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FD-METHOD FOR A NONLINEAR EIGENVALUE PROBLEM WITH DISCONTINUOUS EIGENFUNCTIONS

FD-МЕТОД ДЛЯ НЕЛІНІЙНОЇ ЗАДАЧІ НА ВЛАСНІ ЗНАЧЕННЯ З РОЗРИВНИМИ ВЛАСНИМИ ФУНКЦІЯМИ

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An algorithm for solution of a nonlinear eigenvalue problem with discontinuous eigenfunctions is developed. The numerical technique is based on a perturbation of the coefficients of differential equation combined with the Adomian decomposition method for the nonlinear term of the equation. The proposed approach provides an exponential convergence rate dependent on the index of the trial eigenvalue and on the transmission coefficient. Numerical examples support the theory.

Розроблено алгоритм для чисельного розв'язування нелінійних задач на власні значення з розривними власними функціями. В основі чисельного методу лежить збурення коефіцієнтів диференціального рівняння в поєднанні з методом декомпозиції Адомяна нелінійної частини рівняння. Запропонований підхід забезпечує експоненціальну швидкість збіжності, яка залежить від порядкового номера власного значення та коефіцієнта трансмісії. Наведені чисельні розрахунки підтверджують теоретичні висновки.

1. Introduction. A functional-discrete method (FD-method) to find numerical solution of boundary-value problems for linear differential equations and, in particular eigenvalue problems, was proposed by V. L. Makarov, in [1 – 3]. The idea of the approach is to approximate the original problem by a recursive sequences of problems with the same differential operator with piecewise constant coefficients and varying right-hand part of the equation dependent on the solutions of previous problems in the recursive sequence. Such a method provides an exponential convergence rate which improves when the eigenvalue index increases. In [4 – 9] this technique is developed for different kinds of boundary conditions. Particularly, in [8, 9] the authors study a linear eigenvalue problem with discontinuous eigenfunctions, or in other words, the eigenvalue transmission problem. Based on the numerical FD-method they establish a qualitative result about dependence of the eigenvalue arrangement on the transmission conditions. In [10] the approach above described, combined with the Adomiane decomposition method [11], is developed for a numerical solution of a nonlinear Sturm – Liouville problem, for which a unique solvability result and basic properties of eigenfunctions are established in [12] and the literature cited therein. In [10] it is shown that for a nonlinear eigenvalue problem, the algorithm based on the FD-approach converges with the same (exponential) characteristics as the algorithms for the linear problems.

In this paper we apply the approach from [10] to develop a numerical algorithm for the nonlinear eigenvalue transmission problem. The technique also provides an exponential convergence rate. However, unlike Dirichlet boundary conditions [10], as the index of the eigenvalue increases, it tends to a constant defined by the transmission coefficient. We also study the influence of the transmission coefficient, matching point, and normalizing conditions on convergence of the algorithm and an arrangement of the eigenvalues.

The paper is organized in the following way. In Section 2 we describe a numerical technique for nonlinear eigenvalue transmission problem with an additional differential normalizing condition (vanishing the first derivative). We prove a convergence theorem which shows the exponential convergence rate. Section 3 is devoted to numerical results. We analyze dependence of the convergence rate on the eigenvalue index, matching point, and the transmission coefficient. In Section 4 we study the nonlinear eigenvalue transmission problem with an additional integral normalizing condition. We obtain a convergence result similar to the result from Section 2, and illustrate the algorithm with numerical examples which confirm the theoretical ones.

2. A differential normalizing condition. Let us consider the following eigenvalue transmission problem:

$$\frac{d^2u_i(x)}{dx^2} + \lambda u_i(x) - N_i(u_i(x)) = 0, \qquad x \in \Omega_i, \tag{1}$$

with the Dirichlet boundary conditions

$$u_1(0) = u_2(1) = 0, \tag{2}$$

the matching conditions

$$\frac{du_i(x^{(1)})}{dx} = r[u(x^{(1)})], \qquad r > 0, \quad i = 1, 2, \tag{3}$$

and the normalizing condition

$$\frac{du_1(0)}{dx} = 1, \tag{4}$$

where $\Omega_1 = (0, x^{(1)}), \quad \Omega_2 = (x^{(1)}, 1)$, $[u(x^{(1)})] = u_2(x^{(1)}) - u_1(x^{(1)})$ is a jump of the function at the matching point $x^{(1)}$, $N_i(u) : \Re^1 \to \Re^1$ is an analytic function with respect to u_i such that

$$N_1(0) = 0, \quad |N_{i,u}^{(k)}(u_i)| = \left|\frac{d^k N_i(u_i)}{du_i^k}\right| \le \overline{N}_{i,u}^{(k)}(|u_i|), \quad u_i \in \mathbb{R}, \tag{5}$$

and $\overline{N}_i(u_i)$ is an analytic function with nonnegative derivatives for $u \ge 0$.

According to the FD-approach, we write the numerical solution of problem (1)–(4) as truncated series:

$$\overset{m}{\lambda}_n = \sum_{j=0}^{m} \lambda_n^{(j)}, \quad \overset{m}{u}_{ni} = \sum_{j=0}^{m} u_{ni}^{(j)}, \qquad i = 1, 2, \tag{6}$$

and we find the terms of the truncated series as follows.

The zero approximation $(\lambda_n^{(0)}, u_{ni}^{(0)}(x), i = 1, 2)$ is a solution of the so-called basic nonperturbed eigenvalue transmission problem

$$\frac{d^2u_{ni}^{(0)}(x)}{dx^2} + \lambda_n^{(0)} u_{ni}^{(0)}(x) = 0, \qquad x \in \Omega_i, \quad i = 1, 2,$$

$$u_{n1}^{(0)}(0) = u_{n2}^{(0)}(1) = 0, \qquad \frac{du_{n1}^{(0)}(0)}{dx} = 1, \qquad \frac{du_{ni}^{(0)}(x^{(1)})}{dx} = r[u_n^{(0)}(x^{(1)})].$$

In [8] it is shown that depending on the transmission point $x^{(1)}$ there can exist the following two kinds of eigenvalues $\lambda_n^{(0)}$:

1) $\lambda_n^{(0)}$ is defined by the formula

$$\lambda_n^{(0)} = \frac{\pi^2(2k+1)^2}{4(x^{(1)})^2} = \frac{\pi^2(2n+1)^2}{4(1-x^{(1)})^2} \tag{7}$$

for $n, k \in \{0\} \cup N$ such that $\dfrac{x^{(1)}}{1-x^{(1)}} = \dfrac{2k+1}{2n+1}$,

2) $\lambda_n^{(0)}$ is a solution of the transcendental equation

$$r\left(\operatorname{tg}\left(\sqrt{\lambda_n^{(0)}}x^{(1)}\right) + \operatorname{tg}\left(\sqrt{\lambda_n^{(0)}}(1-x^{(1)})\right)\right) + \sqrt{\lambda_n^{(0)}} = 0, \tag{8}$$

if $\dfrac{x^{(1)}}{1-x^{(1)}} \neq \dfrac{2k+1}{2n+1}$.

