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MAXIMUM RECOVERABLE WORK AND PSEUDOFREE ENERGIES FOR A RIGID HEAT CONDUCTOR*

МАКСИМАЛЬНА ВІДНОВЛЕНА РОБОТА ТА ПСЕВДОВІЛЬНІ ЕНЕРГІЇ ДЛЯ ТВЕРДОГО ПРОВІДНИКА ТЕПЛА

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In this paper we consider the linear theory for a rigid heat conductor with memory effects for the heat flux in order to derive explicit formulae of the minimum free energy, which is related to the maximum recoverable work we can obtain by a given state of the body. Two equivalent forms of this work are given in the frequency domain. Finally, two different expressions of a thermodynamic potential, called pseudofree energy, are introduced for this material.

Розглядається лінійна теорія для твердого провідника тепла з ефектами запам'ятовування теплового потоку з метою знаходження явних формул для мінімуму вільної енергії, що пов'язана з максимальною відновленою роботою, яку можна отримати з даного стану тіла. Наведено дві еквівалентні форми цієї роботи в частотній області. Введено два різних вирази для термодинамічного потенціалу, названі псевдовільною енергією.

1. Introduction. A non linear model for a rigid heat conductor was developed by Gurtin and Pipkin in [1] on the basis of Coleman's results concerning materials with memory effects [2]. In [1] the authors have also derived a linearization of their theory, thus obtaining for isotropic materials a constitutive equation for the heat flux expressed in terms of the history of the temperature gradient. Such a relation is a generalization of the Cattaneo – Maxwell equation [3], since this appears as a special case of the theory. Many authors have considered such a constitutive equation to study its consequences in thermal problems. We remember, in particular, the approximate theory of thermodynamics developed for Gurtin and Pipkin's model in [4], where some thermodynamic potentials have been derived and used to prove stability and domain of dependence results.

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Recently, the problem of obtaining expressions for the maximum recoverable work, we can obtain from a given state of the material, has been considered by several authors, especially for linear viscoelastic solids (see, for example, [5–10] and [11–13]). A particular importance have the articles [14] and [15] for the interesting methods used for such studies.

In [16] the problem of finding an explicit expression of the minimum free energy, which is related to the maximum recoverable work, has been studied for a rigid heat conductor. To describe the behaviour of any simple material we must consider its states and processes; in particular, the states are expressed in [17] by means of the temperature and of the integrated history of the temperature gradient, which was introduced in [1] and considered also in [4].

In this paper we consider some results obtained in [4] to describe the behaviour of a rigid heat conductor; in particular, we assume a linear relation for the internal energy as a function of the temperature and the linear functional derived in [1] for the heat flux. Thus, we consider the linearization of the Clausius–Duhem inequality in the form derived in [4] in order to examine the effects of the internal energy on the expression of the thermal work; then, we study the problem of determining explicit formulae for the minimum free energy, expressed in the form considered in [4] and there called pseudofree energy. Two expressions of this pseudofree energy are examined, the first of which was already studied in [4], the second one is a new functional, recently introduced and studied, in particular, for linear viscoelastic solids (see, for example, [10]).

In Section 2, fundamental relations of the considered linear theory are introduced, together with the linearization of the local form of the Second Law of Thermodynamics. In Section 3, states and processes are defined and an equivalence between states is defined. In Section 4, the notion of the thermal work is introduced and some of its expressions are written for some particular cases; moreover, another equivalence relation between states is defined in terms of the work. In Sections 5 and 6 two equivalent forms of the maximum recoverable work are derived. In Section 7, we give the expressions of two functionals which can be considered as pseudofree energies for these materials.

2. Preliminaries. Let $\mathcal{B}$ be a rigid heat conductor occupying a fixed and bounded domain Ω in the Euclidean three-dimensional space. We suppose that Ω is a regular domain, that is simply-connected with a smooth boundary, whose unit outward normal is denoted by $\mathbf{n}$.

We regard $\mathcal{B}$ as a homogeneous, isotropic and endowed of memory for the heat flux, within the linear theory of thermodynamics developed in [1] and studied also in [4], and we assume the constitutive equations

$$e(\mathbf{x},t) = \alpha_0 \vartheta(\mathbf{x},t), \qquad \mathbf{q}(\mathbf{x},t) = -\int_0^{+\infty} k(s)\mathbf{g}^t(\mathbf{x},s)ds \tag{1}$$

for the internal energy e and the heat flux $\mathbf{q}$. In these linearized relations ϑ denotes the relative temperature, with respect to the absolute reference temperature Θ_0 uniform in $\bar{\Omega}$, the heat flux relaxation $k : \mathbf{R}^+ \equiv [0,+\infty) \to \mathbf{R}$ is such that $k \in L^1(\mathbf{R}^+) \cap H^1(\mathbf{R}^+)$, while $\mathbf{g} = \nabla\vartheta$ is the temperature gradient, whose history up to time t is defined by $\mathbf{g}^t(\mathbf{x},s) = \mathbf{g}(\mathbf{x},t-s)\ \forall s \in \mathbf{R}^+$; finally, we denote by $\mathbf{x}$ the position vector in Ω.

The function k is defined by

$$k(t) = k_0 + \int\limits_0^t \dot{k}(\tau) d\tau \qquad \forall t \in \mathbf{R}^+, \tag{2}$$

where $k_0 = k(0)$, its initial value, and $\lim\limits_{t \to +\infty} k(t) = 0$ [4, 18]; moreover, in the expression of e there is a coefficient $\alpha_0 > 0$ because of physical observation.

In [4], starting from the Clausius – Duhem inequality, the authors have derived its linearization, which involves second order approximations for the free energy and the entropy, because of the first order ones assumed for e and $\mathbf{q}$. Thus, by introducing the function $\psi = \Theta_0(e - \Theta_0\eta)$, they have obtained the following inequality:

$$\dot{\psi}(\mathbf{x}, t) \leq \dot{e}(\mathbf{x}, t)\vartheta(\mathbf{x}, t) - \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x}, t). \tag{3}$$

The function now considered is called pseudofree energy, since, even if its values haven't the dimensions of an energy, its properties closely resemble those of the canonical free energy $e - \Theta_0\eta$. We observe that the introduction of the factor Θ_0 in the definition of the pseudofree energy yields the elimination of the factor $1/\Theta_0$ involved by the linearization in the scalar product $\mathbf{q} \cdot \mathbf{g}$, and, therefore, in (3) the reference temperature does not appear.

From the inequality (3), which expresses the linearized local form of the Second Law of Thermodynamics, it follows the equality

$$\dot{\psi}(\mathbf{x}, t) + D(\mathbf{x}, t) = \dot{e}(\mathbf{x}, t)\vartheta(\mathbf{x}, t) - \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{g}(\mathbf{x}, t), \tag{4}$$

where we have introduced $D(\mathbf{x}, t)$, called the internal dissipation function, which must be nonnegative because of the same Second Law.

