kinetic term and power-law potential are considered. Exact analytic solutions are found in some special cases. A number of models have the field solutions with infinite behavior in the past or even singular behavior at finite redshifts. We point out that introduction of the cosmological scalar field involves some degeneracy leading to lower precision in determination of Ω_m. To remove this degeneracy additional information is needed besides the data on large-scale geometry.

КОСМОЛОГІЧНІ СКАЛЯРНІ ПОЛЯ, ЩО ІМІТУЮТЬ КОСМОЛОГІЧНУ ΛCDM-МОДЕЛЬ, Жданов В. І., Іващенко Г. Ю. — Досліджено космологічні моделі зі скалярним полем (темна енергія без космологічної сталої), що імітують стандартну космологічну ΛCDM-модель, тобто приводять до тієї ж самої геометрії та еволюції параметра Габбла (з тією ж залежністю фотометричної відстані та відстані за кутовим діаметром від z). Вивчено асимптотичну поведінку розв'язків у випадку просторово-плоского Всесвіту з беззіткнювальною матерією і різними лагранжіанами скалярного поля. Розглянуто випадки степеневого кінетичного члена та степеневого потенціалу. Знайдено аналітичні розв'язки для деяких спеціальних випадків. Низка моделей мають розв'язки з нескінченною поведінкою в минулому або навіть сингулярністю на скінченних червоних зміщеннях. Відзначено, що введення космологічного скалярного поля призводить до виродження і меншої точності у визначенні параметра Ω_m. Для усунення цього виродження необхідна додаткова інформація, окрім даних про геометрію Всесвіту.

КОСМОЛОГИЧЕСКИЕ СКАЛЯРНЫЕ ПОЛЯ, ИМИТИРУЮЩИЕ КОСМОЛОГИЧЕСКУЮ ΛCDM-МОДЕЛЬ, Жданов В. И., Иващенко А. Ю. — Исследованы космологические модели со скалярным полем (темная энергия без космологической постоянной), которые имитируют стандарт-

ную ΛCDM космологическую модель, с той же геометрией и эволюцией параметра Хаббла (с той же зависимостью фотометрического расстояния и расстояния по угловому диаметру от z). Изучено ассимптотическое поведение решений в случае пространственно плоской Вселенной с бесстолкновительной материей и различными лагранжианами скалярного поля. Рассмотрены случаи степенного кинетического члена и степенного потенциала. Найдены аналитические решения для некоторых частных случаев. Ряд моделей имеют решения с бесконечным поведением в прошлом или сингулярностью для конечных красных смещений. Отмечается, что введение космологического скалярного поля приводит к некоторому вырождению и меньшей точности в определении параметра Ω_m. Чтобы устранить это вырождение, необходима дополнительная информация, помимо данных о геометрии Вселенной.

INTRODUCTION

Standard ΛCDM cosmological model explains the wealth of experimental data on CMB anisotropy and Ia type supernovae [3, 8, 13], though some facts require modification of the dark matter equation of state [6, 11] without any revision of the cosmological constant. On the other hand, theoretical and experimental developments necessitate modification of the Standard model leading to ideas of inflation, cosmological fields, branes in extra dimensions etc (see, e. g., [10, 11, 14, 15]. Introduction of scalar fields as a key element of the dark energy seems to be one of the most simple and natural ways to launch primordial inflation and to explain the cosmological coincidences [10, 16]. As soon as the scalar field was introduced in cosmology, the question about reconstruction of the field Lagrangian from observational data arose [1, 20]. A number of interesting examples have been considered, which establish some correspondence between different models that may be used to explain observational data on equal terms (see [1, 7, 16, 17, 20]) and references therein). The solution of this problem appears to be unstable: small errors in the experimental data lead to considerable changes in the potential. Introduction of non-canonical kinetic term like k-essence models [4, 5, 9, 12, 18] obviously creates additional ambiguities in reconstruction of the scalar field Lagrangian.

