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Volodymyr Lyubashenko

Institute of Mathematics, NAS of Ukraine, Kyiv
E-mail: lub@imath.kiev.ua

Oleksandr Manzyuk

Institute of Mathematics, NAS of Ukraine, Kyiv
E-mail: manzyuk@mathematik.uni-kl.de

Unital A_∞-categories

Ми доводимо, що три означення унітальності для A_∞-категорій запропоновані Любашенком, Концевичем і Сойбельманом, та Фукая є еквівалентними.

We prove that three definitions of unitality for A_∞-categories suggested by Lyubashenko, by Kontsevich and Soibelman, and by Fukaya are equivalent.

Keywords: *A_∞-category, unital A_∞-category, weak unit*

1. Introduction

Over the past decade, A_∞-categories have experienced a resurgence of interest due to applications in symplectic geometry, deformation theory, non-commutative geometry, homological algebra, and physics.

The notion of A_∞-category is a generalization of Stasheff's notion of A_∞-algebra [11]. On the other hand, A_∞-categories generalize differential graded categories. In contrast to differential graded categories, composition in A_∞-categories is associative only up to homotopy that satisfies certain equation up to another homotopy, and so on. The notion of A_∞-category appeared in the work of Fukaya on Floer homology [1] and was related to mirror symmetry by Kontsevich [5]. Basic concepts of the theory of A_∞-categories have been developed by Fukaya [2], Keller [4], Lefèvre-Hasegawa [7], Lyubashenko [8], Soibelman [10].

The definition of A_∞-category does not assume the existence of identity morphisms. The use of A_∞-categories without identities requires caution: for example, there is no a sensible notion of isomorphic objects, the notion of equivalence does not make sense, etc. In order to develop a comprehensive theory of A_∞-categories, a notion of unital A_∞-category, i.e., A_∞-category with identity morphisms (also called units), is necessary. The obvious notion of strictly unital A_∞-category, despite its technical advantages, is not quite satisfactory: it is not homotopy invariant, meaning that it does not translate along homotopy equivalences. Different definitions of (weakly) unital A_∞-category have been suggested by Lyubashenko [8, Definition 7.3], by Kontsevich and Soibelman [6, Definition 4.2.3], and by Fukaya [2, Definition 5.11]. We prove that these definitions are equivalent. The main ingredient of the proofs is the Yoneda Lemma for unital (in the sense of Lyubashenko) A_∞-categories proven in [9, Appendix A].

2. Preliminaries

We follow the notation and conventions of [8], sometimes without explicit mentioning. Some of the conventions are recalled here.

Throughout, $\Bbbk$ is a commutative ground ring. A graded $\Bbbk$-module always means a $\mathbb{Z}$-graded $\Bbbk$-module.

A *graded quiver* $\mathcal{A}$ consists of a set $\operatorname{Ob}\mathcal{A}$ of objects and a graded $\Bbbk$-module $\mathcal{A}(X,Y)$, for each $X, Y \in \operatorname{Ob}\mathcal{A}$. A *morphism of graded quivers* $f : \mathcal{A} \to \mathcal{B}$ of degree n consists of a function $\mathrm{Ob} f : \mathrm{Ob}\mathcal{A} \to \mathrm{Ob}\mathcal{B}$, $X \mapsto Xf$, and a $\Bbbk$-linear map $f = f_{X,Y} : \mathcal{A}(X,Y) \to \mathcal{B}(Xf,Yf)$ of degree n, for each $X, Y \in \mathrm{Ob}\mathcal{A}$.

For a set S, there is a category $\mathscr{Q}/S$ defined as follows. Its objects are graded quivers whose set of objects is S. A morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathscr{Q}/S$ is a morphism of graded quivers of degree 0 such that $\mathrm{Ob} f = \mathrm{id}_S$. The category $\mathscr{Q}/S$ is monoidal. The tensor product of graded quivers $\mathcal{A}$ and $\mathcal{B}$ is a graded quiver $\mathcal{A} \otimes \mathcal{B}$ such that

$$(\mathcal{A} \otimes \mathcal{B})(X,Z) = \bigoplus_{Y \in S} \mathcal{A}(X,Y) \otimes \mathcal{B}(Y,Z), \quad X, Z \in S.$$

The unit object is the *discrete quiver* $\Bbbk S$ with $\mathrm{Ob}\Bbbk S = S$ and

$$(\Bbbk S)(X,Y) = \begin{cases} \Bbbk & \text{if } X = Y, \\ 0 & \text{if } X \neq Y, \end{cases} \quad X, Y \in S.$$

Note that a map of sets $f : S \to R$ gives rise to a morphism of graded quivers $\Bbbk f : \Bbbk S \to \Bbbk R$ with $\mathrm{Ob}\Bbbk f = f$ and $(\Bbbk f)_{X,Y} = \mathrm{id}_\Bbbk$ is $X = Y$ and $(\Bbbk f)_{X,Y} = 0$ if $X \neq Y$, $X, Y \in S$.

An *augmented graded cocategory* is a graded quiver $\mathcal{C}$ equipped with the structure of on augmented counital coassociative coalgebra in the monoidal category $\mathscr{Q}/\mathrm{Ob}\mathcal{C}$. Thus, $\mathcal{C}$ comes with a comultiplication $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, a counit $\varepsilon : \mathcal{C} \to \Bbbk\mathrm{Ob}\mathcal{C}$, and an augmentation $\eta : \Bbbk\mathrm{Ob}\mathcal{C} \to \mathcal{C}$, which are morphisms in $\mathscr{Q}/\mathrm{Ob}\mathcal{C}$ satisfying the usual axioms. A *morphism of augmented graded cocategories* $f : \mathcal{C} \to \mathcal{D}$ is a morphism of graded quivers of degree 0 that preserves the comultiplication, counit, and augmentation.

The main example of an augmented graded cocategory is the following. Let $\mathcal{A}$ be a graded quiver. Denote by $T\mathcal{A}$ the direct sum of graded quivers $T^n\mathcal{A}$, where $T^n\mathcal{A} = \mathcal{A}^{\otimes n}$ is the n-fold tensor product of $\mathcal{A}$ in $\mathscr{Q}/\mathrm{Ob}\mathcal{A}$; in particular,

$T^0\mathcal{A} = \Bbbk\mathrm{Ob}\mathcal{A}$, $T^1\mathcal{A} = \mathcal{A}$, $T^2\mathcal{A} = \mathcal{A}\otimes\mathcal{A}$, etc. The graded quiver $T\mathcal{A}$ is an augmented graded cocategory in which the comultiplication is the so called 'cut' comultiplication $\Delta_0 : T\mathcal{A} \to T\mathcal{A}\otimes T\mathcal{A}$ given by

$$f_1\otimes\cdots\otimes f_n \mapsto \sum_{k=0}^{n} f_1\otimes\cdots\otimes f_k \bigotimes f_{k+1}\otimes\cdots\otimes f_n,$$

the counit is given by the projection $\mathrm{pr}_0 : T\mathcal{A} \to T^0\mathcal{A} = \Bbbk\mathrm{Ob}\mathcal{A}$, and the augmentation is given by the inclusion $\mathrm{in}_0 : \Bbbk\mathrm{Ob}\mathcal{A} = T^0\mathcal{A} \hookrightarrow T\mathcal{A}$.