The corresponding eigenfunctions are the following:

$$u_{n1}^{(0)}(x) = \frac{\sin\left(\sqrt{\lambda_n^{(0)}}x\right)}{\sqrt{\lambda_n^{(0)}}}, \qquad u_{n2}^{(0)}(x) = \varphi_n \frac{\sin\left(\sqrt{\lambda_n^{(0)}}(1-x)\right)}{\sqrt{\lambda_n^{(0)}}}, \tag{9}$$

where

$$\varphi_n = \begin{cases} (-1)^{n-k}, & \dfrac{x^{(1)}}{1-x^{(1)}} = \dfrac{2k+1}{2n+1}, \\[2ex] -\dfrac{\cos\left(\sqrt{\lambda_n^{(0)}}x^{(1)}\right)}{\cos\left(\sqrt{\lambda_n^{(0)}}(1-x^{(1)})\right)}, & \dfrac{x^{(1)}}{1-x^{(1)}} \neq \dfrac{2k+1}{2n+1}. \end{cases}$$

The corrections $u_{ni}^{(j+1)}(x), j = 0, 1, 2, \ldots$, are solutions of the system of transmission problems for linear nonhomogenous differential equations

$$\frac{d^2u_{ni}^{(j+1)}(x)}{dx^2} + \lambda_n^{(0)} u_{ni}^{(j+1)}(x) = -F_{ni}^{(j+1)}(x), \qquad x \in \Omega_i, \quad i = 1, 2, \tag{10}$$

$$u_{n1}^{(j+1)}(0) = u_{n2}^{(j+1)}(1) = 0, \quad \frac{du_{n1}^{(j+1)}(0)}{dx} = 0, \quad \frac{du_{ni}^{(j+1)}(x^{(1)})}{dx} = r[u_n^{(j+1)}(x^{(1)})],$$

where

$$F_{ni}^{(j+1)}(x) = -\sum_{s=0}^{j} \lambda_n^{(j+1-s)} u_{ni}^{(s)}(x) + A_{ni}^{(j)}(u_{ni}^{(0)}, u_{ni}^{(1)}, \ldots, u_{ni}^{(j)})$$

and the Adomian polynomials $A_{ni}^{(j)}(u_{ni}^{(0)}, \ldots, u_{ni}^{(j)})$ are defined in the following way:

$$A_{ni}^{(j)}(u_{ni}^{(0)}, \ldots, u_{ni}^{(j)}) =$$

$$= \sum_{\alpha_1+\cdots+\alpha_j=j} N_{iu_i}^{(\alpha_1)}(u_{ni}^{(0)}(x)) \frac{[u_{ni}^{(1)}(x)]^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!} \cdots \frac{[u_{ni}^{(j-1)}(x)]^{\alpha_{j-1}-\alpha_j}}{(\alpha_{j-1}-\alpha_j)!} \frac{[u_{ni}^{(j)}(x)]^{\alpha_j}}{(\alpha_j)!}, \quad j > 0,$$

$$A_{ni}^{(0)}(u_{ni}^{(0)}) = N(u_{ni}^{(0)}).$$

The solvability condition for equation (10) yields

$$\lambda_n^{(j+1)} =$$

$$= \frac{\left\{ -\sum\limits_{p=1}^{j} \lambda_n^{(j+1-p)} \sum\limits_{i=1}^{2} \int\limits_{\Omega_i} u_{ni}^{(p)}(\xi) u_{ni}^{(0)}(\xi) d\xi + \sum\limits_{i=1}^{2} \int\limits_{\Omega_i} A_{ni}^{(j)}(u_{ni}^{(0)}(\xi), \ldots, u_{ni}^{(j)}(\xi)) u_{ni}^{(0)}(\xi) d\xi \right\}}{\|u_n^{(0)}\|^2}, \tag{11}$$

where $\|u_n\| = \sqrt{\sum_{i=1}^{2} \int_{\Omega_i} u_{ni}^2(x)dx}$ is the $L_2(\Omega_1 \times \Omega_2)$-norm and

$$\|u_n^{(0)}\| = \begin{cases} \dfrac{\sqrt{2}(1-x^{(1)})}{\pi(2n+1)} = \dfrac{\sqrt{2}x^{(1)}}{\pi(2k+1)}, & \dfrac{x^{(1)}}{1-x^{(1)}} = \dfrac{2k+1}{2n+1}, \\ \dfrac{|\cos\sqrt{\lambda_n^{(0)}}x^{(1)}|}{\sqrt{2\lambda_n^{(0)}}} \sqrt{\dfrac{x^{(1)}}{\cos^2\sqrt{\lambda_n^{(0)}}x^{(1)}} + \dfrac{1-x^{(1)}}{\cos^2\sqrt{\lambda_n^{(0)}}(1-x^{(1)})} + \dfrac{1}{r}}, & \dfrac{x^{(1)}}{1-x^{(1)}} \neq \dfrac{2k+1}{2n+1}. \end{cases}$$