Introduction of the additional field may lead to a revision of some cosmological parameters, even if we retain the large-scale geometry of the FRW Universe described by the redshift dependence of the Hubble parameter $H(z)$. Present constraints on Ω_Λ, Ω_m within ~2—3 % accuracy [3, 8, 13] rely upon measurements of $H(z)$ within the framework of the ΛCDM cosmological model, which use the data on Ia type supernovae magnitude-redshift dependence and WMAP data that restrict the position of the first peak in the CMB anisotropy power spectrum. The redshift-space distortions used to constrain the cosmological parameters from LSS surveys (see, e.g., [2, 7]) also have geometrical origin. When additional degrees of freedom due to the scalar field are introduced, the same $H(z)$ dependence as in case of the ΛCDM cosmological model might be preserved. At the same time this leads to a reduction of Ω_m if some independent information on this parameter is used. Of course, there are the other ways to study the dark matter content (galactic rotation curves, virial mass estimates, gravitational lensing), however they determine Ω_m with much lower accuracy. If the scalar field is introduced leaving somehow the large-scale geometry unchanged, we cannot separate its contribution into the cosmological density from the other forms of matter with the same precision as in the ΛCDM model.

We keep in mind this problem in the present paper dealing with reconstruction of the scalar field Lagrangians that yield exactly the Hubble diagram of the standard ΛCDM model.

BASIC EQUATIONS

Here we present the basic relations of the scalar field cosmology. Details may be found in [4, 9, 14, 16]. We confine ourselves to the case of the critical cosmological density, which means that the space-time metric is spatially flat [2]

$$ds^2 = dt^2 - a^2(t)[d\chi^2 + \chi^2(d\theta^2 + \sin^2(\theta)d\varphi^2)], \tag{1}$$

($c = 1$), the scale factor a can be related to the redshift z:

$$1 + z = a(t_0)/a(t). \tag{2}$$

The observable photometric distance is

$$D_{ph}(z) = (1+z)\int_0^z [H(\zeta)]^{-1} d\zeta,$$

where $H(z) = \dot{a}(t)/a(t)$ is the Hubble parameter at the cosmological epoch t. Therefore one may consider $H(z)$ as an experimentally measurable function yielding $a(t)$ up to an unessential constant factor.

We consider a cosmological model, where the main contribution to the cosmological density is due to the scalar field plus the matter with zero pressure $p = 0$. The mass density ρ_m of the pressureless matter varies as a function of the redshift as follows:

$$\rho_m = (1+z)^3 \Omega_m \rho_{cr}, \tag{3}$$

where $\rho_{cr} = 3H_0^2/(8\pi G)$, $\Omega_m = \rho_m(t_0)/\rho_{cr}$, $H_0 = H(0)$ being the modern value of the Hubble parameter and t_0 is the modern epoch. The scalar field Lagrangian

$$L = L(S, \varphi), \quad S = \frac{1}{2} g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi \tag{4}$$

leads to the Friedmann cosmological equations, which in case of spatially flat Universe are

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + \dot{\varphi}^2\frac{\partial L}{\partial S} + 2L\right), \tag{5}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_m + \dot{\varphi}^2\frac{\partial L}{\partial S} - L\right), \tag{6}$$

$$S \equiv \frac{\dot{\varphi}^2}{2}.$$

Taking (3) into account we have $dz/dt = -(1+z)H(z)$. This enables us to rewrite (5), (6) in terms of observable quantities z, H:

$$S\frac{\partial L}{\partial S}=\frac{(1+z)}{16\pi G}\frac{dH^2}{dz}-\frac{\Omega_m}{2}(1+z)^3\rho_{cr}, \tag{7}$$

$$L(S,\varphi)=\frac{(1+z)^4}{8\pi G}\frac{d}{dz}\left[\frac{H^2}{(1+z)^3}\right], \tag{8}$$

$$S=\frac{(1+z)^2}{2}H^2\left(\frac{d\varphi}{dz}\right)^2. \tag{9}$$

SCALAR FIELD MODEL YIELDING HUBBLE DIAGRAM OF ΛCDM MODEL

At present the ΛCDM cosmological model is in a good agreement with all the observational data having relevance to the space-time geometry. In case of spatially flat cosmological model with pressureless matter (without the scalar field) the cosmological equations yield

$$H^2(z)=H_0^2h^2(z),$$
$$h(z)\equiv[\Omega_m^0(1+z)^3+1-\Omega_m^0]^{1/2}, \tag{10}$$

where the values $H_0\approx 72$ km$\cdot$s^{-1}Mpc^{-1}, $\Omega_m^0\approx 0.3$ are obtained by fitting the data with the standard ΛCDM model [3, 8, 13]. Note that in presence of the scalar field the real content Ω_m of the pressureless matter in the cosmological density will be less than Ω_m^0.

