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Reparametrizations of vector fields and their shift maps

Let M be a smooth manifold, F be a smooth vector field on M, and $(\mathbf{F}_t)$ be the local flow of F. Denote by $Sh(F)$ the subset of $C^\infty(M,M)$ consisting of maps $h: M \to M$ of the following form:

$$h(x) = \mathbf{F}_{\alpha(x)}(x),$$

where α runs over all smooth functions $M \to \mathbb{R}$ which can be substituted into $\mathbf{F}$ instead of t. This space often contains the identity component of the group of diffeomorphisms preserving orbits of F. In this note it is shown that $Sh(F)$ is not changed under reparametrizations of F, that is for any smooth strictly positive function $\mu : M \to (0, +\infty)$ we have that $Sh(F) = Sh(\mu F)$. As an application it is proved that F can be reparametrized to induce a circle action on M if and only if there exists a smooth function $\mu : M \to (0, +\infty)$ such that $\mathbf{F}(x, \mu(x)) \equiv x$.

Keywords: *Reparametrization of a flow, shift map, circle action*

1. Introduction

Let M be a smooth manifold and F be a smooth vector field on M tangent to ∂M. For each $x \in M$ its *integral trajectory* with respect to F is a unique mapping $o_x : \mathbb{R} \supset (a_x, b_x) \to M$ such that $o_x(0) = x$ and $\frac{d}{dt} o_x = F(o_x)$, where $(a_x, b_x) \subset \mathbb{R}$ is the maximal interval on which a map with the previous two properties can be defined. The image of o_x will be denoted by the same symbol o_x and also called the *orbit* of x. It follows that from standard theorems in ODE the following subset of $M \times \mathbb{R}$

$$\mathsf{dom}(F) = \bigcup_{x \in M} x \times (a_x, b_x),$$

is an open, connected neighbourhood of $M \times 0$ in $M \times \mathbb{R}$. Then the *local flow* of F is the following map

$$\mathbf{F} : M \times \mathbb{R} \supset \mathsf{dom}(F) \to M, \qquad \mathbf{F}(x,t) = \mathbf{F}_x(t).$$

It is well known that if M is compact, or F has compact support, then $\mathbf{F}$ is defined on all of M.

Denote by $\mathsf{func}(F) \subset C^\infty(M,\mathbb{R})$ the subset consisting of functions $\alpha : M \to \mathbb{R}$ whose graph $\Gamma_\alpha = \{(x, \alpha(x)) : x \in M\}$ is contained in $\mathsf{dom}(F)$. Then we can define the following map

$$\varphi : C^\infty(M,\mathbb{R}) \supset \mathsf{func}(F) \longrightarrow C^\infty(M,M),$$

$$\varphi(\alpha)(x) = \mathbf{F}(x, \alpha(x)).$$

This map will be called the *shift map* along orbits of F and its image in $C^\infty(M,M)$ will be denoted by $Sh(F)$.

It is easy to see, [1, Lm. 2], that φ is $\mathsf{S}^{r,r}$-continuous for all $r \geq 0$, that is continuous between the corresponding S^r Whitney topologies of $\mathsf{func}(F)$ and $C^\infty(M,M)$.

Moreover, if the set Σ_F of singular points of F is nowhere dense, then φ is locally injective, [1, Pr. 14]. Therefore it is natural to know whether it is a homeomorphism with respect to some Whitney topologies, and, in particular, whether it is $\mathsf{S}^{r,s}$-open, i.e. open as a map from S^r topology of $\mathsf{func}(F)$ into S^s topology of the image $Sh(F)$, for some $r, s \geq 0$. These problems and their applications were treated e.g. in [1–3].

In this note we prove the following theorems describing the behaviour of the image of shift maps under reparametrizations and pushforwards.