Then we can write the solution of the nonhomogeneous problem as

$$u_{ni}^{(j+1)}(x) = \hat{u}_{ni}^{(j+1)}(x), \tag{12}$$

where

$$\hat{u}_{n1}^{(j+1)}(x) = \\ = \int\limits_0^x \frac{\sin\left(\sqrt{\lambda_n^{(0)}}(x-\xi)\right)}{\sqrt{\lambda_n^{(0)}}} \left\{ -\sum_{p=0}^{j} \lambda_n^{(j+1-p)} u_{n1}^{(p)}(\xi) + A_{n1}^{(j)}(u_{n1}^{(0)}(\xi), u_{n1}^{(1)}(\xi), \ldots, u_{n1}^{(j)}(\xi)) \right\} d\xi, \tag{13}$$

$$x \in (0, x^{(1)}),$$

$$\hat{u}_{n2}^{(j+1)}(x) = \int\limits_0^{x^{(1)}} \left[\frac{\sin\left(\sqrt{\lambda_n^{(0)}}(x-\xi)\right)}{\sqrt{\lambda_n^{(0)}}} + \frac{\cos\left(\sqrt{\lambda_n^{(0)}}(x^{(1)}-x)\right)\cos\left(\sqrt{\lambda_n^{(0)}}(x^{(1)}-\xi)\right)}{r} \right] \times \\ \times \left\{ -\sum_{p=0}^{j} \lambda_n^{(j+1-p)} u_{n1}^{(p)}(\xi) + A_{n1}^{(j)}(u_{n1}^{(0)}(\xi), u_{n1}^{(1)}(\xi), \ldots, u_{n1}^{(j)}(\xi)) \right\} d\xi + \\ + \int\limits_{x^{(1)}}^{x} \frac{\sin\left(\sqrt{\lambda_n^{(0)}}(x-\xi)\right)}{\sqrt{\lambda_n^{(0)}}} \left\{ -\sum_{p=0}^{j} \lambda_n^{(j+1-p)} u_{n2}^{(p)}(\xi) + A_{n2}^{(j)}(u_{n2}^{(0)}(\xi), u_{n2}^{(1)}(\xi), \ldots, u_{n2}^{(j)}(\xi)) \right\} d\xi, \tag{14}$$

$$x \in (x^{(1)}, 1).$$

Hence, the numerical algorithm for the nonlinear eigenvalue transmission problem with differential normalizing condition (1) – (4) consists of finding eigenvalues of the basic eigenvalue transmission problem according to (7) (or (8)), (9), evaluating (11) – (14) for $j = 0, 1, 2, \ldots, m$ and (6).

The error of the algorithm can be estimated as

$$\|u_n - \overset{m}{u}_n\|_\infty \leq \sum_{j=m+1}^{\infty} \|u_n^{(j)}\|_\infty, \\ |\lambda_n - \overset{m}{\lambda}_n| \leq \sum_{j=m+1}^{\infty} |\lambda_n^{(j)}|. \tag{15}$$

From (13), (14), (11) we receive the estimates

$$\|u_n^{(j+1)}\|_\infty \le \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r}\right)\left\{\sum_{p=0}^{j} |\lambda_n^{(j+1-p)}| \|u_n^{(p)}\|_\infty + \right.$$

$$\left. + \sum_{\alpha_1+\ldots+\alpha_j=j} \overline{N}^{(\alpha_1)}(\|u_n^{(0)}\|_\infty) \frac{\|u_n^{(1)}\|_\infty^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!} \cdots \frac{\|u_n^{(j-1)}\|_\infty^{\alpha_{j-1}-\alpha_j}}{(\alpha_{j-1}-\alpha_j)!} \frac{\|u_n^{(j)}\|_\infty^{\alpha_j}}{\alpha_j!}\right\},$$

$$|\lambda_n^{(j+1)}| \le \left\{\sum_{p=1}^{j} |\lambda_n^{(j+1-p)}| \|u_n^{(p)}\|_\infty + \right.$$

$$\left. + \sum_{\alpha_1+\ldots+\alpha_j=j} \overline{N}^{(\alpha_1)}(\|u_n^{(0)}\|_\infty) \frac{\|u_n^{(1)}\|_\infty^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!} \cdots \frac{\|u_n^{(j-1)}\|_\infty^{\alpha_{j-1}-\alpha_j}}{(\alpha_{j-1}-\alpha_j)!} \frac{\|u_n^{(j)}\|_\infty^{\alpha_j}}{\alpha_j!}\right\} / \|u_n^{(0)}\|.$$

Introducing the new variables

$$u_{j+1} = \frac{\|u_n^{(j+1)}\|_\infty}{\|u_n^{(0)}\|} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r}\right)^{-(j+1)}, \qquad \mu_{j+1} = \frac{|\lambda_n^{(j+1)}|}{\|u_n^{(0)}\|} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r}\right)^{-j}$$

and their numerical majorants

$$u_{j+1} \le \overline{u}_{j+1}, \quad \overline{u}_0 = \frac{\|u_n^{(0)}\|_\infty}{\|u_n^{(0)}\|}, \qquad \mu_{j+1} \le \overline{\mu}_{j+1}.$$

we obtain the following majorazing system of equations:

$$\overline{u}_{j+1} = \sum_{p=0}^{j} \overline{\mu}_{j+1-p}\overline{u}_p + \overline{A}^{(j)}(\overline{N}, \overline{u}_1, \ldots, \overline{u}_j),$$

$$\overline{\mu}_{j+1} = \overline{u}_{j+1} - \overline{\mu}_{j+1}u_0, \qquad j = 0, 1, \ldots,$$

which leads to

$$\overline{u}_{j+1} = \sum_{p=1}^{j} \overline{u}_{j+1-p}\overline{u}_p + (1+\overline{u}_0)\overline{A}^{(j)}(\overline{N}, \overline{u}_0, \ldots, \overline{u}_j).$$

Analogously to [10] introducing the generating function for the sequence $\{\overline{u}_j\}$ by

$$f(z) = \sum_{j=0}^{\infty} z^j \overline{u}_j, \tag{16}$$

from the last equation we get

$$(f(z) - \overline{u}_0)^2 - (f(z) - \overline{u}_0) + z(1 + \overline{u}_0)\overline{N}(f(z)) = 0.$$

Considering z as a variable depending on f, we obtain the following expression:

$$z = \frac{(f - \overline{u}_0)(1 + \overline{u}_0 - f)}{(1 + \overline{u}_0)\overline{N}(f)}.$$

Since $z \geq 0$, we have $\overline{u}_0 \leq f \leq 1 + \overline{u}_0$. It is easy to see that on the interval $(\overline{u}_0, 1 + \overline{u}_0)$ there exists a unique extremum for $z(f)$,

$$z_{\max} = R = z(f_{\max}), \tag{17}$$

where $f_{\max}$ satisfies $z'(f_{\max}) = 0$.

Thus series (16) converges, that is, there exists a positive generating function and

$$R^j \overline{u}_j \leq \frac{c}{j^{1+\varepsilon}},$$

where c and ε are some positive constants.