The expression $(1)_2$ of the heat flux, by integrating by parts, assumes the form

$$\mathbf{q}(\mathbf{x}, t) = \int\limits_0^{+\infty} k'(s)\bar{\mathbf{g}}^t(\mathbf{x}, s) ds, \tag{5}$$

where we have introduced the integrated history of $\mathbf{g}$, that is a function $\bar{\mathbf{g}}^t(\mathbf{x}, \cdot) : \mathbf{R} \to \mathbf{R}^3$ defined by

$$\bar{\mathbf{g}}^t(\mathbf{x}, s) = \int\limits_{t-s}^{t} \mathbf{g}(\mathbf{x}, \tau) d\tau = \int\limits_0^s \mathbf{g}^t(\mathbf{x}, \lambda) d\lambda.$$

It is useful for what follows to introduce the formal Fourier transform of any function $f : \mathbf{R} \to \mathbf{R}^n$,

$$f_F(\omega) = \int\limits_{-\infty}^{+\infty} f(s) e^{-i\omega s} ds = f_-(\omega) + f_+(\omega) \qquad \forall \omega \in \mathbf{R},$$

where

$$f_-(\omega) = \int_{-\infty}^{0} f(s)e^{-i\omega s}ds, \qquad f_+(\omega) = \int_{0}^{+\infty} f(s)e^{-i\omega s}ds, \tag{6}$$

together with the half-range Fourier sine and cosine transforms

$$f_s(\omega) = \int_{0}^{+\infty} f(s)\sin(\omega s)ds, \qquad f_c(\omega) = \int_{0}^{+\infty} f(s)\cos(\omega s)ds.$$

We remember that if f is a real-valued function, then the complex conjugate $(f_F(\omega))^* = f(-\omega)$; moreover, if f is defined on $\mathbf{R}^+$ we can consider f_+, f_s and f_c, while for a function defined on $\mathbf{R}^-$ the definition of f_- is used. Finally, we have the following relations:

$$f_F(\omega) = f_c(\omega) - if_s(\omega), \quad f_F(\omega) = 2f_c(\omega), \quad f_F(\omega) = -if_s(\omega), \tag{7}$$

when f, defined on $\mathbf{R}^+$, is extended to $\mathbf{R}$ by considering its usual extension, that is by identifying f with a function defined on $\mathbf{R}$ which is equal to zero on the strictly negative reals $\mathbf{R}^{--}$, or its extension by means of an even extension, i.e., $f(s) = f(-s)\ \forall s \in \mathbf{R}^{--}$, or its extension by using an odd function that is when $f(s) = -f(-s)\ \forall s \in \mathbf{R}^{--}$ respectively. Moreover, if f and f' belong to $L^1(\mathbf{R}^+) \cap L^2(\mathbf{R}^+)$, then

$$f'_s(\omega) = -\omega f_c(\omega).$$

Under these hypotheses, for the kernel k we have the conditions [4, 18]

$$k_c(\omega) = -\frac{1}{\omega}k'_s(\omega) > 0 \qquad \forall\omega \neq 0, \qquad k(0) = -\frac{2}{\pi}\int_{0}^{+\infty} \frac{k'_s(\omega)}{\omega}d\omega > 0, \tag{8}$$

because of the thermodynamic restrictions on the constitutive equation and using the inverse half-range Fourier transform.

Furthermore, if $k'' \in L^2(R^+)$ and $\mid \mathbf{k}'(0) \mid < +\infty$, then

$$\sup_{\omega\in\mathbf{R}} |\omega k'_s(\omega)| < +\infty, \qquad \lim_{\omega\to\infty} \omega k'_s(\omega) = -\lim_{\omega\to\infty} \omega^2 k_c(\omega) = k'(0) \le 0.$$

We assume

$$k_c(0) > 0, \qquad k'(0) < 0. \tag{9}$$

The complex z-plane $\mathbf{C}$ has an important role in what follows and we define its subsets

$$\mathbf{C}^- = \left\{z \in \mathbf{C};\ \operatorname{Im} z \in \mathbf{R}^-\right\}, \quad \mathbf{C}^{(-)} = \left\{z \in \mathbf{C};\ \operatorname{Im} z \in \mathbf{R}^{--}\right\};$$

analogous meanings are given to $\mathbf{C}^+$ and $\mathbf{C}^{(+)}$, related to $\mathbf{R}^+$ and $\mathbf{R}^{++} = (0, +\infty)$. We observe that $\mathbf{C}^{\pm}$ include $\mathbf{R}$ and $\mathbf{C}^{(\pm)}$ exclude it. Thus, $f_\pm$, given by (6), can be defined for $z \in \mathbf{C}$ and are analytic functions in $\mathbf{C}^{(\mp)}$, but, assuming their analyticity on $\mathbf{R}$, they become analytic in $\mathbf{C}^{\mp}$

[15]. Finally, we shall use the notation $f_{(\pm)}(z)$ to denote that the zeros and the singularities of f are only in $\mathbf{C}^{\pm}$.

3. States and processes. A rigid heat conductor characterized by the constitutive equations (1) can be consider as a simple material and hence its behaviour is described by means of states and processes [1, 19–21].

Thus, we choose as thermodynamic state the couple

$$\sigma(t) = (\vartheta(t), \bar{\mathbf{g}}^t)$$

and denote by Σ the set of the possible states of the body.

Then, we define as thermodynamic process the piecewise continuous map $P : [0, d) \to \to \mathbf{R} \times \mathbf{R}^3$ given by

$$P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau)) \qquad \forall \tau \in [0, d),$$

where d is termed the duration of the process. The set of thermodynamic processes is denoted by Π. If $P \in \Pi$ also its restriction $P_{[\tau_1, \tau_2)}$ to any interval $[\tau_1, \tau_2) \subset [0, d)$ belongs to Π; moreover, given two processes $P_j \in \Pi$, $j = 1, 2$, with durations d_j, $j = 1, 2$, their composition

$$P_1 * P_2(\tau) = \begin{cases} P_1(\tau) & \forall \tau \in [0, d_1), \\ P_2(\tau - d_1) & \forall \tau \in [d_1, d_1 + d_2), \end{cases}$$

also belongs to Π.

The state transition function $\rho : \Sigma \times \Pi \to \Sigma$ maps any initial state $\sigma^i \in \Sigma$ and $P \in \Pi$ into the final state $\sigma^f = \rho(\sigma^i, P) \in \Sigma$. Thus, given $\sigma^i = \sigma(0)$ and a process $P_{[0,\tau)}$, the final state is $\sigma(\tau) = \rho(\sigma(0), P_{[0,\tau)})$; moreover, the pair (σ, P) is called a cycle if $\sigma(d) = \rho(\sigma(0), P) = \sigma(0)$.