The graded quiver $T\mathcal{A}$ admits also the structure of a graded category, i.e., the structure of a unital associative algebra in the monoidal category $\mathscr{Q}/\mathrm{Ob}\mathcal{A}$. The multiplication $\mu : T\mathcal{A}\otimes T\mathcal{A} \to T\mathcal{A}$ removes brackets in tensors of the form $(f_1\otimes\cdots\otimes f_m)\bigotimes(g_1\otimes\cdots\otimes g_n)$. The unit $\eta : \Bbbk\mathrm{Ob}\mathcal{A} \to T\mathcal{A}$ is given by the inclusion $\mathrm{in}_0 : \Bbbk\mathrm{Ob}\mathcal{A} = T^0\mathcal{A} \hookrightarrow T\mathcal{A}$.

For a graded quiver $\mathcal{A}$, denote by $s\mathcal{A}$ its *suspension*, the graded quiver given by $\mathrm{Ob}s\mathcal{A} = \mathrm{Ob}\mathcal{A}$ and $(s\mathcal{A}(X,Y))^n = \mathcal{A}(X,Y)^{n+1}$, for each $n \in \mathbb{Z}$ and $X, Y \in \mathrm{Ob}\mathcal{A}$. An A_∞-*category* is a graded quiver $\mathcal{A}$ equipped with a differential $b : Ts\mathcal{A} \to Ts\mathcal{A}$ of degree 1 such that $(Ts\mathcal{A}, \Delta_0, \mathrm{pr}_0, \mathrm{in}_0, b)$ is an *augmented differential graded cocategory*. In other terms, the equations

$$b^2 = 0, \quad b\Delta_0 = \Delta_0(b\otimes 1 + 1\otimes b), \quad b\mathrm{pr}_0 = 0, \quad \mathrm{in}_0 b = 0$$

hold true. Denote by

$$b_{mn} \overset{\text{def}}{=} \big[T^m s\mathcal{A} \xrightarrow{\mathrm{in}_m} Ts\mathcal{A} \xrightarrow{b} Ts\mathcal{A} \xrightarrow{\mathrm{pr}_n} T^n s\mathcal{A}\big]$$

matrix coefficients of b, for $m, n \geqslant 0$. Matrix coefficients b_{m1} are called *components* of b and abbreviated by b_m. The above equations imply that $b_0 = 0$ and that b is unambiguously determined by its components via the formula

$$b_{mn} = \sum_{\substack{p+k+q=m \\ p+1+q=n}} 1^{\otimes p} \otimes b_k \otimes 1^{\otimes q} : T^m s\mathcal{A} \to T^n s\mathcal{A}, \quad m, n \geqslant 0.$$

The equation $b^2 = 0$ is equivalent to the system of equations

$$\sum_{p+k+q=m} (1^{\otimes p} \otimes b_k \otimes 1^{\otimes q}) b_{p+1+q} = 0 : T^m s\mathcal{A} \to s\mathcal{A}, \quad m \geqslant 1.$$

For A_∞-categories $\mathcal{A}$ and $\mathcal{B}$, an A_∞*-functor* $f : \mathcal{A} \to \mathcal{B}$ is a morphism of augmented differential graded cocategories $f : Ts\mathcal{A} \to Ts\mathcal{B}$. In other terms, f is a morphism of augmented graded cocategories and preserves the differential, meaning that $fb = bf$. Denote by

$$f_{mn} \overset{\text{def}}{=} \bigl[T^m s\mathcal{A} \xrightarrow{\text{in}_m} Ts\mathcal{A} \xrightarrow{f} Ts\mathcal{B} \xrightarrow{\text{pr}_n} T^n s\mathcal{B}\bigr]$$

matrix coefficients of f, for $m, n \geqslant 0$. Matrix coefficients f_{m1} are called *components* of f and abbreviated by f_m. The condition that f is a morphism of augmented graded cocategories implies that $f_0 = 0$ and that f is unambiguously determined by its components via the formula

$$f_{mn} = \sum_{i_1+\dots+i_n=m} f_{i_1} \otimes \dots \otimes f_{i_n} : T^m s\mathcal{A} \to T^n s\mathcal{B}, \quad m, n \geqslant 0.$$

The equation $fb = bf$ is equivalent to the system of equations

$$\sum_{i_1+\dots+i_n=m} (f_{i_1} \otimes \dots \otimes f_{i_n}) b_n = \sum_{p+k+q=m} (1^{\otimes p} \otimes b_k \otimes 1^{\otimes q}) f_{p+1+q} : T^m s\mathcal{A} \to s\mathcal{B},$$

for $m \geqslant 1$. An A_∞-functor f is called *strict* if $f_n = 0$ for $n > 1$.

3. Definitions

3.1. **Definition** (cf. [2,4]). An A_∞-category $\mathcal{A}$ is *strictly unital* if, for each $X \in \operatorname{Ob}\mathcal{A}$, there is a $\Bbbk$-linear map ${}_X\mathbf{i}_0^{\mathcal{A}} : \Bbbk \to (s\mathcal{A})^{-1}(X,X)$, called a *strict unit*, such that the following conditions are satisfied: ${}_X\mathbf{i}_0^{\mathcal{A}} b_1 = 0$, the chain maps $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{A}})b_2, -({}_X\mathbf{i}_0^{\mathcal{A}} \otimes 1)b_2 : s\mathcal{A}(X,Y) \to s\mathcal{A}(X,Y)$ are equal to the identity map, for each $X, Y \in \operatorname{Ob}\mathcal{A}$, and $(\dots \otimes \mathbf{i}_0^{\mathcal{A}} \otimes \dots)b_n = 0$ if $n \geqslant 3$.

For example, differential graded categories are strictly unital.