Theorem 1. *Let $\mu : M \to \mathbb{R}$ be any smooth function and $G = \mu F$ be the vector field obtained by the multiplication F by μ. Then*

$$Sh(G) \subset Sh(F). \tag{1}$$

Suppose that $\mu \neq 0$ on all of M. Then

$$Sh(\mu F) = Sh(F).$$

In this case the shift mapping $\varphi : \mathsf{func}(F) \to Sh(F)$ of F is $\mathsf{S}^{r,s}$-open for some $r, s \geq 0$, if and only if so is the shift mapping $\psi : \mathsf{func}(G) \to Sh(G)$ of G.

Theorem 2. *Let $z \in M$, $\alpha : (M, z) \to \mathbb{R}$ be a germ of smooth function at z, and $f : M \to M$ be a germ of smooth map defined by $f(x) = \mathbf{F}(x, \alpha(x))$. Suppose that f is a germ of diffeomorphism at z. Then*

$$f_* F = (1 + F(\alpha)) \cdot F, \tag{2}$$

where $f_* F = Tf \circ F \circ f^{-1}$ is the vector field induced by f, and $F(\alpha)$ is the derivative of α along F. Thus $f_* F$ is just a reparametrization of F.

If $\alpha : M \to \mathbb{R}$ is defined on all of M and $f = \varphi(\alpha)$ is a diffeomorphism of M, then

$$Sh(f_* F) = Sh(F).$$

Further in §3 we will apply these results to circle actions. In particular, we prove that F can be reparametrized to induce a circle action on M if and only if there exists a smooth function $\mu : M \to (0, +\infty)$ such that $\mathbf{F}(x, \mu(x)) \equiv x$, see Corollary 1.

2. Proofs of Theorems 1 and 2

These theorems are based on the following well-known statement, see e.g. [4,5,8] for its variants in the category of measurable maps.

Lemma 1. *Let $G = \mu F$ and $\mathbf{G} : \mathsf{dom}(G) \to M$ be the local flow of G. Then there exists a smooth function $\alpha : \mathsf{dom}(G) \to \mathbb{R}$ such that*

$$\mathbf{G}(x, s) = \mathbf{F}(x, \alpha(x, s)).$$

In fact,

$$\alpha(x,s) = \int_0^s \mu(\mathbf{G}(x,\tau))d\tau. \tag{3}$$

In particular, for each $\gamma \in \mathsf{func}(G)$ *we have that*

$$\mathbf{G}(x,\gamma(x)) = \mathbf{F}(x,\alpha(x,\gamma(x))), \tag{4}$$

whence $Sh(G) \subset Sh(F)$.

Proof. Put $\mathcal{G}(x,s) = \mathbf{F}(x,\alpha(x,s))$, where α is defined by (3). We have to show that $\mathbf{G} = \mathcal{G}$.

Notice that a flow $\mathbf{G}$ of a vector field G is a *unique* mapping that satisfies the following ODE with initial condition:

$$\left.\frac{\partial \mathbf{G}(x,s)}{\partial s}\right|_{s=0} = G(x) = F(x)\mu(x), \qquad \mathbf{G}(x,0) = x.$$

Notice that

$$\alpha(x,0) = 0, \qquad \alpha'_s(x,0) = \mu(\mathbf{G}(x,0)) = \mu(x).$$

In particular, $\mathcal{G}(x,0) = \mathbf{F}(x,\alpha(x,0)) = x$. Therefore it remains to verify that

$$\left.\frac{\partial \mathcal{G}(x,s)}{\partial s}\right|_{s=0} = F(x)\cdot\mu(x). \tag{5}$$

We have:

$$\frac{\partial \mathcal{G}}{\partial s}(x,s) = \frac{\partial \mathbf{F}}{\partial s}(x,\alpha(x,s)) = \left.\frac{\partial \mathbf{F}(x,t)}{\partial t}\right|_{t=\alpha(x,s)} \cdot \alpha'_s(x,s). \tag{6}$$

Substituting $s = 0$ in (6) we get (5). □

Proof of Theorem 1. Eq. (1) is established in Lemma 1.