Let $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ be a process applied at time $t = 0$ to the initial state $\sigma(0) = = (\vartheta_*(0), \mathbf{g}_*^0)$, then $\tau \equiv t \in [0, d)$ and the state $\sigma(t)$ is given by

$$\vartheta(t) = \vartheta_*(0) + \int_0^t \dot{\vartheta}_P(s) ds, \quad \bar{\mathbf{g}}^t(s) = \begin{cases} \displaystyle\int_{t-s}^t \mathbf{g}_P(\xi) d\xi, & 0 \le s < t, \\ \displaystyle\int_0^t \mathbf{g}_P(\xi) d\xi + \bar{\mathbf{g}}_*^0(s - t), & s \ge t. \end{cases} \tag{10}$$

If the process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ $\forall \tau \in [0, d)$ is applied at time $t > 0$ to the initial state $\sigma^i(t) = (\vartheta_i(t), \bar{\mathbf{g}}_i^t)$, we have the subsequent states characterized by the temperature

$$\vartheta_P(\tau) \equiv \vartheta(t + \tau) = \vartheta(t) + \int_0^\tau \dot{\vartheta}_P(\eta) d\eta \tag{11}$$

together with the prolongation of the integrated history $\bar{\mathbf{g}}_i^t$ defined by

$$\bar{\mathbf{g}}(t+d-s) = (\mathbf{g}_P * \bar{\mathbf{g}}_i)^{t+d}(s) =$$

$$= \begin{cases} \int\limits_{d-s}^{d} \mathbf{g}_P(\xi)d\xi = \bar{\mathbf{g}}_P^d(s), & 0 \le s < d, \\ \int\limits_{0}^{d} \mathbf{g}_P(\xi)d\xi + \int\limits_{t-(s-d)}^{t} \mathbf{g}_i(\xi)d\xi = \bar{\mathbf{g}}_P^d(d) + \bar{\mathbf{g}}_i^t(s-d), & s \ge d. \end{cases} \tag{12}$$

Definition 1. *Two states $\sigma_j = (\vartheta_j, \bar{\mathbf{g}}_j^t)$, $j = 1, 2$, are said to be equivalent if*

$$e(\rho(\sigma_1, P_{[0,\tau)})) = e(\rho(\sigma_2, P_{[0,\tau)})), \;\; \tilde{\mathbf{q}}((\mathbf{g}_P * \mathbf{g}_1)^{t+\tau}) = \tilde{\mathbf{q}}((\mathbf{g}_P * \mathbf{g}_2)^{t+\tau}) \quad \forall \tau > 0 \tag{13}$$

for any process $P \in \Pi$.

Theorem 1. *For a rigid heat conductor characterized by (1), two states $\sigma_j = (\vartheta_j, \bar{\mathbf{g}}_j^t)$, $j = 1, 2$, are equivalent if and only if*

$$\vartheta_1(t) = \vartheta_2(t), \qquad \int\limits_0^{+\infty} k'(\xi+\tau)\left[\bar{\mathbf{g}}_1^t(\xi) - \bar{\mathbf{g}}_2^t(\xi)\right] d\xi = \mathbf{0} \quad \forall \tau > 0. \tag{14}$$

Proof. Taking into account (11), (12) it is easy to prove that (14) implies (13) and vice-versa.

Given the state $\sigma(t) = (\vartheta(t), \bar{\mathbf{g}}^t)$, taking account of (11), (12), from (1) we have

$$e(t+\tau) = \alpha_0 \left[\vartheta(t) + \int\limits_0^{\tau} \dot{\vartheta}_P(\eta)d\eta\right],$$

$$\mathbf{q}(t+\tau) = \int\limits_0^{\tau} k'(s)\bar{\mathbf{g}}_P^{\tau}(s)ds + \int\limits_{\tau}^{+\infty} k'(s)[\bar{\mathbf{g}}_P^{\tau}(\tau) + \bar{\mathbf{g}}^t(s-\tau)]ds,$$

by applying any process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ of duration d.

It is easy to show that such a state $\sigma(t) = (\vartheta(t), \bar{\mathbf{g}}^t)$ is equivalent to the zero state $\sigma_0 = (0, \bar{\mathbf{0}}^{\dagger})$, where the relative temperature is null and $\bar{\mathbf{0}}^{\dagger}$ denotes the zero integrated history, i.e., $\bar{\mathbf{0}}^{\dagger}(s) = \mathbf{g}^t(s) = \mathbf{0}\ \forall s \in \mathbf{R}^+$, if

$$\vartheta(t) = 0, \qquad \int\limits_{\tau}^{+\infty} k'(s)\bar{\mathbf{g}}^t(s-\tau)ds = \int\limits_0^{+\infty} k'(\xi+\tau)\bar{\mathbf{g}}^t(\xi)d\xi = \mathbf{0}.$$

Thus, it follows that the two states σ_j, $j = 1, 2$, are equivalent if their difference $\sigma_1 - \sigma_2 = (\vartheta_1 - \vartheta_2, \bar{\mathbf{g}}_1^t - \bar{\mathbf{g}}_2^t)$ is a state equivalent to $\sigma_0 = (0, \bar{\mathbf{0}}^{\dagger})$.

4. The thermal work. The linearized form (3) of the Clausius – Duhem inequality allows us to introduce the thermal power expressed by the following expression:

$$w(t) = \dot{e}(t)\vartheta(t) - \mathbf{q}(t) \cdot \mathbf{g}(t).$$

Thus, the thermal work done on a process $\mathrm{P}(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ applied for every $\tau \in [0, d)$, starting from the initial state $\sigma(t) = (\vartheta(t), \bar{\mathbf{g}}^t)$ at time t, is expressed by

$$W(\sigma, P) = \tilde{W}(\vartheta(t), \bar{\mathbf{g}}^t; \dot{\vartheta}_P, \mathbf{g}_P) = \int_0^d [\dot{e}(t+\tau)\vartheta_P(\tau) - \mathbf{q}(t+\tau) \cdot \mathbf{g}_P(\tau)] d\tau. \tag{15}$$

Taking into account (1), (11) and (12), (15) becomes

$$\tilde{W}(\vartheta(t), \bar{\mathbf{g}}^t; \dot{\vartheta}_P, \mathbf{g}_P) = \int_0^d [\alpha_0 \dot{\vartheta}_P(\tau)\vartheta_P(\tau) - \mathbf{q}((\mathbf{g}_P * \bar{\mathbf{g}})^{t+\tau}) \cdot \mathbf{g}_P(\tau)] d\tau =$$

$$= \frac{1}{2}\alpha_0[\vartheta^2(t+d) - \vartheta^2(t)] - \int_0^d \int_0^{+\infty} k'(s)(\mathbf{g}_P * \bar{\mathbf{g}})^{t+\tau}(s) ds \cdot \mathbf{g}(\tau) d\tau. \tag{16}$$

In order to distinguish the part of the work done only during the application of the process, let us consider the particular case related to the initial state $\sigma_0 = (0, \bar{\mathbf{0}}^\dagger)$ and apply to this state a process P of duration d at time $t = 0$. The ensuing fields are given by (10), which now reduce to

$$\vartheta_0(t) = \int_0^t \dot{\vartheta}_P(s) ds, \quad (\mathbf{g}_P * \bar{\mathbf{0}}^\dagger)^t(s) = \begin{cases} \bar{\mathbf{g}}_0^t(s), & 0 \le s < t, \\ \bar{\mathbf{g}}_0^t(t), & s \ge t. \end{cases} \tag{17}$$

Definition 2. *A process $P = (\dot{\vartheta}_P, \mathbf{g}_P)$, of duration d, applied at time $t = 0$ and related to $\vartheta_0(t)$ and $(\mathbf{g}_P * \bar{\mathbf{0}}^\dagger)^t$ expressed by (17), is called a finite work process if*

$$\tilde{W}(0, \bar{\mathbf{0}}^\dagger; \dot{\vartheta}_P, \mathbf{g}_P) = \int_0^d [\alpha_0 \dot{\vartheta}_0(t)\vartheta_0(t) - \mathbf{q}((\mathbf{g}_0 * \bar{\mathbf{0}}^\dagger)^t) \cdot \mathbf{g}_0(t)] dt < +\infty. \tag{18}$$