From this we receive

$$\|u_n^{(j+1)}\|_\infty \leq \frac{c}{(j+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^{j+1}.$$

Analogously we obtain

$$|\lambda_n^{(j+1)}| \leq \frac{c}{R(j+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^{j}.$$

Hence, series (6) converge and (15) implies the estimates

$$\begin{aligned} \|u_n - \overset{m}{u}_n\|_\infty &\leq \frac{c}{(m+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^{m+1}, \\ |\lambda_n - \overset{m}{\lambda}_n| &\leq \frac{c}{R(m+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^{m} \end{aligned} \tag{18}$$

provided that

$$\frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) < 1. \tag{19}$$

Thus, we have proven the following convergence result.

Theorem 1. *Let condition (5) hold and let the solution of the basic eigenvalue problem, $\lambda_n^{(0)}$, satisfy (19), where R is defined by (17). Under these assumptions, the numerical algorithm (6), (7) (or (8)), (9), (11) – (14) converges exponentially to (1) – (4) with estimates (18).*

Corollary 1. *Since, according to [8],*

$$\sqrt{\lambda_n^{(0)}} = \begin{cases} \dfrac{\pi(2n+1)}{2(1-x^{(1)})}, & \dfrac{x^{(1)}}{1-x^{(1)}} = \dfrac{2k+1}{2n+1}, \\ \dfrac{\pi(2n+1)}{2(1-x^{(1)})} + \dfrac{2r}{\pi(2n+1)} + O\left(n^{-2}\right), & \dfrac{x^{(1)}}{1-x^{(1)}} \neq \dfrac{2k+1}{2n+1}, \end{cases}$$

the convergence improves with the increase of the index of the trial eigenpair and tends to a constant defined by the transmission coefficient r.

3. Numerical results. We consider the problem

$$\frac{d^2u_i(x)}{dx^2} + \lambda u_i(x) - u_i^3(x) = 0, \qquad x \in \Omega_i, \quad i = 1, 2,$$

$$u_1(0) = u_2(1) = 0, \qquad \frac{du_1(0)}{dx} = 1, \qquad \frac{du_i(x^{(1)})}{dx} = r[u(x^{(1)})].$$

To estimate the error of the algorithm, we compare the numerical solution obtained by the FD-method, $\overset{m}{\lambda}_n$, with the solution obtained by the bisection method using the Maple procedure dverc 78, λ_n^{ex}, with the accuracy 10^{-8}.

1. Firstly, let us consider the case $x^{(1)} = 1/3$. According to our theory all roots of the basic problem depend on r. The results of calculations for λ_1, λ_3 and λ_6 for $r = 1$ are given in Tables 1 – 3. The error for the eigenvalues with different indexes in the logarithmic scale is depicted in Fig. 1. From Fig. 1 we note that the absolute value of the deviation of the approximate eigenvalue from the exact one, $\delta_n(m) = |\overset{m}{\lambda}_n - \lambda_n^{\text{ex}}|$, obey the following functional dependence:

$$\log \delta_n(m) \approx \alpha_n m + c,$$

which confirms the exponential convergence rate. From Tables and Fig. 1 we can also see that the convergence rate improves with the increase of the index of the eigenvalue and tends to a constant. Dependence of the error on the transmission coefficient for λ_6 is depicted in

Table 1

m	$\overset{m}{\lambda}_1$	$\lvert\overset{m}{\lambda}_1 - \lambda_1^{\text{ex}*}\rvert$
0	7,4165088739016625	$6,3420 \cdot 10^{-1}$
1	8,0958853583842360	$4,5174 \cdot 10^{-2}$
2	8,0462992047046045	$4,4126 \cdot 10^{-3}$
3	8,0512031756222008	$4,9133 \cdot 10^{-4}$
4	8,0506527365158858	$5,9109 \cdot 10^{-5}$

$^*\lambda_1^{\text{ex}} = 8,05071184559989\ldots$

Table 2

m	$\overset{m}{\lambda}_3$	$\lvert\overset{m}{\lambda}_3 - \lambda_3^{\text{ex}^*}\rvert$
0	53,2835555483183284	$3,3503 \cdot 10^{-1}$
1	53,6275661821205418	$6,3441 \cdot 10^{-3}$
2	53,6337909108476428	$1,1941 \cdot 10^{-4}$
3	53,6339077207744245	$2,6050 \cdot 10^{-6}$

$^*\,\lambda_3^{\text{ex}} = 53,6339103257738684\ldots$

Table 3

m	$\overset{m}{\lambda}_6$	$\lvert\overset{m}{\lambda}_6 - \lambda_6^{\text{ex}^*}\rvert$
0	452,5368547373059060	$3,9380 \cdot 10^{-1}$
1	452,9320089941100867	$2,3104 \cdot 10^{-3}$
2	452,9296847981662003	$1,3732 \cdot 10^{-5}$
3	452,9296985957231688	$6,5348 \cdot 10^{-8}$

$^*\,\lambda_6^{\text{ex}} = 452,9296985303755375\ldots$

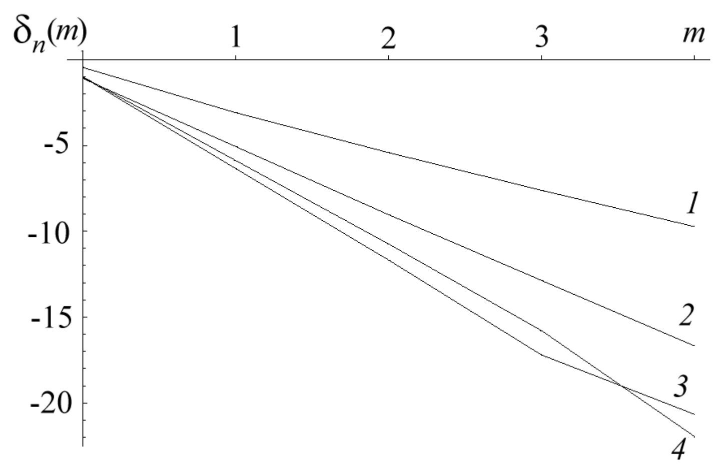

Fig. 1. Dependence of the absolute error on the discretization parameter m for eigenvalues with different indexes: $\delta_n(m) = \ln(|\overset{m}{\lambda}_n - \lambda_n^{\text{ex}}|)$, $x^{(1)} = 1/3$, $r = 1$. The curves are shown for different values of n, $n = 1$ (*1*), $n = 3$ (*2*), $n = 8$ (*3*), $n = 5$ (*4*).