Suppose that $\mu \neq 0$ on all of M. Then $F = \frac{1}{\mu}G$, and $\frac{1}{\mu}$ is smooth on all of M. Hence again by Lemma 1 $Sh(F) \subset Sh(G)$, and thus $Sh(F) = Sh(G)$.

To prove the last statement define a map $\xi : \mathsf{func}(G) \to \mathsf{func}(F)$ by

$$\xi(\gamma)(x) = \alpha(x, \gamma(x)) = \int_0^s \mu(\mathbf{G}(x,\tau))d\tau, \qquad \gamma \in \mathsf{func}(G).$$

Then (4) means that the following diagram is commutative:

$$\begin{array}{ccc} \mathsf{func}(G) & \xrightarrow{\xi} & \mathsf{func}(F) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Sh(G) & = & Sh(F) \end{array}$$

We claim that ξ is a homeomorphism with respect to S^r topologies for all $r \geq 0$. Indeed, evidently ξ is $\mathsf{S}^{r,r}$-continuous. Put

$$\beta(x,s) = \int_0^s \frac{d\tau}{\mu(\mathbf{F}(x,\tau))}. \tag{7}$$

Then the inverse map $\xi^{-1} : \mathsf{func}(F) \to \mathsf{func}(G)$ is given by

$$\xi^{-1}(\delta)(x) = \beta(x, \delta(x)) = \int_0^{\delta(x)} \frac{d\tau}{\mu(\mathbf{F}(x,\tau))}, \qquad \delta \in \mathsf{func}(F), \tag{8}$$

and is also $\mathsf{S}^{r,r}$-continuous. Hence ψ is $\mathsf{S}^{r,s}$-open iff so is φ. Theorem 1 is completed.

Proof of Theorem 2. First we reduce the situation to the case $\alpha(z) = 0$. Suppose that $a = \alpha(z) \neq 0$ and let $\beta(x) = \alpha(x) - a$. Define the following germ of diffeomorphisms $g = \mathbf{F}_{-a} \circ f$ at z:

$$g(x) = \mathbf{F}(\mathbf{F}(x, \alpha(x)), -a) = \mathbf{F}(x, \alpha(x) - a) = \mathbf{F}(x, \beta(x)).$$

Then $g(z) = z$, and $\beta(z) = 0$.

Since $\mathbf{F}$ preserves F, i.e. $(\mathbf{F}_t)_* F = F$ for all $t \in \mathbb{R}$, we obtain that

$$f_* F = f_* (\mathbf{F}_{-a})_* F = (f \circ \mathbf{F}_{-a})_* F = g_* F.$$

Moreover, $F(\alpha) = F(\beta)$. Therefore it suffices to prove our statement for g.

If z is a singular point of F, i.e. $F = 0$, then both parts of (2) vanish. Therefore we can assume that z is a regular point of F. Then there are local coordinates $(x_1, \dots, x_n)$ at $z = 0 \in \mathbb{R}^n$ in which $F(x) = \frac{\partial}{\partial x_1}$ and

$$\mathbf{F}(x_1, \dots, x_n, t) = (x_1 + t, x_2, \dots, x_n).$$

Then $g(x_1, \dots, x_n) = (x_1 + \beta(x), x_2, \dots, x_n)$, whence

$$Tg \circ F \circ g^{-1} = \begin{pmatrix} 1+\beta'_{x_1} & \beta'_{x_2} & \cdots & \beta'_{x_n} \\ 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ 0 \\ \cdots \\ 0 \end{pmatrix} =$$
$$= (1 + \beta'_{x_1})F = (1 + F(\beta))F.$$

Suppose now that α is defined on all of M and f is a diffeomorphism of all of M. Then by [1] the function $\mu = 1 + F(\alpha) \neq 0$ on all of M, whence by Theorem 1 $Sh(\mu F) = Sh(F)$.