Lemma 1. *The work (18) is such that*

$$\tilde{W}(0, \bar{\mathbf{0}}^\dagger; \dot{\vartheta}_P, \mathbf{g}_P) > 0.$$

Proof. From (18), taking account of (17), we have

$$\tilde{W}(0, \bar{\mathbf{0}}^\dagger; \dot{\vartheta}_P, \mathbf{g}_P) = \frac{1}{2}\alpha_0 \vartheta_0^2(d) -$$

$$- \int_0^d \left[\int_0^t k'(s)\bar{\mathbf{g}}_0^t(s) ds + \int_t^{+\infty} k'(s)\bar{\mathbf{g}}_0^t(t) ds \right] \cdot \mathbf{g}_0(t) dt,$$

whence, by integrating by parts in the first integral and evaluating the second one, we get

$$\tilde{W}(0,\bar{\mathbf{0}}^{\dagger};\dot{\vartheta}_P,\mathbf{g}_P) = \frac{1}{2}\alpha_0\vartheta_0^2(d) + \int_0^d\int_0^t k(s)\mathbf{g}_0^t(s)ds\cdot\mathbf{g}_0(t)dt. \tag{19}$$

The last integral, applying Plancherel's theorem, on supposing the functions equal to zero for any $t > d$, can be written as

$$\int_0^{+\infty}\int_0^t k(s)\mathbf{g}_0^t(s)ds\cdot\mathbf{g}_0(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} k_F(\omega)\mathbf{g}_{0_F}(\omega)\cdot(\mathbf{g}_{0_F}(\omega))^* d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} k_c(\omega)[\mathbf{g}_{0_c}^2(\omega)+\mathbf{g}_{0_s}^2(\omega)]d\omega > 0,$$

since the Fourier transforms of the functions, equal to zero on $\mathbf{R}^{--}$, are given by $(7)_1$, where the cosine and sine transforms are even and odd functions, respectively. Thus, by virtue of $(8)_1$, it follows that the work (19) is positive.

The lemme is proved.

Any process $P(\tau) = (\dot{\vartheta}_P(\tau),\mathbf{g}_P(\tau))$, defined for any $\tau\in[0,d)$ with $d < +\infty$ and applied at time $t = 0$, can be extended on $\mathbf{R}^+$ by putting $P(\tau) = (0,\mathbf{0})\ \forall\tau\ge d$; we can assume $\vartheta_P(\tau) = 0$ for any $\tau > d$ and obtain from (18)

$$\tilde{W}(0,\bar{\mathbf{0}}^{\dagger};\dot{\vartheta}_P,\mathbf{g}_P) = \frac{1}{2}\alpha_0\vartheta_0^2(d) -$$

$$- \int_0^{+\infty}\left[\int_0^{\tau} k'(s)\bar{\mathbf{g}}_P^{\tau}(s)ds + \int_{\tau}^{+\infty} k'(s)\bar{\mathbf{g}}_P^{\tau}(\tau)ds\right]\cdot\mathbf{g}_P(\tau)d\tau =$$

$$= \frac{1}{2}\alpha_0\vartheta_0^2(d) + \int_0^{+\infty}\int_0^{\tau} k(\tau-\xi)\mathbf{g}_P(\xi)\cdot\mathbf{g}_P(\tau)d\xi d\tau =$$

$$= \frac{1}{2}\alpha_0\vartheta_0^2(d) + \frac{1}{2}\int_0^{+\infty}\int_0^{+\infty} k(\mid\tau-\xi\mid)\mathbf{g}_P(\xi)\cdot\mathbf{g}_P(\tau)d\xi d\tau.$$

Application of Plancherel's theorem, using $(7)_2$, yields

$$\tilde{W}(0,\bar{\mathbf{0}}^{\dagger};\dot{\vartheta}_P,\mathbf{g}_P) = \frac{1}{2}\alpha_0\vartheta_0^2(d) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} k_c(\omega)\mathbf{g}_{P+}(\omega)\cdot(\mathbf{g}_{P+}(\omega))^* d\omega.$$

With such a result, following Gentili [14], the finite work processes can be characterized by the function space

$$\tilde{H}_k(\mathbf{R}^+, \mathbf{R}^3) = \left\{ \mathbf{g} : \mathbf{R}^+ \to \mathbf{R}^3;\ \int\limits_{-\infty}^{+\infty} k_c(\omega)\mathbf{g}_+(\omega) \cdot (\mathbf{g}_{P+}(\omega))^* d\omega < +\infty \right\},$$

from which, by means of a completion with the norm induced by the inner product $(\mathbf{g}_1, \mathbf{g}_2)_k = \int_{-\infty}^{+\infty} k_c(\omega)\mathbf{g}_{1+}(\omega) \cdot (\mathbf{g}_{2+}(\omega))^* d\omega$, we get an Hilbert space, $H_k(\mathbf{R}^+, \mathbf{R}^3)$.

Now, let the process P of duration d be applied at time $t > 0$ to the initial state $\sigma(t) = (\vartheta(t), \bar{\mathbf{g}}^t)$. We suppose that $P(\tau) = (0, \mathbf{0})\ \forall \tau \geq d$ and that $\vartheta_P(\tau) = 0\ \forall \tau > d$. The work done on P is given by $(16)_1$, which becomes

$$W(\sigma(t), P) = \tilde{W}(\vartheta(t), \bar{\mathbf{g}}^t; \dot{\vartheta}_P, \mathbf{g}_P) = \frac{1}{2}\alpha_0[\vartheta_P^2(d) - \vartheta_P^2(0)] -$$

$$- \int\limits_0^{+\infty} \left\{ \int\limits_0^{\tau} k'(s)\bar{\mathbf{g}}_P^{\tau}(s)ds + \int\limits_{\tau}^{+\infty} k'(s)[\bar{\mathbf{g}}_P^{\tau}(\tau) + \bar{\mathbf{g}}^t(s-\tau)]ds \right\} \cdot \mathbf{g}_P(\tau)d\tau =$$

$$= \frac{1}{2}\alpha_0[\vartheta_P^2(d) - \vartheta_P^2(0)] +$$

$$+ \int\limits_0^{+\infty} \left[\int\limits_0^{\tau} k(\tau-\xi)\mathbf{g}_P(\xi)d\xi - \int\limits_0^{+\infty} k'(\tau+\eta)\bar{\mathbf{g}}^t(\eta)d\eta \right] \cdot \mathbf{g}_P(\tau)d\tau. \tag{20}$$

Putting

$$\mathbf{I}^t(\tau, \bar{\mathbf{g}}^t) = \int\limits_0^{+\infty} k'(\tau+s)\bar{\mathbf{g}}^t(s)ds \qquad \forall \tau \geq 0, \tag{21}$$

from $(20)_2$ we get

$$W(\sigma(t), P) = \frac{1}{2}\alpha_0[\vartheta_P^2(d) - \vartheta_P^2(0)] +$$

$$+ \int\limits_0^{+\infty} \left[\frac{1}{2} \int\limits_0^{+\infty} k(\mid \tau - \xi \mid)\mathbf{g}_P(\xi)d\xi - \mathbf{I}^t(\tau, \bar{\mathbf{g}}^t) \right] \cdot \mathbf{g}_P(\tau)d\tau =$$

$$= \frac{1}{2}\alpha_0[\vartheta^2(t+d) - \vartheta^2(t)] + \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} k_c(\omega)\mathbf{g}_{P+}(\omega)\cdot(\mathbf{g}_{P+}(\omega))^* d\omega -$$

$$- \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \mathbf{I}_+^t(\omega, \bar{\mathbf{g}}^t)\cdot(\mathbf{g}_{P+}(\omega))^* d\omega \tag{22}$$

by virtue of Plancherel's theorem.