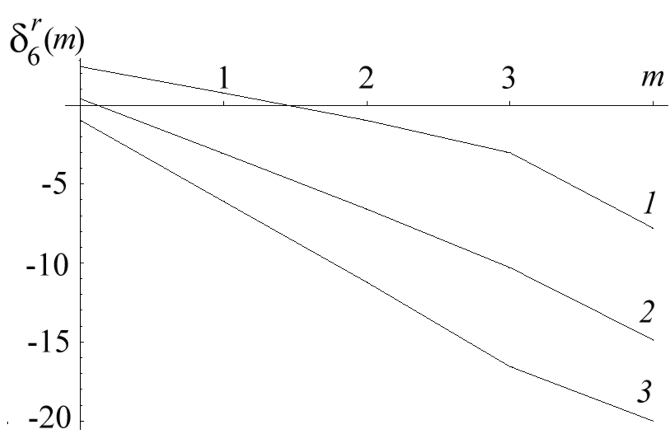

Fig. 2. Dependence of the absolute error on the discretization parameter m for λ_6 with different values of the transmission coefficient: $\delta_6^r(m) = \ln(|\overset{m}{\lambda}_6 - \lambda_6^{\text{ex}}|)$, $x^{(1)} = 1/3$. The curves are shown for different values of r, $r = 1/5$ (*1*), $r = 1/2$ (*2*), $r = 1$ (*3*).

Fig. 2. As it follows from Fig. 2, convergence rate improves with the increase of the transmission coefficient r.

2. Let us consider the case $x^{(1)} = 1/4$. According to our theory there are two kinds of the eigenvalues $\lambda_n^{(0)}$ for the basic eigenvalue transmission problem:

2.1) eigenvalues dependent on r with $n = 4k-2$, $k = 1, 2, \ldots$;

2.2) eigenvalues not dependent on r with $n = 4k-3, 4k-1, 4k$, $k = 1, 2, \ldots$.

For $r = 1$ numerical results are presented in Tables 4–6. The error of the algorithm for eigenvalues with different indexes in the logarithmic scale is depicted in Fig. 3. From Tables and Fig. 3 we conclude that the convergence rate is exponential. For the eigenvalues with dependent on r zero approximation $\lambda_n^{(0)}$, $n = 1, 4, 9$, it improves with the increase of the index of eigenvalue and tends to a constant. For the eigenvalues with not dependent on r zero approximation $\lambda_n^{(0)}$, $n = 2, 6, 10$, it also improves with the increase of the index of the eigenvalue, but does not tends to a constant. We note that the numerical eigenvalues $\overset{m}{\lambda}_n$ of the original problem with zero approximation not dependent on r also do not depend on r. The reason for this is the type of nonlinearity, that is, in this case the term on $1/r$ for $\hat{u}_{n2}^{(j+1)}(x)$ identically equals to zero,

Table 4

m	$\overset{m}{\lambda}_1$	$\lvert\overset{m}{\lambda}_1 - \lambda_1^{\text{ex}^*}\rvert$
0	6,2315467060814359	$9,1749 \cdot 10^{-1}$
1	7,1939908990839667	$4,4960 \cdot 10^{-2}$
2	7,1463227453492218	$2,7085 \cdot 10^{-3}$
3	7,1492041124602485	$1,7288 \cdot 10^{-4}$

$^*\lambda_1^{\text{ex}} = 7,1490312331582256\ldots$

Table 5

m	$\overset{m}{\lambda}_2$	$\lvert\overset{m}{\lambda}_2 - \lambda_2^{\mathrm{ex}^*}\rvert$
0	39,4784176043544344	$1,8995 \cdot 10^{-2}$
1	39,4974153262903728	$2,6658 \cdot 10^{-6}$
2	39,4974126598608147	$6,5919 \cdot 10^{-10}$
3	39,49741266053293244	$1,2932 \cdot 10^{-11}$

$^*\,\lambda_2^{\mathrm{ex}} = 39,4974126605298376\ldots$

Table 6

m	$\overset{m}{\lambda}_4$	$\lvert\overset{m}{\lambda}_4 - \lambda_4^{\mathrm{ex}^*}\rvert$
0	112,4437999204786364	$5,3781 \cdot 10^{-1}$
1	112,9769575287073761	$4,6492 \cdot 10^{-3}$
2	112,9815829113849929	$2,3778 \cdot 10^{-5}$
3	112,9816068711603011	$1,8141 \cdot 10^{-7}$
4	112,9816066967542689	$7,0034 \cdot 10^{-9}$

$^*\,\lambda_4^{\mathrm{ex}} = 112,9816066897508219\ldots$

and hence, the expected convergence is $O\left(1/\sqrt{\lambda_n^{(0)}}\right) = O\left(1/n\right)$. Dependence of the error on the transmission coefficient for λ_1 is depicted in Fig. 4. As it follows from Fig. 4 the convergence rate improves with the increase of the transmission coefficient r.

Hence, we can conclude that numerical results confirm the theoretical ones.

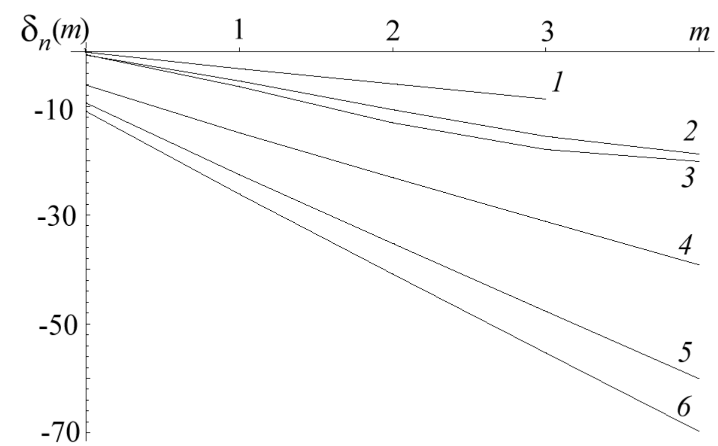

Fig. 3. Dependence of the absolute error on the discretization parameter m for eigenvalues with different indexes: $\delta_n(m) = \ln\left(\lvert\overset{m}{\lambda}_n - \lambda_n^{\mathrm{ex}}\rvert\right)$, $x^{(1)} = 1/4$, $r = 1$. The curves are shown for different values of n, $n = 1$ (*1*), $n = 4$ (*2*), $n = 9$ (*3*), $n = 2$ (*4*), $n = 6$ (*5*), $n = 10$ (*6*).