3.2. **Definition** (Lyubashenko [8, Definition 7.3]). An A_∞-category $\mathcal{A}$ is *unital* if, for each $X \in \operatorname{Ob}\mathcal{A}$, there is a $\Bbbk$-linear map ${}_X\mathbf{i}_0^{\mathcal{A}} : \Bbbk \to (s\mathcal{A})^{-1}(X,X)$, called a *unit*, such that the following conditions hold: ${}_X\mathbf{i}_0^{\mathcal{A}} b_1 = 0$ and the chain maps $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{A}})b_2, -({}_X\mathbf{i}_0^{\mathcal{A}} \otimes 1)b_2 : s\mathcal{A}(X,Y) \to s\mathcal{A}(X,Y)$ are homotopic to the identity map, for each $X, Y \in \operatorname{Ob}\mathcal{A}$. An arbitrary homotopy between $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{A}})b_2$ and the identity map is called a *right unit homotopy*. Similarly, an arbitrary homotopy between $-({}_X\mathbf{i}_0^{\mathcal{A}} \otimes 1)b_2$ and the identity map is called a *left unit homotopy*. An A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ between unital A_∞-categories is *unital* if the cycles ${}_X\mathbf{i}_0^{\mathcal{A}} f_1$ and ${}_{Xf}\mathbf{i}_0^{\mathcal{B}}$ are cohomologous, i.e., differ by a boundary, for each $X \in \operatorname{Ob}\mathcal{A}$.

Clearly, a strictly unital A_∞-category is unital.

With an arbitrary A_∞-category $\mathcal{A}$ a strictly unital A_∞-category $\mathcal{A}^{\mathsf{su}}$ with the same set of objects is associated. For each $X, Y \in \operatorname{Ob}\mathcal{A}$, the graded $\Bbbk$-module $s\mathcal{A}^{\mathsf{su}}(X,Y)$ is given by

$$s\mathcal{A}^{\mathsf{su}}(X,Y) = \begin{cases} s\mathcal{A}(X,Y) & \text{if } X \neq Y, \\ s\mathcal{A}(X,X) \oplus \Bbbk\, {}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{su}}} & \text{if } X = Y, \end{cases}$$

where ${}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{su}}}$ is a new generator of degree -1. The element ${}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{su}}}$ is a strict unit by definition, and the natural embedding $e : \mathcal{A} \hookrightarrow \mathcal{A}^{\mathsf{su}}$ is a strict A_∞-functor.

3.3. **Definition** (Kontsevich–Soibelman [6, Definition 4.2.3])**.** A *weak unit* of an A_∞-category $\mathcal{A}$ is an A_∞-functor $U : \mathcal{A}^{\mathsf{su}} \to \mathcal{A}$ such that

$$\big[\mathcal{A} \overset{e}{\hookrightarrow} \mathcal{A}^{\mathsf{su}} \xrightarrow{U} \mathcal{A}\big] = \mathrm{id}_{\mathcal{A}}.$$

3.4. **Proposition.** *Suppose that an A_∞-category $\mathcal{A}$ admits a weak unit. Then the A_∞-category $\mathcal{A}$ is unital.*

Proof. Let $U : \mathcal{A}^{\mathsf{su}} \to \mathcal{A}$ be a weak unit of $\mathcal{A}$. For each $X \in \mathrm{Ob}\mathcal{A}$, denote by ${}_X\mathbf{i}_0^{\mathcal{A}}$ the element ${}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{su}}}U_1 \in s\mathcal{A}(X,X)$ of degree -1. It follows from the equation $U_1b_1 = b_1U_1$ that ${}_X\mathbf{i}_0^{\mathcal{A}}b_1 = 0$. Let us prove that ${}_X\mathbf{i}_0^{\mathcal{A}}$ are unit elements of $\mathcal{A}$.

For each $X, Y \in \mathrm{Ob}\mathcal{A}$, there is a $\Bbbk$-linear map

$$h = (1 \otimes {}_Y\mathbf{i}_0)U_2 : s\mathcal{A}(X,Y) \to s\mathcal{A}(X,Y)$$

of degree -1. The equation

$$(1 \otimes b_1 + b_1 \otimes 1)U_2 + b_2U_1 = U_2b_1 + (U_1 \otimes U_1)b_2 \tag{3.1}$$

implies that

$$-b_1h + 1 = hb_1 + (1 \otimes {}_Y\mathbf{i}_0^{\mathcal{A}})b_2 : s\mathcal{A}(X,Y) \to s\mathcal{A}(X,Y),$$

thus h is a right unit homotopy for $\mathcal{A}$. For each $X, Y \in \mathrm{Ob}\mathcal{A}$, there is a $\Bbbk$-linear map

$$h' = -({}_X\mathbf{i}_0 \otimes 1)U_2 : s\mathcal{A}(X,Y) \to s\mathcal{A}(X,Y)$$

of degree -1. Equation (3.1) implies that

$$b_1h' - 1 = -h'b_1 + ({}_X\mathbf{i}_0^{\mathcal{A}} \otimes 1)b_2 : s\mathcal{A}(X,Y) \to s\mathcal{A}(X,Y),$$

thus h' is a left unit homotopy for $\mathcal{A}$. Therefore, $\mathcal{A}$ is unital. □

3.5. **Definition** (Fukaya [2, Definition 5.11]). An A_∞-category $\mathcal{C}$ is called *homotopy unital* if the graded quiver

$$\mathcal{C}^+ = \mathcal{C} \oplus \Bbbk\mathcal{C} \oplus s\Bbbk\mathcal{C}$$

(with $\operatorname{Ob}\mathcal{C}^+ = \operatorname{Ob}\mathcal{C}$) admits an A_∞-structure b^+ of the following kind. Denote the generators of the second and the third direct summands of the graded quiver $s\mathcal{C}^+ = s\mathcal{C} \oplus s\Bbbk\mathcal{C} \oplus s^2\Bbbk\mathcal{C}$ by ${}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}} = 1s$ and $\mathbf{j}_X^{\mathcal{C}} = 1s^2$ of degree respectively -1 and -2, for each $X \in \operatorname{Ob}\mathcal{C}$. The conditions on b^+ are:

(1) for each $X \in \operatorname{Ob}\mathcal{C}$, the element ${}_X\mathbf{i}_0^{\mathcal{C}} \overset{\text{def}}{=} {}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}} - \mathbf{j}_X^{\mathcal{C}} b_1^+$ is contained in $s\mathcal{C}(X, X)$;
(2) $\mathcal{C}^+$ is a strictly unital A_∞-category with strict units ${}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}}$, $X \in \operatorname{Ob}\mathcal{C}$;
(3) the embedding $\mathcal{C} \hookrightarrow \mathcal{C}^+$ is a strict A_∞-functor;
(4) $(s\mathcal{C} \oplus s^2\Bbbk\mathcal{C})^{\otimes n} b_n^+ \subset s\mathcal{C}$, for each $n > 1$.

In particular, $\mathcal{C}^+$ contains the strictly unital A_∞-category $\mathcal{C}^{\mathsf{su}} = \mathcal{C} \oplus \Bbbk\mathcal{C}$. A version of this definition suitable for filtered A_∞-algebras (and filtered A_∞-categories) is given by Fukaya, Oh, Ohta and Ono in their book [3, Definition 8.2].