Definition 3. *Two states $\sigma_j(t) = (\vartheta_j(t), \bar{\mathbf{g}}_j^t)$, $j = 1, 2$, are termed w-equivalent if and only if the equality*

$$W(\sigma_1(t), P) = W(\sigma_2(t), P) \tag{23}$$

holds for every process $P : [0, \tau) \to \mathbf{R}\times\mathbf{R}^3$ and for every $\tau > 0$.

Theorem 2. *For any rigid heat conductor described by the constitutive equation (1), two equivalent states are equivalent in the sense of Definition 1 if and only if they are w-equivalent.*

Proof. If two states $\sigma_j(t) = (\vartheta_j(t), \bar{\mathbf{g}}_j^t)$, $j = 1, 2$, are equivalent, then, for every process $P(\tau) = (\dot{\vartheta}_P(\tau), \mathbf{g}_P(\tau))$ and for every $\tau > 0$, (13), as well as (14), hold and hence the two works done on the same P by starting from the initial states $\sigma_j(t)$, $j = 1, 2$, coincide because of (16).

Now, on assuming that the two states are w-equivalent, we see that (23) holds and, taking into account $(16)_1$ with the second term expressed by means of its expression given by $(22)_1$, reduces to

$$\int\limits_0^{+\infty} \alpha_0\dot{\vartheta}_P(\tau)[\vartheta_1(t) - \vartheta_2(t)]d\tau = \int\limits_0^{+\infty} [\mathbf{I}^t(\tau, \bar{\mathbf{g}}_1^t) - \mathbf{I}^t(\tau, \bar{\mathbf{g}}_2^t)]\cdot\mathbf{g}_P(\tau)d\tau,$$

whence, because of the arbitrarinesses of $\dot{\vartheta}_P$ and $\mathbf{g}_P$ in P, the expressions in the square brackets must vanish, that is, by virtue of (21), we have (14), which expresses the equivalence of the two states.

The theorem is proved.

5. Formulation of the maximum recoverable work. The amount of energy which is available at a given state σ of a body is related to the maximum work we can obtain from the fixed state [11].

Definition 4. *The maximum work obtained by starting from a given σ of the body $\mathcal{B}$ is given by*

$$W_R(\sigma) = \sup\{-W(\sigma, P) : \ P \in \Pi\},$$

where Π denotes the set of finite work processes.

We observe that $W_R(\sigma)$ is nonnegative, since the null process, which belongs to Π, gives a null work and it is also bounded from above as a consequence of the thermodynamics. Moreover, this work coincides with the minimum free energy [11, 14, 15] that is

$$\psi_m(\sigma) = W_R(\sigma).$$

In order to derive an expression for this quantity, we consider the work done on a process $P \in \Pi$, of duration $d > 0$, applied at a fixed time $t > 0$ when the state is $\sigma(t) = (\vartheta(t), \bar{\mathbf{g}}^t)$. We can extend on $\mathbf{R}^+$ the process by defining $P = (0, \mathbf{0})$ $\forall \tau \in [d, +\infty)$ and we also assume $\vartheta_P(d) = 0$. To determine the maximum of $-W(\sigma, P)$ we consider the expression $(16)_1$ or $(22)_1$ of the work written for the process P characterized by

$$\dot{\vartheta}_P(\tau) = \dot{\vartheta}^{(m)}(\tau) + \gamma\varphi(\tau), \quad \mathbf{g}_P(\tau) = \mathbf{g}^{(m)}(\tau) + \varepsilon\mathbf{v}(\tau) \quad \forall \tau \in \mathbf{R}^+,$$

where the subscript $^{(m)}$ denotes the process which yields the said maximum, while γ and ε are two real parameters and $(\varphi(\tau), \mathbf{v}(\tau))$ are arbitrary functions with $\varphi(0) = 0$, $\mathbf{v}(0) = \mathbf{0}$. Thus, we have

$$-\tilde{W}(\vartheta(t), \bar{\mathbf{g}}^t; \dot{\vartheta}^{(m)} + \gamma\varphi, \mathbf{g}^{(m)} + \varepsilon\mathbf{v}) =$$

$$= -\int_0^{+\infty} \alpha_0[\dot{\vartheta}^{(m)}(\tau) + \gamma\varphi(\tau)]\left\{\vartheta(t) + \int_0^{\tau}[\dot{\vartheta}^{(m)}(\xi) + \gamma\varphi(\xi)]d\xi\right\}d\tau -$$

$$-\int_0^{+\infty}\left\{\frac{1}{2}\int_0^{+\infty} k(|\ \tau - \xi\ |)[\mathbf{g}^{(m)}(\xi) + \varepsilon\mathbf{v}(\xi)]d\xi - \mathbf{I}^t(\tau, \bar{\mathbf{g}}^t)\right\}\cdot$$

$$\cdot [\mathbf{g}^{(m)}(\tau) + \varepsilon\mathbf{v}(\tau)]d\tau,$$

whence we get the following system:

$$\frac{\partial}{\partial\gamma}[-W(\sigma, P)]\ |_{\gamma=0} = -\int_0^{+\infty}\alpha_0\left\{\varphi(\tau)[\vartheta(t) + \int_0^{\tau}[\dot{\vartheta}^{(m)}(\xi)d\xi] + \right.$$

$$\left. + \ \dot{\vartheta}^{(m)}(\tau)\int_0^{\tau}\varphi(\xi)d\xi\right\}d\tau = 0, \tag{24}$$

$$\frac{\partial}{\partial\varepsilon}[-W(\sigma, P)]\ |_{\varepsilon=0} = -\int_0^{+\infty}\left[\int_0^{+\infty} k(|\ \tau - \xi\ |)\mathbf{g}^{(m)}(\xi)d\xi - \mathbf{I}^t(\tau, \bar{\mathbf{g}}^t)\right]\cdot\mathbf{v}(\tau)d\tau = \mathbf{0}.$$

Integrating by parts with respect to τ in the last term and taking account of (11) it is easy to see that the first equation of (24) is identically satisfied. The second one, for the arbitrariness of $\mathbf{v}$ yields the relation

$$\int_0^{+\infty} k(|\ \tau - \xi\ |)\mathbf{g}^{(m)}(v)d\xi = \mathbf{I}^t(\tau, \bar{\mathbf{g}}^t) \qquad \forall\tau \in \mathbf{R}^+, \tag{25}$$

which is an integral equation of the Wiener – Hopf type of the first kind, which in our particular case is solvable by virtue of the thermodynamic properties of the integral kernel and some theorems of factorization.