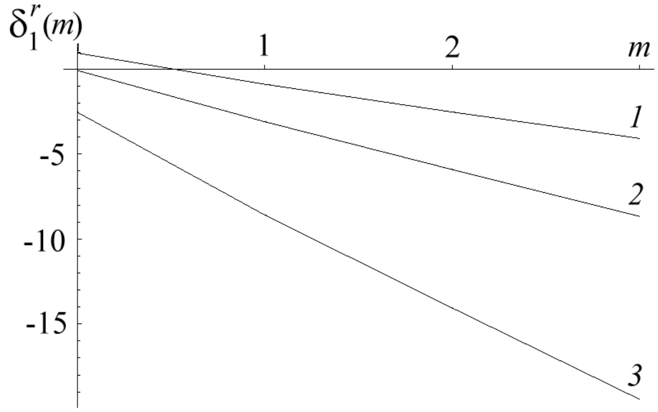

Fig. 4. Dependence of the absolute error on the discretization parameter m for λ_1 with different values of the transmission coefficient: $\delta_1^r(m) = \ln(|\overset{m}{\lambda}_1 - \lambda_1^{\rm ex}|)$, $x^{(1)} = 1/4$. The curves are shown for different values of r, $r = 1/2$ (*1*), $r = 1$ (*2*), $r = 100$ (*3*).

4. An integral normalizing condition. We consider problem (1) – (3) with integral normalizing condition

$$\|u\|^2 = \sum_{i=1}^{2} \int\limits_{\Omega_i} u_{ni}^2(x)dx = M. \tag{20}$$

The solution of the corresponding basic linear eigenvalue transmission problem,

$$\frac{d^2u_{ni}^{(0)}(x)}{dx^2} + \lambda_n^{(0)} u_{ni}^{(0)}(x) = 0, \qquad x \in \Omega_i, \quad i = 1, 2,$$

$$u_{n1}^{(0)}(0) = u_{n2}^{(0)}(1) = 0, \quad \frac{du_{ni}^{(0)}(x^{(1)})}{dx} = r[u_n^{(0)}(x^{(1)})],$$

$$\|u_n^{(0)}\| = \sqrt{M}$$

is the following:

$$\lambda_n^{(0)} = \begin{cases} \dfrac{\pi^2(2k+1)^2}{4(x^{(1)})^2} = \dfrac{\pi^2(2n+1)^2}{4(1-x^{(1)})^2}, & \dfrac{x^{(1)}}{1-x^{(1)}} = \dfrac{2k+1}{2n+1}, \\ \lambda_n^{(0)} : r(\mathrm{tg}(\sqrt{\lambda_n^{(0)}}x^{(1)}) + \mathrm{tg}(\sqrt{\lambda_n^{(0)}}(1-x^{(1)}))) + \sqrt{\lambda_n^{(0)}} = 0, & \dfrac{x^{(1)}}{1-x^{(1)}} \neq \dfrac{2k+1}{2n+1}, \end{cases} \tag{21}$$

$k, n \in \{0\} \cup N$,

$$u_{n1}^{(0)}(x) = \sqrt{2M}\varphi_1 \sin(\sqrt{\lambda_n^{(0)}}x), \quad u_{n2}^{(0)}(x) = \sqrt{2M}\varphi_2 \sin(\sqrt{\lambda_n^{(0)}}(1-x)), \tag{22}$$

where

$$\varphi_1=\begin{cases}1, & \dfrac{x^{(1)}}{1-x^{(1)}}=\dfrac{2k+1}{2n+1},\\[2ex] \dfrac{1}{\cos(\sqrt{\lambda_n^{(0)}}x^{(1)})}\left(\dfrac{x^{(1)}}{\cos^2(\sqrt{\lambda_n^{(0)}}x^{(1)})}+\right.\\ \left.+\dfrac{1-x^{(1)}}{\cos^2(\sqrt{\lambda_n^{(0)}}(1-x^{(1)}))}+\dfrac{1}{r}\right)^{-1/2}, & \dfrac{x^{(1)}}{1-x^{(1)}}\neq\dfrac{2k+1}{2n+1},\end{cases}$$

$$\varphi_2=\begin{cases}(-1)^{n-k}, & \dfrac{x^{(1)}}{1-x^{(1)}}=\dfrac{2k+1}{2n+1},\\[2ex] -\dfrac{1}{\cos(\sqrt{\lambda_n^{(0)}}x^{(1)})}\left(\dfrac{x^{(1)}}{\cos^2(\sqrt{\lambda_n^{(0)}}x^{(1)})}+\right.\\ \left.+\dfrac{1-x^{(1)}}{\cos^2(\sqrt{\lambda_n^{(0)}}(1-x^{(1)}))}+\dfrac{1}{r}\right)^{-1/2}, & \dfrac{x^{(1)}}{1-x^{(1)}}\neq\dfrac{2k+1}{2n+1}.\end{cases}$$

We find the terms $u_{ni}^{(j+1)}(x)$, $j=0,1,2,\ldots$, from the recursive system of transmission problems for the linear nonhomogeneous differential equations:

$$\frac{d^2u_{ni}^{(j+1)}}{dx^2}+\lambda_n^{(0)}u_{ni}^{(j+1)}=-F_{ni}^{(j+1)}(x),\quad x\in\Omega_i,\quad i=1,2, \tag{23}$$

$$u_{n1}^{(j+1)}(0)=u_{n2}^{(j+1)}(1)=0,\qquad \frac{du_{ni}^{(j+1)}(x^{(1)})}{dx}=r[u_n^{(j+1)}(x^{(1)})],$$

$$\sum_{i=1}^{2}\int\limits_{\Omega_i}u_{ni}^{(0)}(x)u_{ni}^{(j+1)}(x)dx=-\frac{1}{2}\sum_{i=1}^{2}\sum_{p=1}^{j}\int\limits_{\Omega_i}u_{ni}^{(p)}(x)u_{ni}^{(j+1-p)}(x)dx, \tag{24}$$

where $F_{ni}^{(j+1)}(x)$ is the same as in the case of the differential normalizing condition.