Let $\mathcal{D}$ be a strictly unital A_∞-category with strict units $\mathbf{i}_0^{\mathcal{D}}$. Then it has a canonical homotopy unital structure $(\mathcal{D}^+, b^+)$. Namely, $\mathbf{j}_X^{\mathcal{D}} b_1^+ = {}_X\mathbf{i}_0^{\mathcal{D}^{\mathsf{su}}} - {}_X\mathbf{i}_0^{\mathcal{D}}$, and b_n^+ vanishes for each $n > 1$ on each summand of $(s\mathcal{D} \oplus s^2\Bbbk\mathcal{D})^{\otimes n}$ except on $s\mathcal{D}^{\otimes n}$, where it coincides with $b_n^{\mathcal{D}}$. Verification of the equation $(b^+)^2 = 0$ is a straightforward computation.

3.6. **Proposition.** *An arbitrary homotopy unital A_∞-category is unital.*

Proof. Let $\mathcal{C} \subset \mathcal{C}^+$ be a homotopy unital category. We claim that the distinguished cycles ${}_X\mathbf{i}_0^{\mathcal{C}} \in \mathcal{C}(X, X)[1]^{-1}$, $X \in \operatorname{Ob}\mathcal{C}$, turn $\mathcal{C}$ into a unital A_∞-category. Indeed, the identity

$$(1 \otimes b_1^+ + b_1^+ \otimes 1) b_2^+ + b_2^+ b_1^+ = 0$$

applied to $s\mathcal{C}\otimes \mathbf{j}^{\mathcal{C}}$ or to $\mathbf{j}^{\mathcal{C}}\otimes s\mathcal{C}$ implies

$$(1\otimes \mathbf{i}_0^{\mathcal{C}})b_2^{\mathcal{C}} = 1 + (1\otimes \mathbf{j}^{\mathcal{C}})b_2^+ b_1^{\mathcal{C}} + b_1^{\mathcal{C}}(1\otimes \mathbf{j}^{\mathcal{C}})b_2^+ \quad : s\mathcal{C}\to s\mathcal{C},$$
$$(\mathbf{i}_0^{\mathcal{C}}\otimes 1)b_2^{\mathcal{C}} = -1 + (\mathbf{j}^{\mathcal{C}}\otimes 1)b_2^+ b_1^{\mathcal{C}} + b_1^{\mathcal{C}}(\mathbf{j}^{\mathcal{C}}\otimes 1)b_2^+ : s\mathcal{C}\to s\mathcal{C}.$$

Thus, $(1\otimes \mathbf{j}^{\mathcal{C}})b_2^+ : s\mathcal{C}\to s\mathcal{C}$ and $(\mathbf{j}^{\mathcal{C}}\otimes 1)b_2^+ : s\mathcal{C}\to s\mathcal{C}$ are unit homotopies. Therefore, the A_∞-category $\mathcal{C}$ is unital. □

The converse of Proposition 3.6 holds true as well.

3.7. **Theorem.** *An arbitrary unital A_∞-category $\mathcal{C}$ with unit elements $\mathbf{i}_0^{\mathcal{C}}$ admits a homotopy unital structure $(\mathcal{C}^+, b^+)$ with $\mathbf{j}^{\mathcal{C}}b_1^+ = \mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}} - \mathbf{i}_0^{\mathcal{C}}$.*

Proof. By [9, Corollary A.12], there exists a differential graded category $\mathcal{D}$ and an A_∞-equivalence $\phi : \mathcal{C}\to\mathcal{D}$. By [9, Remark A.13], we may choose $\mathcal{D}$ and ϕ such that $\operatorname{Ob}\mathcal{D} = \operatorname{Ob}\mathcal{C}$ and $\operatorname{Ob}\phi = \mathrm{id}_{\operatorname{Ob}\mathcal{C}}$. Being strictly unital $\mathcal{D}$ admits a canonical homotopy unital structure $(\mathcal{D}^+, b^+)$. In the sequel, we may assume that $\mathcal{D}$ is a strictly unital A_∞-category equivalent to $\mathcal{C}$ via ϕ with the mentioned properties. Let us construct simultaneously an A_∞-structure b^+ on $\mathcal{C}^+$ and an A_∞-functor $\phi^+ : \mathcal{C}^+\to\mathcal{D}^+$ that will turn out to be an equivalence.

Let us extend the homotopy isomorphism $\phi_1 : s\mathcal{C}\to s\mathcal{D}$ to a chain quiver map $\phi_1^+ : s\mathcal{C}^+\to s\mathcal{D}^+$. The A_∞-equivalence $\phi : \mathcal{C}\to\mathcal{D}$ is a unital A_∞-functor, i.e., for each $X\in\operatorname{Ob}\mathcal{C}$, there exists $v_X\in\mathcal{D}(X,X)[1]^{-2}$ such that ${}_X\mathbf{i}_0^{\mathcal{D}} - {}_X\mathbf{i}_0^{\mathcal{C}}\phi_1 = v_X b_1$. In order that ϕ^+ be strictly unital, we define ${}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}}\phi_1^+ = {}_X\mathbf{i}_0^{\mathcal{D}^{\mathsf{su}}}$. We should have

$$\begin{aligned}\mathbf{j}_X^{\mathcal{C}}\phi_1^+ b_1^+ = \mathbf{j}_X^{\mathcal{C}} b_1^+\phi_1^+ &= {}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}}\phi_1^+ - {}_X\mathbf{i}_0^{\mathcal{C}}\phi_1 \\ &= {}_X\mathbf{i}_0^{\mathcal{D}^{\mathsf{su}}} - {}_X\mathbf{i}_0^{\mathcal{D}} + {}_X\mathbf{i}_0^{\mathcal{D}} - {}_X\mathbf{i}_0^{\mathcal{C}}\phi_1 = (\mathbf{j}_X^{\mathcal{C}} + v_X)b_1^+,\end{aligned}$$

so we define $\mathbf{j}_X^{\mathcal{C}}\phi_1^+ = \mathbf{j}_X^{\mathcal{D}} + v_X$.

We claim that there is a homotopy unital structure $(\mathcal{C}^+, b^+)$ of $\mathcal{C}$ satisfying the four conditions of Definition 3.5 and an A_∞-functor $\phi^+ : \mathcal{C}^+ \to \mathcal{D}^+$ satisfying four parallel conditions:

(1) the first component of ϕ^+ is the quiver morphism ϕ_1^+ constructed above;
(2) the A_∞-functor ϕ^+ is strictly unital;
(3) the restriction of ϕ^+ to $\mathcal{C}$ gives ϕ;
(4) $(s\mathcal{C} \oplus s^2\Bbbk\mathcal{C})^{\otimes n}\phi_n^+ \subset s\mathcal{D}$, for each $n > 1$.

Notice that in the presence of conditions (2) and (3) the first condition reduces to $\mathbf{j}_X^{\mathcal{C}}(\phi^+)_1 = \mathbf{j}_X^{\mathcal{D}} + v_X$, for each $X \in \operatorname{Ob}\mathcal{C}$.