With the solution $\mathbf{g}^{(m)}$ of (25) we can write the expression of the maximum recoverable work, which, using (22) and (25), assumes the form

$$W_R(\sigma) = \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2}\int\limits_0^{+\infty}\int\limits_0^{+\infty} k(|\ \tau-\xi\ |)\mathbf{g}^{(m)}(\xi)\cdot\mathbf{g}^{(m)}(\tau)d\xi d\tau =$$

$$= \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} k_c(\omega)\mathbf{g}_+^{(m)}(\omega)\cdot(\mathbf{g}_+^{(m)}(\omega))^* d\omega, \tag{26}$$

by means of Plancherel's theorem.

The solution of (25) has been already derived in [17], by considering the factorization of the kernel

$$k_c(\omega) = k_{(+)}(\omega)k_{(-)}(\omega), \tag{27}$$

which allows us to obtain from the Fourier transform of (25), modified as

$$\int\limits_0^{+\infty} k(|\ \tau-\xi\ |)\mathbf{g}^{(m)}(\xi)d\xi = \mathbf{I}^t(\tau,\bar{\mathbf{g}}^t) + \mathbf{r}(\tau),$$

where

$$\mathbf{r}(\tau) = \begin{cases} \int\limits_{-\infty}^{+\infty} k(|\ \tau-\xi\ |)\mathbf{g}^{(m)}(\xi)d\xi & \forall\tau\in\mathbf{R}^-, \\ \mathbf{0} & \forall\tau\in\mathbf{R}^{++}, \end{cases}$$

the relation

$$k_+(\omega)\mathbf{g}_+^{(m)}(\omega) = \frac{1}{2k_{(-)}(\omega)}[\mathbf{I}_+^t(\omega,\bar{\mathbf{g}}^t) + \mathbf{r}_-(\omega)]. \tag{28}$$

The Plemelj formulae give [16]

$$\frac{1}{2k_{(-)}(\omega)}\mathbf{I}_+^t(\omega,\bar{\mathbf{g}}^t) = \mathbf{P}_{(-)}^t(\omega) - \mathbf{P}_{(+)}^t(\omega), \tag{29}$$

where

$$\mathbf{P}^t(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{\mathbf{I}_+^t(\omega,\bar{\mathbf{g}}^t)/[2k_{(-)}(\omega)]}{\omega - z}d\omega, \quad \mathbf{P}_{(\pm)}^t(\omega) = \lim_{\beta\to 0^\mp}\mathbf{P}^t(\omega+i\beta),$$

and hence from (28) we get

$$k_{(+)}(\omega)\mathbf{g}_+^{(m)}(\omega) + \mathbf{P}_{(+)}^t(\omega) = \mathbf{P}_{(-)}^t(\omega) + \frac{1}{2k_{(-)}(\omega)}\mathbf{r}_-(\omega) = \mathbf{0}$$

because of the analyticity of the first hand-side in $\mathbf{C}^-$ and of the second one in $\mathbf{C}^+$, which vanish at infinity. Thus, in particular, we have

$$\mathbf{g}_+^{(m)}(\omega) = -\frac{\mathbf{P}_{(+)}^t(\omega)}{k_{(+)}(\omega)},$$

which allows us to derive from $(26)_2$ the required expression

$$\psi_m(\sigma(t)) = \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mid \mathbf{P}_{(+)}^t(\omega) \mid^2 \, d\omega. \tag{30}$$

6. Another formulation for ψ_m. An equivalent expression of the minimum free energy can be deduced by considering the relation between $\mathbf{P}_{(+)}^t(\omega)$ and $\bar{\mathbf{g}}^t(\omega)$.

Let us extend $k'(s)$ on $\mathbf{R}^{--}$ with an odd function by putting

$$k'^{(0)}(\xi) = \begin{cases} k'(\xi) & \forall\xi \geq 0, \\ -k'(-\xi) & \forall\xi < 0, \end{cases}$$

whose Fourier's transform, on account of $(7)_3$ and $(8)_1$, is given by

$$k_F'^{(0)}(\omega) = -2ik_s'(\omega) = 2i\omega k_c(\omega);$$

moreover, $\bar{\mathbf{g}}^t$ is extended on $\mathbf{R}^{--}$ by means of its usual extension, i.e., $\bar{\mathbf{g}}^t(s) = \mathbf{0}\ \forall s < 0$.

Thus, (21) becomes

$$\mathbf{I}^t(\tau, \bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k'^{(0)}(\tau + s)\bar{\mathbf{g}}^t(s)ds \quad \forall\tau \geq 0$$

and can be extended on $\mathbf{R}$ as follows:

$$\mathbf{I}^{t(\mathbf{R})}(\tau, \bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k'^{(0)}(\tau + \xi)\bar{\mathbf{g}}^t(\xi)d\xi = \begin{cases} \mathbf{I}^t(\tau, \bar{\mathbf{g}}^t) & \forall\tau \geq 0, \\ \mathbf{I}^{t(N)}(\tau, \bar{\mathbf{g}}^t) & \forall\tau < 0, \end{cases} \tag{31}$$

where we have put

$$\mathbf{I}^{t(N)}(\tau, \bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k'^{(0)}(\tau + \xi)\bar{\mathbf{g}}^t(\xi)d\xi \quad \forall\tau < 0.$$

Then, we consider the function $\bar{\mathbf{g}}_N^t(s) = \bar{\mathbf{g}}^t(-s)\ \forall s \leq 0$ and its extension $\bar{\mathbf{g}}_N^t(s) = \mathbf{0}\ \forall s > 0$, for which we have

$$\bar{\mathbf{g}}_{N_F}^t(\omega) = \bar{\mathbf{g}}_{N_-}^t(\omega) = \left(\bar{\mathbf{g}}_+^t(\omega)\right)^*. \tag{32}$$

Therefore, (31) with (32) and $(8)_1$ give

$$\mathbf{I}^{t(\mathbf{R})}(\tau, \bar{\mathbf{g}}^t) = \int_{-\infty}^{+\infty} k'^{(0)}(\tau - s)\bar{\mathbf{g}}^t_N(s)ds,$$

from which we have

$$\mathbf{I}^{t(\mathbf{R})}_F(\omega, \bar{\mathbf{g}}^t) = -2ik'_s(\omega)\left(\bar{\mathbf{g}}^t_+(\omega)\right)^* = 2ik_c(\omega)\left(\bar{\mathbf{g}}^t_+(\omega)\right)^* \tag{33}$$

but also from (31) we obtain

$$\mathbf{I}^{t(\mathbf{R})}_F(\omega, \bar{\mathbf{g}}^t) = \mathbf{I}^{t(N)}_-(\omega, \bar{\mathbf{g}}^t) + \mathbf{I}^t_+(\omega, \bar{\mathbf{g}}^t). \tag{34}$$