The solvability condition for nonhomogeneous equation (23) leads to the expression for $\lambda_n^{(j+1)}$,

$$\lambda_n^{(j+1)}=\frac{\left\{-\sum\limits_{p=1}^{j}\lambda_n^{(j+1-p)}\sum\limits_{i=1}^{2}\int\limits_{\Omega_i}u_{ni}^{(p)}(\xi)u_{ni}^{(0)}(\xi)d\xi+\sum\limits_{i=1}^{2}\int\limits_{\Omega_i}A_{ni}^{(j)}(u_{ni}^{(0)}(\xi),\ldots,u_{ni}^{(j)}(\xi))u_{ni}^{(0)}(\xi)d\xi\right\}}{M}. \tag{25}$$

So, the numerical algorithm for the eigenvalue transmission problem with integral normalizing condition (1)–(3), (20) consists of finding a solution of the basic problem according to (21), (22), finding a solution of the nonlinear problems (23), (24), and calculating (25) for $j=0,1,\ldots,m$, and then calculating (6).

In order to estimate the convergence rate of the algorithm according to formulas (15), let us write the solution of the nonhomogeneous problem (23) as follows:

$$u_{ni}^{(j+1)}(x) = B_n^{(j+1)} u_{ni}^{(0)}(x) + \hat{u}_{ni}^{(j+1)}(x), \qquad i = 1, 2, \tag{26}$$

where $u_{ni}^{(0)}(x)$ is defined by (22), and $\hat{u}_{ni}^{(j+1)}(x)$ is defined according to (13) or (14), correspondingly.

Substitution (26) in (24) leads to

$$MB_n^{(j+1)} = -\sum_{i=1}^{2} \int\limits_{\Omega_i} u_{ni}^{(0)}(x)\hat{u}_{ni}^{(j+1)}(x)dx - \frac{1}{2}\sum_{p=1}^{j}\sum_{i=1}^{2}\int\limits_{\Omega_i} u_{ni}^{(p)}(x)u_{ni}^{(j+1-p)}(x)dx, \tag{27}$$

from which

$$|B_n^{(j+1)}| \le \frac{\|u_n^{(0)}\|}{M}\|\hat{u}_n^{(j+1)}\|_\infty + \frac{1}{2M}\sum_{p=1}^{j}\|u_n^{(p)}\|\|u_n^{(j+1-p)}\|.$$

Taking into account the last expression, from (26) we get

$$\|u_n^{(j+1)}\|_\infty \le \left(1 + \frac{\|u_n^{(0)}\|_\infty}{\sqrt{M}}\right)\|\hat{u}_n^{(j+1)}\|_\infty + \frac{\|u_n^{(0)}\|_\infty}{2M}\sum_{p=1}^{j}\|u_n^{(p)}\|_\infty\|u_n^{(j+1-p)}\|_\infty$$

and, together with (13) and (14), we receive the estimate

$$\|u_n^{(j+1)}\|_\infty \le \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r}\right)\left(1 + \frac{\|u_n^{(0)}\|_\infty}{\sqrt{M}}\right)\left\{\sum_{p=0}^{j}|\lambda_n^{(j+1-p)}|\|u_n^{(p)}\|_\infty + \|A_n^{(j)}\|\right\} + \\ + \frac{\|u_n^{(0)}\|_\infty}{2M}\sum_{p=1}^{j}\|u_n^{(p)}\|_\infty\|u_n^{(j+1-p)}\|_\infty. \tag{28}$$

From (25) we get that

$$|\lambda_n^{(j+1)}| \le \frac{1}{\sqrt{M}}\left(\sum_{p=1}^{j}|\lambda_n^{(j+1-p)}|\|u_n^{(p)}\|_\infty + \|A_n^{(j)}\|\right). \tag{29}$$

Then, the recursive system of inequalities (28), (29) leads to the majorating system of equations,

$$\begin{aligned} u_{j+1} &= \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r}\right)\left(1 + \frac{u_0}{\sqrt{M}}\right)\left\{\sum_{p=0}^{j}\mu_{j+1-p}\,u_p + \|A_n^{(j)}\|\right\} + \frac{u_0}{2M}\sum_{p=1}^{j}u_p\,u_{j+1-p}, \\ \mu_{j+1} &= \frac{1}{\sqrt{M}}\left(\sum_{p=0}^{j}\mu_{j+1-p}\,u_p + \|A_n^{(j)}\| - \mu_{j+1}u_0\right), \qquad j = 0, 1, \ldots, \end{aligned} \tag{30}$$

with

$$\|u_n^{(j)}\|_\infty \le u_j, \quad u_0 = \|u_n^{(0)}\|_\infty, \quad \text{and} \quad |\lambda_n^{(j)}| \le \mu_j.$$

Substituting

$$U_j = \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r}\right)^j u_j, \quad M_j = \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r}\right)^{j-1} \mu_j \tag{31}$$

from (30) we obtain

$$\begin{aligned} U_{j+1} &= \left(1 + \frac{u_0}{\sqrt{M}}\right)\left\{\sum_{p=0}^{j} M_{j+1-p}U_p + \|A_n^{(j)}\|\right\} + \frac{u_0}{2M}\sum_{p=1}^{j} U_p U_{j+1-p}, \\ M_{j+1} &= \frac{1}{\sqrt{M}+u_0}\left(\sum_{p=0}^{j} M_{j+1-p}U_p + \|A_n^{(j)}\|\right), \qquad j = 0, 1, \ldots, \end{aligned} \tag{32}$$

with $U_0 = u_0 = \|u_n^{(0)}\|_\infty$.

To solve system (32), we use the method of generating functions,

$$f(z) = \sum_{j=0}^{\infty} z^j U_j, \quad g(z) = \sum_{j=0}^{\infty} z^j M_{j-1}.$$

Relations (32) lead to the following system:

$$f(z) - u_0 = \left(1 + \frac{u_0}{\sqrt{M}}\right) z[g(z)f(z) + \overline{N}(f(z))] + \frac{u_0}{2M}[f(z) - u_0]^2,$$

$$g(z) = \frac{1}{\sqrt{M}+u_0}[g(z)f(z) + \overline{N}(f(z))].$$

From this,

$$z = z(\overline{f}) = \frac{\sqrt{M}\overline{f}\left(1 - \frac{u_0}{2M}\overline{f}\right)(\sqrt{M} - \overline{f})}{(\sqrt{M}+u_0)^2\overline{N}(\overline{f}+u_0)},$$

where $\overline{f} = f(z) - u_0$.