Components of the (1,1)-coderivation $b^+ : Ts\mathcal{C}^+ \to Ts\mathcal{C}^+$ of degree 1 and of the augmented graded cocategory morphism $\phi^+ : Ts\mathcal{C}^+ \to Ts\mathcal{D}^+$ are constructed by induction. We already know components b_1^+ and ϕ_1^+. Given an integer $n \geqslant 2$, assume that we have already found components b_m^+, ϕ_m^+ of the sought b^+ and ϕ^+ for $m < n$ such that the equations

$$(3.2) \quad ((b^+)^2)_m = 0 \qquad : T^m s\mathcal{C}^+(X,Y) \to s\mathcal{C}^+(X,Y),$$

$$(3.3) \quad (\phi^+ b^+)_m = (b^+\phi^+)_m : T^m s\mathcal{C}^+(X,Y) \to s\mathcal{D}^+(Xf,Yf)$$

are satisfied for all $m < n$. Define b_n^+, ϕ_n^+ on direct summands of $T^n s\mathcal{C}^+$ which contain a factor $\mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}}$ by the requirement of strict unitality of $\mathcal{C}^+$ and ϕ^+. Then equations (3.2), (3.3) hold true for $m = n$ on such summands. Define b_n^+, ϕ_n^+ on the direct summand $T^n s\mathcal{C} \subset T^n s\mathcal{C}^+$ as $b_n^{\mathcal{C}}$ and ϕ_n. Then equations (3.2), (3.3) hold true for $m = n$ on the summand $T^n s\mathcal{C}$. It remains to construct those components of b^+ and ϕ^+ which have $\mathbf{j}^{\mathcal{C}}$ as one of their arguments.

Extend $b_1 : s\mathcal{C} \to s\mathcal{C}$ to $b_1' : s\mathcal{C}^+ \to s\mathcal{C}^+$ by $\mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}} b_1' = 0$ and $\mathbf{j}^{\mathcal{C}} b_1' = 0$. Define $b_1^- = b_1^+ - b_1' : s\mathcal{C}^+ \to s\mathcal{C}^+$. Thus, $b_1^-\big|_{s\mathcal{C}^{\mathsf{su}}} = 0$, $\mathbf{j}^{\mathcal{C}} b_1^- = \mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}} - \mathbf{i}_0^{\mathcal{C}}$ and $b_1^+ = b_1' + b_1^-$. Introduce for $0 \leqslant k \leqslant n$

the graded subquiver $\mathcal{N}_k \subset T^n(s\mathcal{C} \oplus s^2\Bbbk\mathcal{C})$ by

$$\mathcal{N}_k = \bigoplus_{p_0+p_1+\dots+p_k+k=n} T^{p_0}s\mathcal{C} \otimes \mathbf{j}^{\mathcal{C}} \otimes T^{p_1}s\mathcal{C} \otimes \dots \otimes \mathbf{j}^{\mathcal{C}} \otimes T^{p_k}s\mathcal{C}$$

stable under the differential $d^{\mathcal{N}_k} = \sum_{p+1+q=n} 1^{\otimes p} \otimes b'_1 \otimes 1^{\otimes q}$, and the graded subquiver $\mathcal{P}_l \subset T^n s\mathcal{C}^+$ by

$$\mathcal{P}_l = \bigoplus_{p_0+p_1+\dots+p_l+l=n} T^{p_0}s\mathcal{C}^{\mathsf{su}} \otimes \mathbf{j}^{\mathcal{C}} \otimes T^{p_1}s\mathcal{C}^{\mathsf{su}} \otimes \dots \otimes \mathbf{j}^{\mathcal{C}} \otimes T^{p_l}s\mathcal{C}^{\mathsf{su}}.$$

There is also the subquiver

$$\mathcal{Q}_k = \bigoplus_{l=0}^{k} \mathcal{P}_l \subset T^n s\mathcal{C}^+$$

and its complement

$$\mathcal{Q}_k^{\perp} = \bigoplus_{l=k+1}^{n} \mathcal{P}_l \subset T^n s\mathcal{C}^+.$$

Notice that the subquiver $\mathcal{Q}_k$ is stable under the differential $d^{\mathcal{Q}_k} = \sum_{p+1+q=n} 1^{\otimes p} \otimes b_1^+ \otimes 1^{\otimes q}$, and $\mathcal{Q}_k^{\perp}$ is stable under the differential $d^{\mathcal{Q}_k^{\perp}} = \sum_{p+1+q=n} 1^{\otimes p} \otimes b'_1 \otimes 1^{\otimes q}$. Furthermore, the image of $1^{\otimes a} \otimes b_1^- \otimes 1^{\otimes c} : \mathcal{N}_k \to T^n s\mathcal{C}^+$ is contained in $\mathcal{Q}_{k-1}$ for all $a, c \geqslant 0$ such that $a+1+c=n$.

Firstly, the components b_n^+, ϕ_n^+ are defined on the graded subquivers $\mathcal{N}_0 = T^n s\mathcal{C}$ and $\mathcal{Q}_0 = T^n s\mathcal{C}^{\mathsf{su}}$. Assume for an integer $0 < k \leqslant n$ that restrictions of b_n^+, ϕ_n^+ to $\mathcal{N}_l$ are already found for all $l < k$. In other terms, we are given $b_n^+ : \mathcal{Q}_{k-1} \to s\mathcal{C}^+$, $\phi_n^+ : \mathcal{Q}_{k-1} \to s\mathcal{D}$ such that equations (3.2), (3.3) hold on $\mathcal{Q}_{k-1}$. Let us construct the restrictions $b_n^+ : \mathcal{N}_k \to s\mathcal{C}$, $\phi_n^+ : \mathcal{N}_k \to s\mathcal{D}$, performing the induction step.