3. Periodic shift maps

Let F be a vector field, and φ be its shift map. The set

$$\ker(\varphi) = \varphi^{-1}(\mathrm{id}_M)$$

will be called the *kernel* of φ, thus $\mathbf{F}(x, \nu(x)) \equiv x$ for all $\nu \in \ker(\varphi)$. Evidently, $0 \in \ker(\varphi)$. Moreover, it is shown in [1, Lm. 5] that $\varphi(\alpha) = \varphi(\beta)$ iff $\alpha - \beta \in \mathsf{func}(F)$.

Suppose that the set Σ_F of singular points of F is nowhere dense in M. Then, [1, Th. 12 & Pr. 14], φ is a locally injective map with respect to any weak or strong topologies, and we have the following two possibilities for $\ker(\varphi)$:

a) **Non-periodic case:** $\ker(\varphi) = \{0\}$, so $\varphi : \mathsf{func}(F) \to Sh(F)$ is a bijection.

b) **Periodic case:** there exists a smooth strictly positive function

$$\theta : M \to (0, +\infty)$$

such that $\mathbf{F}(x, \theta(x)) \equiv x$ and $\ker(\varphi) = \{n\theta\}_{n\in\mathbb{Z}}$.

In this case $\mathsf{func}(F) = C^\infty(M, \mathbb{R})$, φ yields a bijection between $C^\infty(M, \mathbb{R})/\ker(\varphi)$ and $Sh(F)$, and for every $\alpha \in C^\infty(M, \mathbb{R})$ we have that

$$\varphi^{-1} \circ \varphi(\alpha) = \alpha + \ker(\varphi) = \{\alpha + k\theta\}_{k\in\mathbb{Z}}.$$

It also follows that every non-singular point x of F is periodic of some period $\mathrm{Per}(x)$,

$$\theta(x) = n_x \mathrm{Per}(x)$$

for some $n_x \in \mathbb{N}$, and in particular, θ is constant along orbits of F. We will call θ the *period function* for φ.

Lemma 2. *Suppose that the shift map φ of F is periodic and let θ be its period function. Let also $\mu : M \to (0, +\infty)$ be any smooth strictly positive function. Put $G = \mu F$. Then the shift map ψ of G is also periodic, and its period function is*

$$\bar{\theta}(x) \overset{(8)}{=\!=} \xi^{-1}(\theta)(x) = \beta(x, \theta(x)) = \int\limits_0^{\theta(x)} \frac{d\tau}{\mu(\mathbf{F}(x,\tau))}. \tag{9}$$

If μ is constant along orbits of F, then the last formula reduces to the following one:

$$\bar{\theta} = \frac{\theta}{\mu}. \tag{10}$$

In particular, for the vector field $G = \theta F$ its period function is equal to $\bar{\theta} \equiv 1$.

Proof. Let $\mathbf{G} : M \times \mathbb{R} \to M$ be the flow of G. We have to show that $\mathbf{G}(x, \bar{\theta}(x)) \equiv x$ for all $x \in M$:

$$\mathbf{G}(x, \bar{\theta}(x)) \overset{(9)}{=\!=\!=} \mathbf{G}\big(x, \beta(x, \theta(x))\big) = \mathbf{F}(x, \theta(x)) \equiv x. \tag{11}$$

Since θ is the *minimal* positive function for which $\mathbf{F}(x,\theta(x)) \equiv x$ and $\mu > 0$, it follows from (9) that so is $\bar{\theta}$ is also the minimal positive function for which (11) holds true. Hence $\bar{\theta}$ is the period function for the shift map of G.

Let us prove (10). Since μ is constant along orbits of F, we have that $\mu(\mathbf{F}(x,\tau)) = \mu(x)$, whence

$$\bar{\theta}(x) = \beta(x,\theta(x)) = \int\limits_0^{\theta(x)} \frac{d\tau}{\mu(\mathbf{F}(x,\tau))} = \int\limits_0^{\theta(x)} \frac{d\tau}{\mu(x)} = \frac{\theta(x)}{\mu(x)}.$$

Lemma is proved. $\square$

3.1. **Circle actions.** Regard S^1 as the group $U(1)$ of complex numbers with norm 1, and let $\exp : \mathbb{R} \to S^1$ be the exponential map defined by $\exp(t) = e^{2\pi i t}$.