Our first step will be to investigate Eqs. (7)—(9) in case of the dependence (10) that will be regarded as the "observational" one. We shall present the examples of Lagrangian that lead to the same dependence (10).

The equations (7)—(9) may be viewed as observational restrictions on the function $L(S,\varphi)$. Obviously, they do not fix this function in a unique way. We shall consider less general Lagrangian

$$L=F(S)-V(\varphi), \tag{11}$$

with subsequent specific choice either of the kinetic term F or the potential V. The restrictions on these functions on account of (7), (8) take on the form

$$S\frac{dF}{dS}=\frac{\rho_{cr}}{2}(1+z)^3(\Omega_m^0-\Omega_m), \tag{12}$$

$$V(\varphi)=F(S)+\rho_{cr}(1-\Omega_m^0). \tag{13}$$

We see from (12) that for an increasing $F(S)$ one must require that $\Omega_m<\Omega_m^0$: the scalar field "eats" part of the cosmological density. For $dF/dS<0$ we have $\Omega_m>\Omega_m^0$. Further we put for definiteness $d\varphi/dz>0$.

Quintessence: canonical kinetic term. For certain known kinetic term $F(S)$ the field $\varphi(z)$ may be obtained from (12); then $V(\varphi)$ is determined parametrically. Now we proceed to concrete examples.

In case of the standard kinetic term $F(S)\equiv S$ Eqs. (12), (13) are easily solved to yield

$$\varphi(z)=\left[\frac{3(\Omega_m^0-\Omega_m)}{8\pi G}\right]^{1/2}\int_0^z d\zeta\left[\frac{(1+\zeta)}{\Omega_m^0(1+\zeta)^3+1-\Omega_m^0}\right]^{1/2}+\varphi(0), \tag{14}$$

$$V(\varphi) = \frac{\rho_{cr}}{2}\,[(1+z)^3(\Omega_m^0 - \Omega_m) + 2(1 - \Omega_m^0)] \tag{15}$$

Note that at present epoch $dV/d\varphi \neq 0$ ($z = 0$).

For $z \gg 1$ we have an exponential growth of $V(\varphi)$

$$V(\varphi) \approx \frac{\rho_{cr}}{2}(\Omega_m^0 - \Omega_m)\exp\left[\left(\frac{24G\Omega_m^0}{\Omega_m^0 - \Omega_m}\right)^{1/2}\varphi\right], \tag{16}$$

but the field grows only logarithmically as a function of the redshift. In case of small deviation of Ω_m from Ω_m^0 the contribution of $V(\varphi)$ remains small for all z in comparison with the cold matter energy density. More general problem of Lagrangian reconstruction including non-spatially-flat case has been considered in [7].

K-essence: F is given, find V. Now we consider equations (12), (13) with a kinetic term of the form

$$F(S) = (a|S|^\alpha + b)^\beta, \tag{17}$$

for definiteness we consider positive a, b, α, β. The left-hand side of (12) is a monotonous function, so S and therefore $d\varphi/dz > 0$ is uniquely defined from (12). This gives the monotonous function $\varphi(z)$ up to the additive constant. The potential $V(\varphi)$ also turns out to be a monotonous function. We denote $\gamma = \alpha\beta$.

For $\gamma < 1$ the field asymptotic behavior is

$$\varphi(z) \sim \frac{1}{1-\gamma}(\Omega_m^0 - \Omega_m)^{1/(2\gamma)}\, z^{3(1-\gamma)/(2\gamma)} + O(1)$$

as $z \to \infty$,

$$V(\varphi) \sim \frac{1}{(\Omega_m^0 - \Omega_m)^{\gamma/(1-\gamma)}}\,\varphi^{2\gamma/(1-\gamma)}, \qquad \varphi \to \infty.$$

For $\gamma = 1$ the potential has exponential behavior like (16). For $\gamma > 1$ the scalar field is bounded $\varphi(z) \to \varphi_1$ for $z \to \infty$, where $\varphi_1 < \infty$, and $V(\varphi) \to \infty$, $\varphi \to \varphi_1$.