Thus, using (27), from (33), (34) and (29) we have

$$\frac{1}{2k_{(-)}(\omega)}\mathbf{I}^{t(\mathbf{R})}_F(\omega, \bar{\mathbf{g}}^t) = \frac{1}{2k_{(-)}(\omega)}\mathbf{I}^{t(N)}_-(\omega, \bar{\mathbf{g}}^t) + \mathbf{P}^t_{(-)}(\omega) - \mathbf{P}^t_{(+)}(\omega),$$

whence it follows that

$$\mathbf{P}^t_{(+)}(\omega) - \mathbf{P}^{(1)^t}_{(+)}(\omega) = \mathbf{P}^t_{(-)}(\omega) - \mathbf{P}^{(1)^t}_{(-)}(\omega) + \frac{1}{2k_{(-)}(\omega)}\mathbf{I}^{t(N)}_-(\omega, \bar{\mathbf{g}}^t), \tag{35}$$

where we have applied the Plemelj formulae, that is

$$\frac{1}{2k_{(-)}(\omega)}\mathbf{I}^{t(\mathbf{R})}_F(\omega, \bar{\mathbf{g}}^t) = \mathbf{P}^{(1)^t}_{(-)}(\omega) - \mathbf{P}^{(1)^t}_{(+)}(\omega),$$

$$\mathbf{P}^{(1)^t}(z) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\mathbf{I}^{t(\mathbf{R})}_F(\omega, \bar{\mathbf{g}}^t)/[2k_{(-)}(\omega)]}{\omega - z}d\omega.$$

From (35), because of the analyticity of both sides, which vanish at infinity, it follows that

$$\mathbf{P}^t_{(+)}(\omega) = \mathbf{P}^{(1)^t}_{(+)}(\omega) = \lim_{z\to\omega^-}\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{i\omega' k_{(+)}(\omega')(\bar{\mathbf{g}}^t_+(\omega'))^*}{\omega' - z}d\omega',$$

whence

$$\left(\mathbf{P}^t_{(+)}(\omega)\right)^* = i\lim_{\eta\to\omega^+}\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\omega' k_{(-)}(\omega')\bar{\mathbf{g}}^t_+(\omega')}{\omega' - \eta}d\omega'. \tag{36}$$

Applying again the Plemelj formulae, we have

$$\mathbf{Q}^t_{(\pm)}(\omega) = \lim_{z\to\omega^\mp}\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\omega' k_{(-)}(\omega')\bar{\mathbf{g}}^t_+(\omega')}{\omega' - z}d\omega', \tag{37}$$

$$\mathbf{Q}^t_{(-)}(\omega) - \mathbf{Q}^t_{(+)}(\omega) = \omega k_{(-)}(\omega)\bar{\mathbf{g}}^t_+(\omega), \tag{38}$$

which allow us to write (36) as follows:

$$\left(\mathbf{P}^t_{(+)}(\omega)\right)^* = i\mathbf{Q}^t_{(-)}(\omega)$$

and, finally, to change (30) in the form

$$\psi_m(\sigma) = \frac{1}{2}\alpha_0\vartheta^2(t) + \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \mid \mathbf{Q}^t_{(-)}(\omega)\mid^2 d\omega.$$

We observe that the quantity now introduced can express the heat flux. In fact, Plancherel's theorem, applied to (5) by using $(7)_3$ for k' and $(8)_1$, yields

$$\mathbf{q}(\bar{\mathbf{g}}^t) = -\frac{i}{\pi}\int\limits_{-\infty}^{+\infty} \omega k_c(\omega)\bar{\mathbf{g}}^t_+(\omega)d\omega = -\frac{i}{\pi}\int\limits_{-\infty}^{+\infty} \omega k_{(+)}(\omega)\left[\mathbf{Q}^t_{(-)}(\omega) - \mathbf{Q}^t_{(+)}(\omega)\right]d\omega =$$

$$= -\frac{i}{\pi}\int\limits_{-\infty}^{+\infty} k_{(+)}(\omega)\mathbf{Q}^t_{(-)}(\omega)d\omega,$$

where we have also considered (27), (38) and the analyticity of $k_{(+)}(\omega)\mathbf{Q}^t_{(+)}(\omega)$ in $\mathbf{C}^-$ which yields a null value of the corresponding integral.

7. Particular pseudofree energies. A first example of thermodynamic potentials for the rigid heat conductor is given by the Graffi–Volterra functional [22–25], which, we now write in the following form:

$$\psi_{\mathcal{G}}(\vartheta(t), \bar{\mathbf{g}}^t) = \frac{1}{2}\alpha_0\vartheta^2(t) - \frac{1}{2}\int\limits_0^{+\infty} k'(s)\bar{\mathbf{g}}^t(s)\cdot\bar{\mathbf{g}}^t(s)ds. \tag{39}$$

We observe that $\psi_{\mathcal{G}}$ is a positive definite quadratic form because of the positiveness of α_0 and if we suppose that $k'(s) < 0$; we also assume $k''(s) \geq 0\ \forall s \in \mathbf{R}^+$.

From (39) we have

$$\dot{\psi}_{\mathcal{G}}(t) = \alpha_0\dot{\vartheta}(t)\vartheta(t) - \int\limits_0^{+\infty} k'(s)\frac{d}{dt}[\bar{\mathbf{g}}^t(s)]\cdot\bar{\mathbf{g}}^t(s)ds =$$

$$= \alpha_0\dot{\vartheta}(t)\vartheta(t) - \mathbf{q}(t)\cdot\mathbf{g}(t) + \int\limits_0^{+\infty} k'(s)\mathbf{g}^t(s)\cdot\bar{\mathbf{g}}^t(s)ds. \tag{40}$$

Here, the last integral becomes

$$\int\limits_0^{+\infty} k'(s)\mathbf{g}^t(s)\cdot\bar{\mathbf{g}}^t(s)ds = \frac{1}{2}\int\limits_0^{+\infty} k'(s)\frac{d}{ds}[\bar{\mathbf{g}}^t(s)\cdot\bar{\mathbf{g}}^t(s)]ds$$

and, integrating by parts, we can write (40) as follows:

$$\dot{\psi}_{\mathcal{G}}(t) + \frac{1}{2}\int_0^{+\infty} k''(s)\bar{\mathbf{g}}^t(s)\cdot\bar{\mathbf{g}}^t(s)ds = \alpha_0\dot{\vartheta}(t)\vartheta(t) - \mathbf{q}(t)\cdot\mathbf{g}(t),$$

which implies that (4) is satisfied with a corresponding dissipation $D_{\mathcal{G}}(t)$ expressed by

$$D_{\mathcal{G}}(t) = \frac{1}{2}\int_0^{+\infty} k''(s)\bar{\mathbf{g}}^t(s)\cdot\bar{\mathbf{g}}^t(s)ds \geq 0$$

for all integrated histories, by virtue of the hypothesis $k''(s) \geq 0\ \forall s \in \mathbf{R}^+$.

A second example is given by a functional expressed in terms of $\mathbf{I}^t$ [10], given by

$$\psi_{\mathcal{F}}(\vartheta, \mathbf{I}^t) = \frac{1}{2}\alpha_0\vartheta^2(t) - \frac{1}{2}\int_0^{+\infty}\frac{1}{k'(\tau)}\mathbf{I}^t_{(1)}(\tau)\cdot\mathbf{I}^t_{(1)}(\tau)d\tau, \tag{41}$$

where $\mathbf{I}^t_{(1)}(\tau)$ is the derivative of (21) with respect to τ, that is,

$$\mathbf{I}^t_{(1)}(\tau) = \frac{d}{d\tau}\mathbf{I}^t(\tau, \bar{\mathbf{g}}^t) = \int_0^{+\infty} k''(\tau + s)\bar{\mathbf{g}}^t(s)ds, \tag{42}$$

and the kernel k is supposed to be such that $k''(s)$ is a nonnegative function of $s \in \mathbf{R}^+$, while $k'(s)$ is a nonpositive function of $s \in \mathbf{R}^+$, in order that (41) gives a nonnegative quantity.