Let $\Gamma : M \times S^1 \to M$ be a smooth action of S^1 on M. Then it yields a smooth $\mathbb{R}$-cation (or a flow) $\mathbf{G} : M \times \mathbb{R} \to M$ given by

$$\mathbf{G}(x,t) = \Gamma(x, \exp(t)). \tag{12}$$

Moreover $\mathbf{G}$ is generated by the following vector field

$$G(x) = \left.\frac{\partial \mathbf{G}(x,t)}{\partial t}\right|_{t=0}.$$

Evidently, any of Γ, $\mathbf{G}$, and G determines two others. In particular, a flow $\mathbf{G}$ on M is of the form (12) for some smooth circle action Γ on M if and only if $\mathbf{G}_1 = \mathrm{id}_M$, i.e. $\mathbf{G}(x,1) \equiv x$ for all $x \in M$.

In other words, the shift map of $\mathbf{G}$ is periodic and its period function is the constant function $\theta \equiv 1$.

As a consequence of Lemma 2 we get the following:

Corollary 1. *Let F be a smooth vector field on M and*

$$\theta : M \to (0, +\infty)$$

be a smooth strictly positive function. Then the following conditions are equivalent:

(a) *the vector field $G = \theta F$ yields a smooth circle action, i.e.* $\mathbf{G}(x, 1) = x$ *for all* $x \in M$*;*
(b) *the shift map φ of F is periodic and θ is its period function, i.e.* $\mathbf{F}(x, \theta(x)) \equiv x$ *for all* $x \in M$*.*

Corollary 2. *Suppose that the shift map φ of F is periodic and let $z \in M$ be a singular point of F. Then there are $k, l \geq 0$ such that $2k+l = \dim M$, non-zero numbers $A_1, \dots, A_k \in \mathbb{R} \setminus \{0\}$, local coordinates $(x_1, y_1, \dots, x_k, y_k, t_1, \dots, t_l)$ at $z = 0 \in \mathbb{R}^{2k+l}$, and in which the linear part of F at 0 is given by*

$$\begin{aligned} j^1_0 F(x_1, y_1, \dots, x_k, y_k, t_1, \dots, t_l) \;=\; & -A_1 y_1 \frac{\partial}{\partial x_1} + A_1 x_1 \frac{\partial}{\partial y_1} + \cdots \\ & -A_k y_k \frac{\partial}{\partial x_k} + A_k x_k \frac{\partial}{\partial y_k}. \end{aligned}$$

Proof. Let θ be the period function for F and $G = \theta F$. Since $\theta > 0$, it follows that $\Sigma_F = \Sigma_G$ and for every $z \in \Sigma_F$ we have that

$$j^1_z G = \theta(z) \cdot j^1_z F.$$

Therefore it suffices to prove our statement for G.

By Corollary 1 G yields a circle action, i.e. $\mathbf{G}_1 = \mathrm{id}_M$, where $\mathbf{G}$ is the flow of G. Then $\mathbf{G}$ yields a linear flow $T_z\mathbf{G}_t$ on the tangent space $T_z M$ such that $T_z\mathbf{G}_1 = \mathrm{id}$. In other words we obtain a linear action (i.e. representation) of the circle group $U(1)$ in the finite-dimensional vector space $T_z M$. Now the result follows from standard theorems about presentations of $U(1)$. □

Remark 1. Suppose that in Corollary 2 $\dim M = 2$. Then we can choose local coordinates (x, y) at $z = 0 \in \mathbb{R}^2$ in which

$$j^1_0 F(x, y) = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$

For this case the normal forms of such vector fields are described in [7].

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