K-essence: V is given, find F. First of all we note that in case of the constant potential $V(\varphi) = V_0$, for any non-trivial dependence $F(S)$, it follows from (13) that S = const. This is incompatible with (12) unless the field is constant, $S = 0$ and $\Omega_m = \Omega_m^0$.

Consider the power-law potential

$$V(\varphi) = A\varphi^n. \tag{18}$$

Then Eq. (13) takes on the form

$$F(S) = \rho_{cr}[(\Omega_m^0 - \Omega_m)\psi^n - (1 - \Omega_m^0)], \tag{19}$$

where $\psi = \varphi/\alpha$, $\alpha = [(\Omega_m^0 - \Omega_m)\rho_{cr}/A]^{1/n}$. After differentiation of (19) with respect to z and combining the result with (12) we have

$$(1+z)^3\frac{d}{dz}\left\{\ln\left[(1+z)h(z)\frac{d\psi}{dz}\right]\right\} = n\psi^{n-1}\frac{d\psi}{dz}, \tag{20}$$

In case of the linear potential ($n = 1$) Eq. (20) leads to a first order linear differential equation with respect to $(d\psi/dz)^{-1}$ yielding an explicit solution

$$\psi(z) = C_1 + \int_0^z dz'\left\{(1+z')h(z')\left[C_2 - \int_0^{z'}\frac{dz''}{(1+z'')^4 h(z'')}\right]\right\}^{-1}, \tag{21}$$

C_1, C_2 are arbitrary constants. Then

$$S=\frac{\alpha^2 H_0^2}{2}\left[C_2-\int_0^z\frac{dz'}{(1+z')^4 h(z')}\right]^{-2}. \tag{22}$$

Equations (19), (22) define F parametrically.

Denote

$$I_0=\int_0^\infty\frac{dz'}{(1+z')^4\,h(z')}\,.$$

— (i) In case of $C_2 > I_0$ the function $S(z)$ is monotonically increasing on z in $(-1,\ \infty)$; it varies within the interval $(0,\ S_{max})$, $S_{max} = (\alpha^2 H_0^2/2)\,[C_2 - I_0]^{-2}$. Therefore F and φ may be defined as single-valued functions of S only on $(0,\ S_{max})$ and we are free for arbitrary choice of $F(S)$ for $S > S_{max}$.

— (ii) In case of $C_2 = I_0$ we have $\varphi(z) \sim z^3$ and $F(S) \sim S^{1/3} \sim z^3$ for $S \to S_{max} = \infty$.

— (iii) If $0 < C_2 < I_0$, then there is a field singularity $\varphi \sim \ln(z_1 - z)$, $F(S) \sim \ln(S)$ for some $z \to z_1$ and, accordingly, singularity of the energy density. The solution of the system (11), (19) cannot be extended for all z in the past. Analogous situation occurs in the future for $C_2 < 0$.

In a general case $n > 0$ it follows from (11) that in a regular point of the function $F(S)$ the derivative $d\varphi/dz$ cannot be zero. Equation (20) may be written as

$$\frac{(1+z)^3}{2S}\frac{dS}{dz}=n\psi^{n-1}\frac{d\psi}{dz}\,,$$

whence $dS/dz \neq 0$; so $S(z)$ and $\varphi(z)$ are monotonous functions and there exists a single-valued inverse function $z(S)$ on some interval. Depending of the initial conditions for (20) the following cases are possible for solutions of (20) for $z > 0$:

— (i) $S(z)$ is bounded for all z;

— (ii) $S(z) \to \infty$ as $z \to \infty$, in this case there is a solution having power-law asymptotics $\psi(z) \sim \left(\frac{1}{n}+\frac{1}{2}\right)^{1/n} z^{3/n}$, $z \to \infty$;