We note that the factor $1/k'(\tau)$, increasing at large τ, is multiplied by factors whose behaviours yield the existence of the integral in (41). Thus, we can introduce the domain

$$H'_{\mathcal{F}}(\mathbf{R}^+) = \left\{\mathbf{I}^t;\ \left|\int_0^{+\infty}\frac{1}{k'(\tau)}\mathbf{I}^t_{(1)}(\tau)\cdot\mathbf{I}^t_{(1)}(\tau)d\tau\right| < +\infty\right\},$$

that is a space very much larger than the domain of $\psi_{\mathcal{G}}$.

We only note that (41) can be written in terms of the integrated history $\bar{\mathbf{g}}^t$, since from (42) we have

$$\psi_{\mathcal{F}}(t) = \frac{1}{2}\alpha_0\vartheta^2(t) - \int_0^{+\infty}\int_0^{+\infty}\left[\int_0^{+\infty}\frac{1}{k'(\tau)}k''(\tau + s_1)k''(\tau + s_2)d\tau\right]\bar{\mathbf{g}}^t(s_1)\cdot$$

$$\cdot\,\bar{\mathbf{g}}^t(s_2)ds_1ds_2 =$$

$$= \frac{1}{2}\alpha_0\vartheta^2(t) + \int_0^{+\infty}\int_0^{+\infty} k_{12}(s_1, s_2)\bar{\mathbf{g}}^t(s_1)\cdot\bar{\mathbf{g}}^t(s_2)ds_1ds_2,$$

where we have denoted by

$$k_{12}(s_1, s_2) = \frac{\partial^2}{\partial s_2 \partial s_1} k(s_1, s_2)$$

the second derivative of the function

$$k(s_1, s_2) = -\int_0^{+\infty} \frac{1}{k'(\tau)} k'(\tau + s_1) k'(\tau + s_2) d\tau,$$

which satisfies

$$k(s_1, 0) = k(s_1), \quad k(0, s_2) = k(s_2), \quad k(0, 0) = k(0).$$

Now, we return to (41) to evaluate the derivative with respect to time,

$$\dot{\psi}_{\mathcal{F}}(t) = \alpha_0 \dot{\vartheta}(t) \vartheta(t) - \int_0^{+\infty} \frac{1}{k'(\tau)} \dot{\mathbf{I}}^t_{(1)}(\tau) \cdot \mathbf{I}^t_{(1)}(\tau) d\tau. \tag{43}$$

Here, taking into account (42), we have

$$\dot{\mathbf{I}}^t_{(1)}(\tau) = \frac{d}{dt} \mathbf{I}^t_{(1)}(\tau) = -k'(\tau) \mathbf{g}(t) + \mathbf{I}^t_{(2)}(\tau),$$

where

$$\mathbf{I}^t_{(2)}(\tau) = \frac{d}{d\tau} \mathbf{I}^t_{(1)}(\tau) = \frac{d^2}{d\tau^2} \mathbf{I}^t(\tau).$$

Hence, (43) becomes

$$\dot{\psi}_{\mathcal{F}}(t) = \alpha_0 \dot{\vartheta}(t) \vartheta(t) + \int_0^{+\infty} \breve{\mathbf{I}}^t_{(1)}(\tau) d\tau \cdot \mathbf{g}(t) - \int_0^{+\infty} \frac{1}{k'(\tau)} \mathbf{I}^t_{(2)}(\tau) \cdot \mathbf{I}^t_{(1)}(\tau) d\tau,$$

where the first integral gives

$$\int_0^{+\infty} \mathbf{I}^t_{(1)}(\tau) d\tau = -\mathbf{I}^t(0, \bar{\mathbf{g}}^t) = -\int_0^{+\infty} k'(s) \bar{\mathbf{g}}^t(s) ds = -\mathbf{q}(t),$$

while the second one, by integrating by parts, yields

$$\int_0^{+\infty} \frac{1}{k'(\tau)} \mathbf{I}^t_{(2)}(\tau) \cdot \mathbf{I}^t_{(1)}(\tau) d\tau = \frac{1}{2} \int_0^{+\infty} \frac{1}{k'(\tau)} \frac{d}{d\tau} [\mathbf{I}^t_{(1)}(\tau) \cdot \mathbf{I}^t_{(1)}(\tau)] d\tau =$$

$$= -\frac{1}{2} \int_0^{+\infty} \frac{d}{d\tau} \left[\frac{1}{k'(\tau)} \right] \mathbf{I}^t_{(1)}(\tau) \cdot \mathbf{I}^t_{(1)}(\tau) d\tau - \frac{1}{2} \frac{1}{k'(0)} \mathbf{I}^t_{(1)}(0) \cdot \mathbf{I}^t_{(1)}(0).$$

Thus, we get

$$\dot{\psi}_{\mathcal{F}}(t) + D_{\mathcal{F}}(t) = \alpha_0 \dot{\vartheta}(t)\vartheta(t) - \mathbf{q}(t) \cdot \mathbf{g}(t),$$

where the corresponding dissipation is given by

$$D_{\mathcal{F}}(t) = -\frac{1}{2} \int\limits_0^{+\infty} \frac{d}{d\tau} \left[\frac{1}{k'(\tau)} \right] \mathbf{I}^t_{(1)}(\tau) \cdot \mathbf{I}^t_{(1)}(\tau) d\tau - \frac{1}{2} \frac{1}{k'(0)} \mathbf{I}^t_{(1)}(0) \cdot \mathbf{I}^t_{(1)}(0) \geq 0; \tag{44}$$

the positiveness of $D_{\mathcal{F}}(t)$ follows from the assumption $(9)_2$ and the following inequality:

$$\frac{d}{d\tau} \left[\frac{1}{\mu'(\tau)} \right] = -\frac{1}{[k'(\tau)]^2} k''(\tau) \leq 0. \tag{45}$$

Obviously, from (44) we get

$$D_{\mathcal{F}}(t) \geq -\frac{1}{2} \int\limits_0^{+\infty} \frac{d}{d\tau} \left[\frac{1}{k'(\tau)} \right] \mathbf{I}^t_{(1)}(\tau) \cdot \mathbf{I}^t_{(1)}(\tau) d\tau \geq 0$$

and from this inequality, on assuming that there exists $\delta \in \mathbf{R}^{++}$ such that

$$k''(\tau) + \delta k'(\tau) \geq 0 \qquad \forall \tau \in \mathbf{R}^+,$$

which, using (45), yields

$$\frac{d}{d\tau} \left[\frac{1}{\mu'(\tau)} \right] = -\frac{k''(\tau)}{[k'(\tau)]^2} \leq \frac{\delta}{k'(\tau)} < 0,$$

it follows that the dissipation also satisfies the inequality

$$D_{\mathcal{F}}(t) \geq \delta \left[\psi_{\mathcal{F}}(t) - \frac{1}{2} \alpha_0 \vartheta^2(t) \right].$$

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