Since $z = z(\overline{f})$ is a positive continuous function on the interval $l = \left(0, \min\left\{\sqrt{M}, \frac{2M}{u_0}\right\}\right)$ and z is equal to zero at the ends of the interval l, there exists

$$z_{\max} = R = z(\overline{f}_{\max}), \tag{33}$$

where $\overline{f}_{\max}$ satisfies the equation $z'(\overline{f}) = 0$. The value $z_{\max} = R$ is obviously the convergence radius for the series $f(z)$. Hence,

$$R^j U_j < \frac{c}{j^{1+\varepsilon}}, \quad R^j M_j < \frac{c}{j^{1+\varepsilon}},$$

where c and ε are positive constants.

Taking into account (31) we obtain the estimates

$$\|u_n^{(j+1)}\|_\infty \le \frac{c}{(j+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^{j+1},$$

$$|\lambda_n^{(j+1)}| \le \frac{c}{R(j+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^{j}.$$

Thus, series (6) converge and from (15) we receive the estimates

$$\begin{aligned} \|u_n - \overset{m}{u}_n\|_\infty &\le \frac{c}{(m+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^{m+1}, \\ |\lambda_n - \overset{m}{\lambda}_n| &\le \frac{c}{R(m+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^{m}, \end{aligned} \tag{34}$$

provided that

$$\frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) < 1. \tag{35}$$

Thus, we have proven the following convergence result.

Theorem 2. *Let conditions (5) and (35) be satisfied. Then the numerical algorithm (21) – (25) converges exponentially to a solution of problem (1) – (3), (20) with estimates (34).*

Corollary 2. *The convergence rate of algorithm (21) – (25) improves with an increase of the index of the trial eigenvalue and tends to a constant defined by the transmission coefficient r.*

Corollary 3. *As it follows from (35), the convergence rate of the algorithm improves with an increase of the transmission coefficient r.*

Example:

$$\frac{d^2 u_i(x)}{dx^2} + \lambda u_i(x) - u_i^3(x) = 0, \quad x \in \Omega_i, \quad i = 1, 2,$$

$$u_1(0) = u_2(1) = 0, \quad \frac{du_i(x^{(1)})}{dx} = r[u(x^{(1)})], \quad \|u\|^2 = M.$$

We consider the case with two kinds of zero approximations of eigenvalues, that is, $\lambda_n^{(0)}$ dependent on the transmission coefficient r and $\lambda_n^{(0)}$ not dependent on r. Let us set $x^{(1)} = 1/4$, $M = 1$. Using solver Maple 9, we find $R < 0{,}016$, that provid convergence of our algorithm for large enough n and r. However, requirement (35) is sufficient and our algorithm converges also for much smaller indexes of the eigenvalues and the transmission coefficient. As in previous calculations, to estimate the error of algorithm, we compare the numerical solution obtained by

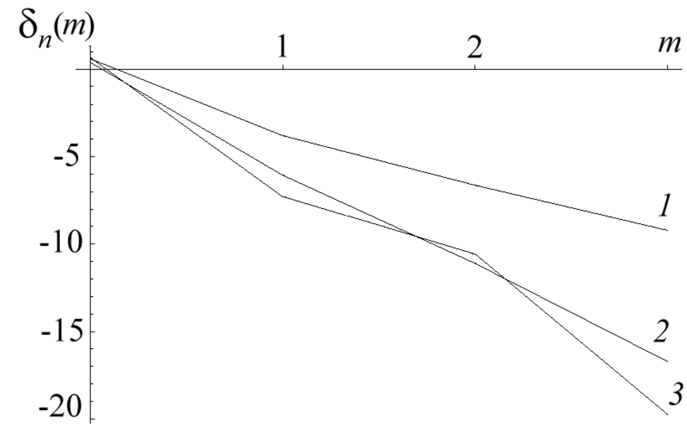

Fig. 5. Dependence of the absolute error on the discretization parameter m for eigenvalues with different indexes: $\delta_n(m) = \ln(|\overset{m}{\lambda}_n - \lambda_n^{\mathrm{ex}}|)$, $x^{(1)} = 1/4, r = 1, M = 1$. The curves are shown for different values of n, $n = 1$ (*1*), $n = 2$ (*2*), $n = 4$ (*3*).

Table 7

m	$\overset{m}{\lambda}_1$	$\lvert\overset{m}{\lambda}_1 - \lambda_1^{\mathrm{ex}^*}\rvert$
0	6,2315467060814359	1,8193
1	8,0735525671370393	$2,2683 \cdot 10^{-2}$
2	8,0495411885163239	$1,3281 \cdot 10^{-3}$
3	8,0509697134406525	$1,0047 \cdot 10^{-4}$

$^*\,\lambda_1^{\mathrm{ex}} = 8,05086923780985\ldots$

Table 8

m	$\overset{m}{\lambda}_2$	$\lvert\overset{m}{\lambda}_2 - \lambda_2^{\mathrm{ex}^*}\rvert$
0	39,4784176043544344	1,4976
1	40,9784176043544344	$2,3596 \cdot 10^{-3}$
2	40,9760428891158172	$1,5094 \cdot 10^{-5}$
3	40,9760579271757605	$5,5591 \cdot 10^{-8}$

$^*\,\lambda_2^{\mathrm{ex}} = 40,97605799284510\ldots$

Table 9

m	$\overset{m}{\lambda}_4$	$\lvert\overset{m}{\lambda}_4 - \lambda_4^{\mathrm{ex}^*}\rvert$
0	112,4437999204786364	1,9720
1	114,4165149249466765	$6,9386 \cdot 10^{-4}$
2	114,4158464832084226	$2,5428 \cdot 10^{-5}$
3	114,4158184347494351	$2,6207 \cdot 10^{-7}$

$^*\,\lambda_4^{\mathrm{ex}} = 112,9816066897508219\ldots$

FD-method with the solution obtained by bisection method using the Maple procedure dverc 78. For $r = 1$, the numerical results are given in Tables 7–9 and depicted in Fig. 5. From Tables and Fig. 5 we note that the convergence rate is exponential. It improves with the increase of the index of the eigenvalue and tends to a constant. The method converges better for the eigenvalues with zero approximation not dependent on r.

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