Introduce a (1,1)-coderivation $\tilde{b} : Ts\mathcal{C}^+ \to Ts\mathcal{C}^+$ of degree 1 by its components $(0, b_1^+, \dots, b_{n-1}^+, \mathrm{pr}_{\mathcal{Q}_{k-1}} \cdot b_n^+|_{\mathcal{Q}_{k-1}}, 0, \dots)$. Introduce also a morphism of augmented graded cocategories

$\tilde{\phi} : Ts\mathcal{C}^+ \to Ts\mathcal{D}^+$ with $\mathrm{Ob}\tilde{\phi} = \mathrm{Ob}\phi$ by its components $(\phi_1^+, \dots, \phi_{n-1}^+, \mathrm{pr}_{\mathcal{Q}_{k-1}} \cdot \phi_n^+|_{\mathcal{Q}_{k-1}}, 0, \dots)$. Here $\mathrm{pr}_{\mathcal{Q}_{k-1}} : T^n s\mathcal{C}^+ \to \mathcal{Q}_{k-1}$ is the natural projection, vanishing on $\mathcal{Q}_{k-1}^\perp$. Then $\lambda \overset{\text{def}}{=} \tilde{b}^2 : Ts\mathcal{C}^+ \to Ts\mathcal{C}^+$ is a (1,1)-coderivation of degree 2 and $\nu \overset{\text{def}}{=} -\tilde{\phi}b^+ + \tilde{b}\tilde{\phi} : Ts\mathcal{C}^+ \to Ts\mathcal{D}^+$ is a $(\tilde{\phi}, \tilde{\phi})$-coderivation of degree 1. Equations (3.2), (3.3) imply that $\lambda_m = 0$, $\nu_m = 0$ for $m < n$. Moreover, λ_n, ν_n vanish on $\mathcal{Q}_{k-1}$. On the complement the n-th components equal

$$\begin{aligned}
\lambda_n &= \sum_{a+r+c=n}^{1<r<n} (1^{\otimes a} \otimes b_r^+ \otimes 1^{\otimes c}) b_{a+1+c}^+ \\
&\quad + \sum_{a+1+c=n} (1^{\otimes a} \otimes b_1^- \otimes 1^{\otimes c}) \tilde{b}_n : \mathcal{Q}_{k-1}^\perp \to s\mathcal{C}^+, \\
\nu_n &= -\sum_{i_1+\dots+i_r=n}^{1<r\leqslant n} (\phi_{i_1}^+ \otimes \dots \otimes \phi_{i_r}^+) b_r^+ \\
&\quad + \sum_{a+r+c=n}^{1<r<n} (1^{\otimes a} \otimes b_r^+ \otimes 1^{\otimes c}) \phi_{a+1+c}^+ \\
&\quad + \sum_{a+1+c=n} (1^{\otimes a} \otimes b_1^- \otimes 1^{\otimes c}) \tilde{\phi}_n : \mathcal{Q}_{k-1}^\perp \to s\mathcal{D}.
\end{aligned}$$

The restriction $\lambda_n|_{\mathcal{N}_k}$ takes values in $s\mathcal{C}$. Indeed, for the first sum in the expression for λ_n this follows by the induction assumption since $r > 1$ and $a+1+c > 1$. For the second sum this follows by the induction assumption and strict unitality if $n > 2$. In the case of $n = 2$, $k = 1$ this is also straightforward. The only case which requires computation is $n = 2$, $k = 2$:

$$(\mathbf{j}^\mathcal{C} \otimes \mathbf{j}^\mathcal{C})(1 \otimes b_1^- + b_1^- \otimes 1)\tilde{b}_2 = \mathbf{j}^\mathcal{C} - (\mathbf{j}^\mathcal{C} \otimes \mathbf{i}_0^\mathcal{C}) b_2^+ - \mathbf{j}^\mathcal{C} - (\mathbf{i}_0^\mathcal{C} \otimes \mathbf{j}^\mathcal{C}) b_2^+,$$

which belongs to $s\mathcal{C}$ by the induction assumption.

Equations (3.2), (3.3) for $m = n$ take the form

(3.4) $$-b_n^+ b_1 - \sum_{a+1+c=n} (1^{\otimes a} \otimes b'_1 \otimes 1^{\otimes c}) b_n^+ = \lambda_n : \mathcal{N}_k \to s\mathcal{C},$$

(3.5)
$$\phi_n^+ b_1 - \sum_{a+1+c=n} (1^{\otimes a} \otimes b'_1 \otimes 1^{\otimes c}) \phi_n^+ - b_n^+ \phi_1 = \nu_n : \mathcal{N}_k \to s\mathcal{D}.$$

For arbitrary objects X, Y of $\mathcal{C}$, equip the graded $\Bbbk$-module $\mathcal{N}_k(X,Y)$ with the differential $d^{\mathcal{N}_k} = \sum_{p+1+q=n} 1^{\otimes p} \otimes b'_1 \otimes 1^{\otimes q}$ and denote by u the chain map

$$\underline{\mathsf{C}}_\Bbbk(\mathcal{N}_k(X,Y), s\mathcal{C}(X,Y)) \to \underline{\mathsf{C}}_\Bbbk(\mathcal{N}_k(X,Y), s\mathcal{D}(X\phi, Y\phi)),$$
$$\lambda \mapsto \lambda\phi_1.$$

Since ϕ_1 is homotopy invertible, the map u is homotopy invertible as well. Therefore, the complex $\operatorname{Cone}(u)$ is contractible, e.g. by [8, Lemma B.1], in particular, acyclic. Equations (3.4) and (3.5) have the form $-b_n^+ d = \lambda_n$, $\phi_n^+ d + b_n^+ u = \nu_n$, that is, the element (λ_n, ν_n) of

$$\underline{\mathsf{C}}^2_\Bbbk(\mathcal{N}_k(X,Y), s\mathcal{C}(X,Y)) \oplus \underline{\mathsf{C}}^1_\Bbbk(\mathcal{N}_k(X,Y), s\mathcal{D}(X\phi, Y\phi)) = \operatorname{Cone}^1(u)$$

has to be the boundary of the sought element (b_n^+, ϕ_n^+) of

$$\underline{\mathsf{C}}^1_\Bbbk(\mathcal{N}_k(X,Y), s\mathcal{C}(X,Y)) \oplus \underline{\mathsf{C}}^0_\Bbbk(\mathcal{N}_k(X,Y), s\mathcal{D}(X\phi, Y\phi)) = \operatorname{Cone}^0(u).$$

These equations are solvable because (λ_n, ν_n) is a cycle in $\operatorname{Cone}^1(u)$. Indeed, the equations to verify $-\lambda_n d = 0$, $\nu_n d +$

$\lambda_n u = 0$ take the form

$$-\lambda_n b_1 + \sum_{p+1+q=n} (1^{\otimes p} \otimes b'_1 \otimes 1^{\otimes q})\lambda_n = 0 : \mathcal{N}_k \to s\mathcal{C},$$

$$\nu_n b_1 + \sum_{p+1+q=n} (1^{\otimes p} \otimes b'_1 \otimes 1^{\otimes q})\nu_n - \lambda_n \phi_1 = 0 : \mathcal{N}_k \to s\mathcal{D}.$$

Composing the identity $-\lambda\tilde{b} + \tilde{b}\lambda = 0 : T^n s\mathcal{C}^+ \to Ts\mathcal{C}^+$ with the projection $\mathrm{pr}_1 : Ts\mathcal{C}^+ \to s\mathcal{C}^+$ yields the first equation. The second equation follows by composing the identity $\nu b^+ + \tilde{b}\nu - \lambda\tilde{\phi} = 0 : T^n s\mathcal{C}^+ \to Ts\mathcal{D}^+$ with $\mathrm{pr}_1 : Ts\mathcal{D}^+ \to s\mathcal{D}^+$.