— (iii) $S(z) \to \infty$, $\varphi(z) \to \infty$ as $z \to z_1$ for some $z_1 < \infty$. In this case (20) yields

$$(1+z_1)^3\frac{d}{dz}\left[\ln\left(\frac{d\psi}{dz}\right)\right]=\frac{d\psi^n}{dz}\,,$$

where we have omitted the bounded terms that are not essential for asymptotical behavior of ψ for $z \to z_1$. After substitution $\psi = (1 + z_1)^{3/n}\xi^{1/n}$ (neglecting an integration constant) we have

$$\ln\left(\frac{\xi^{(1-n)/n}}{n}\frac{d\xi}{dz}\right)=\xi,$$

whence

$$\int_z^\infty \xi^{(1-n)/n}e^{-\xi}\,d\xi=C_3(z_1-z),$$

C_3 is an integration constant.

The asymptotic expansion of the left-hand side integral in powers of ξ^{-1} is

$$\int_z^\infty \xi^{(1-n)/n} e^{-\xi}\, d\xi = \xi^{(1-n)/n} e^{-\xi}\left(1 + \frac{1-n}{n\xi} + \frac{(1-n)(1-2n)}{n^2\xi^2} + \ldots\right),$$

whence we get

$$\psi(z) = [-(1+z_1)^3 \ln(z_1 - z)]^{1/n}\left\{1 + O\left[\frac{\ln(\ln(z_1 - z))}{\ln(z_1 - z)}\right]\right\}.$$

DISCUSSION

We considered the FRW cosmological models without Λ-term but with the scalar field and the pressureless matter in case of spatially flat Universe. We study the scalar field Lagrangians $L = F(S) - V(\varphi)$ yielding the same dependence $H(z)$ as in the ΛCDM model, which provides the same dependence of photometric distance and angular diameter distance upon redshift. The $H(z)$ dependence restricts the potential $V(\varphi)$ up to integration constants, provided that the kinetic term $F(S)$ be given, and vise versa. The exact analytic expressions in parametric form are presented for V if $F(S) \equiv S$; and for F in case of the linear potential $V(\varphi)$.

If the scalar field is introduced, then some part of cosmological density Ω_{field} will be due to this field. If Ω_m^0 (the dark matter content) is derived within the standard model on account the data on large-scale geometry, then introduction of the scalar field leads to the relation $\Omega_m + \Omega_{field} = \Omega_m^0$ and therefore to some change of the real content of the dark matter. Therefore Ω_m and Ω_{field} cannot be separated on account of purely geometric data without additional information constraining these parameters separately. In fact the same dependencies of the Hubble parameter, photometric distance and angular diameter distance upon redshift may take place for different relations of Ω_m and Ω_{field}. This degeneracy can be removed using independent data about Ω_m, which, e.g., are related to the large-scale structure and/or the whole CMB anisotropy spectrum, however in this case the accuracy of determination of Ω_m will be worse in comparison with pure ΛCDM model. It should be pointed out that this degeneracy is due to the unknown Lagrangian; this problem does not arise if we specify the appropriate Lagrangian form up to some parameter set; e.g., in the case of a canonical kinetic term and $V(\varphi) = \alpha\varphi^2 + \beta\varphi^3 + \gamma\varphi^4$ the model parameters can be constrained uniquely (within experimental errors) from the observed $H(z)$ dependence. On the other hand, it would be highly improbable that realistic Lagrangian must have exactly one of the forms describes in Section 3 so as to mimic the ΛCDM model. However, a rigorous approach must rule out such possibility as well.

The examples considered above when the kinetic term is specified show the infinite behavior of the field in the past though the relative content of the field energy density remains typically of the same order as at the present epoch ($\sim \Omega_m^0 - \Omega_m$). The other set of examples shows singular behavior for finite z and thus suggests possibility of new physical situation (phase transitions?) in the early Universe.

At the end we note that though the above considerations deal with the case of spatially flat Universe and pressureless matter, main qualitative aspects (degeneracy in determination of Ω_m and existence of singular behavior in the past) will remain in a more general case dealing with more general dark matter equation of state or $H(z)$ dependence.

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