Thus, the required restrictions of b_n^+, ϕ_n^+ to $\mathcal{N}_k$ (and to $\mathcal{Q}_k$) exist and satisfy the required equations. We proceed by induction increasing k from 0 to n and determining b_n^+, ϕ_n^+ on the whole $\mathcal{Q}_n = T^n s\mathcal{C}^+$. Then we replace n with $n+1$ and start again from $T^{n+1}s\mathcal{C}$. Thus the induction on n goes through. □

3.8. **Remark.** Let $(\mathcal{C}^+, b^+)$ be a homotopy unital structure of an A_∞-category $\mathcal{C}$. Then the embedding A_∞-functor $\iota : \mathcal{C} \to \mathcal{C}^+$ is an equivalence. Indeed, it is bijective on objects. By [8, Theorem 8.8] it suffices to prove that $\iota_1 : s\mathcal{C} \to s\mathcal{C}^+$ is homotopy invertible. And indeed, the chain quiver map $\pi_1 : s\mathcal{C}^+ \to s\mathcal{C}$, $\pi_1|_{s\mathcal{C}} = \mathrm{id}$, ${}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}}\pi_1 = {}_X\mathbf{i}_0^{\mathcal{C}}$, $\mathbf{j}_X^{\mathcal{C}}\pi_1 = 0$, is homotopy inverse to ι_1. Namely, the homotopy $h : s\mathcal{C}^+ \to s\mathcal{C}^+$, $h|_{s\mathcal{C}} = 0$, ${}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{su}}}h = \mathbf{j}_X^{\mathcal{C}}$, $\mathbf{j}_X^{\mathcal{C}}h = 0$, satisfies the equation $\mathrm{id}_{s\mathcal{C}^+} - \pi_1 \cdot \iota_1 = hb_1^+ + b_1^+ h$.

The equation between A_∞-functors

$$\bigl[\mathcal{C} \xrightarrow{\iota^{\mathcal{C}}} \mathcal{C}^+ \xrightarrow{\phi^+} \mathcal{D}^+\bigr] = \bigl[\mathcal{C} \xrightarrow{\phi} \mathcal{D} \xrightarrow{\iota^{\mathcal{D}}} \mathcal{D}^+\bigr]$$

obtained in the proof of Theorem 3.7 implies that ϕ^+ is an A_∞-equivalence as well. In particular, ϕ_1^+ is homotopy invertible.

The converse of Proposition 3.4 holds true as well, however its proof requires more preliminaries. It is deferred until Section 5.

4. Double coderivations

4.1. **Definition.** For A_∞-functors $f, g : \mathcal{A} \to \mathcal{B}$, a *double (f,g)-coderivation* of degree d is a system of $\Bbbk$-linear maps

$$r : (Ts\mathcal{A} \otimes Ts\mathcal{A})(X,Y) \to Ts\mathcal{B}(Xf, Yg), \quad X, Y \in \operatorname{Ob}\mathcal{A},$$

of degree d such that the equation

$$r\Delta_0 = (\Delta_0 \otimes 1)(f \otimes r) + (1 \otimes \Delta_0)(r \otimes g) \tag{4.1}$$

holds true.

Equation (4.1) implies that r is determined by a system of $\Bbbk$-linear maps $r\mathrm{pr}_1 : Ts\mathcal{A} \otimes Ts\mathcal{A} \to s\mathcal{B}$ with components of degree d

$$r_{n,m} : s\mathcal{A}(X_0, X_1) \otimes \dots \otimes s\mathcal{A}(X_{n+m-1}, X_{n+m}) \\ \to s\mathcal{B}(X_0 f, X_{n+m} g),$$

for $n, m \geqslant 0$, via the formula

$$\begin{gathered} r_{n,m;k} = (r|_{T^n s\mathcal{A} \otimes T^m s\mathcal{A}})\mathrm{pr}_k : T^n s\mathcal{A} \otimes T^m s\mathcal{A} \to T^k s\mathcal{B}, \\ r_{n,m;k} = \sum_{\substack{i_1+\dots+i_p+i=n, \\ j_1+\dots+j_q+j=m}}^{p+1+q=k} f_{i_1} \otimes \dots \otimes f_{i_p} \otimes r_{i,j} \otimes g_{j_1} \otimes \dots \otimes g_{j_q}. \end{gathered} \tag{4.2}$$

This follows from the equation

$$r\Delta_0^{(l)} = \sum_{p+1+q=l} (\Delta_0^{(p+1)} \otimes \Delta_0^{(q+1)})(f^{\otimes p} \otimes r \otimes g^{\otimes q}) : \\ Ts\mathcal{A} \otimes Ts\mathcal{A} \to (Ts\mathcal{B})^{\otimes l}, \tag{4.3}$$

which holds true for each $l \geqslant 0$. Here $\Delta_0^{(0)} = \varepsilon$, $\Delta_0^{(1)} = \mathrm{id}$, $\Delta_0^{(2)} = \Delta_0$ and $\Delta_0^{(l)}$ means the cut comultiplication iterated $l-1$ times.

Double (f,g)-coderivations form a chain complex, which we are going to denote by $(\mathscr{D}(\mathcal{A},\mathcal{B})(f,g), B_1)$. For each $d \in \mathbb{Z}$, the component $\mathscr{D}(\mathcal{A},\mathcal{B})(f,g)^d$ consists of double (f,g)-coderivations of degree d. The differential B_1 of degree 1 is given by

$$rB_1 \overset{\text{def}}{=} rb - (-)^d(1 \otimes b + b \otimes 1)r,$$

for each $r \in \mathscr{D}(\mathcal{A},\mathcal{B})(f,g)^d$. The component $[rB_1]_{n,m}$ of rB_1 is given by

(4.4)
$$\sum_{\substack{i_1+\dots+i_p+i=n,\\ j_1+\dots+j_q+j=m}} (f_{i_1} \otimes \dots \otimes f_{i_p} \otimes r_{ij} \otimes g_{j_1} \otimes \dots \otimes g_{j_q})b_{p+1+q} \\ - (-)^r \sum_{a+k+c=n} (1^{\otimes a} \otimes b_k \otimes 1^{\otimes c+m})r_{a+1+c,m} \\ - (-)^r \sum_{u+t+v=m} (1^{\otimes n+u} \otimes b_t \otimes 1^{\otimes v})r_{n,u+1+v},$$

for each $n, m \geqslant 0$. An A_∞-functor $h : \mathcal{B} \to \mathcal{C}$ gives rise to a chain map

$$\mathscr{D}(\mathcal{A},\mathcal{B})(f,g) \to \mathscr{D}(\mathcal{A},\mathcal{C})(fh,gh), \quad r \mapsto rh.$$

The component $[rh]_{n,m}$ of rh is given by

(4.5)
$$\sum_{\substack{i_1+\dots+i_p+i=n,\\ j_1+\dots+j_q+j=m}} (f_{i_1} \otimes \dots \otimes f_{i_p} \otimes r_{i,j} \otimes g_{j_1} \otimes \dots \otimes g_{j_q})h_{p+1+q},$$

for each $n, m \geqslant 0$. Similarly, an A_∞-functor $k : \mathcal{D} \to \mathcal{A}$ gives rise to a chain map

$$\mathscr{D}(\mathcal{A},\mathcal{B})(f,g) \to \mathscr{D}(\mathcal{D},\mathcal{B})(kf,kg), \quad r \mapsto (k \otimes k)r.$$

The component $[(k\otimes k)r]_{n,m}$ of $(k\otimes k)r$ is given by

$$(4.6)\qquad \sum_{\substack{i_1+\dots+i_p=n\\ j_1+\dots+j_q=m}} (k_{i_1}\otimes\dots\otimes k_{i_p}\otimes k_{j_1}\otimes\dots\otimes k_{j_q})r_{p,q},$$

for each $n,m\geqslant 0$. Proofs of these facts are elementary and are left to the reader.

Let $\mathcal{C}$ be an A_∞-category. For each $n\geqslant 0$, introduce a morphism

$$\nu_n=\sum_{i=0}^{n}(-)^{n-i}(1^{\otimes i}\otimes\varepsilon\otimes 1^{\otimes n-i}):(Ts\mathcal{C})^{\otimes n+1}\to(Ts\mathcal{C})^{\otimes n},$$

in $\mathscr{Q}/\mathrm{Ob}\mathcal{C}$. In particular, $\nu_0=\varepsilon:Ts\mathcal{C}\to\Bbbk\mathrm{Ob}\mathcal{C}$. Denote $\nu=\nu_1=(1\otimes\varepsilon)-(\varepsilon\otimes 1):Ts\mathcal{C}\otimes Ts\mathcal{C}\to Ts\mathcal{C}$ for the sake of brevity.

4.2. **Lemma.** *The map $\nu:Ts\mathcal{C}\otimes Ts\mathcal{C}\to Ts\mathcal{C}$ is a double $(1,1)$-coderivation of degree 0 and $\nu B_1=0$.*

Proof. We have:

$$\begin{aligned}(\Delta_0\otimes 1)&(1\otimes\nu)+(1\otimes\Delta_0)(\nu\otimes 1)\\ &=(\Delta_0\otimes 1)(1\otimes 1\otimes\varepsilon)-(\Delta_0\otimes 1)(1\otimes\varepsilon\otimes 1)\\ &\qquad\qquad+(1\otimes\Delta_0)(1\otimes\varepsilon\otimes 1)-(1\otimes\Delta_0)(\varepsilon\otimes 1\otimes 1)\\ &=(\Delta_0\otimes\varepsilon)-(\varepsilon\otimes\Delta_0)=((1\otimes\varepsilon)-(\varepsilon\otimes 1))\Delta_0=\nu\Delta_0,\end{aligned}$$

due to the identities

$$(\Delta_0\otimes 1)(1\otimes\varepsilon\otimes 1)=1\otimes 1=(1\otimes\Delta_0)(1\otimes\varepsilon\otimes 1):\\ Ts\mathcal{C}\otimes Ts\mathcal{C}\to Ts\mathcal{C}\otimes Ts\mathcal{C}.$$

This computation shows that $\nu:Ts\mathcal{C}\otimes Ts\mathcal{C}\to Ts\mathcal{C}$ is a double $(1,1)$-coderivation. Its only non-vanishing components are ${}_{X,Y}\nu_{1,0}=1:s\mathcal{C}(X,Y)\to s\mathcal{C}(X,Y)$ and ${}_{X,Y}\nu_{0,1}=1:s\mathcal{C}(X,Y)\to s\mathcal{C}(X,Y)$, $X,Y\in\mathrm{Ob}\mathcal{C}$.

Since νB_1 is a double $(1,1)$-coderivation of degree 1, the equation $\nu B_1 = 0$ is equivalent to its particular case $\nu B_1 \mathrm{pr}_1 = 0$, i.e., for each $n, m \geqslant 0$

$$\sum_{\substack{0\leqslant i\leqslant n,\\ 0\leqslant j\leqslant m}} (1^{\otimes n-i} \otimes \nu_{i,j} \otimes 1^{\otimes m-j}) b_{n-i+1+m-j} \\ - \sum_{a+k+c=n} (1^{\otimes a} \otimes b_k \otimes 1^{\otimes c+m}) \nu_{a+1+c,m} \\ - \sum_{u+t+v=m} (1^{\otimes n+u} \otimes b_t \otimes 1^{\otimes v}) \nu_{n,u+1+v} = 0 : \\ T^n s\mathcal{C} \otimes T^m s\mathcal{C} \to s\mathcal{C}.$$

It reduces to the identity

$$\chi(n > 0) b_{n+m} - \chi(m > 0) b_{n+m} \\ - \chi(m = 0) b_n + \chi(n = 0) b_m = 0,$$

where $\chi(P) = 1$ if a condition P holds and $\chi(P) = 0$ if P does not hold. □

Let $\mathcal{C}$ be a strictly unital A_∞-category. The strict unit $\mathbf{i}_0^{\mathcal{C}}$ is viewed as a morphism of graded quivers $\mathbf{i}_0^{\mathcal{C}} : \Bbbk \mathrm{Ob}\mathcal{C} \to s\mathcal{C}$ of degree -1, identity on objects. For each $n \geqslant 0$, introduce a morphism of graded quivers

$$\xi_n = \bigl[(Ts\mathcal{C})^{\otimes n+1} \xrightarrow{1\otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1 \otimes \cdots \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1} \\ Ts\mathcal{C} \otimes s\mathcal{C} \otimes Ts\mathcal{C} \otimes \cdots \otimes s\mathcal{C} \otimes Ts\mathcal{C} \xrightarrow{\mu^{(2n+1)}} Ts\mathcal{C}\bigr],$$

of degree $-n$, identity on objects. Here $\mu^{(2n+1)}$ denotes composition of $2n+1$ composable arrows in the graded category $Ts\mathcal{C}$. In particular, $\xi_0 = 1 : Ts\mathcal{C} \to Ts\mathcal{C}$. Denote $\xi = \xi_1 = (1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(3)} : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C}$ for the sake of brevity.

4.3. **Lemma.** *The map $\xi : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C}$ is a double $(1,1)$-coderivation of degree -1 and $\xi B_1 = \nu$.*

Proof. The following identity follows directly from the definitions of μ and Δ_0:

$$\mu\Delta_0 = (\Delta_0 \otimes 1)(1 \otimes \mu) + (1 \otimes \Delta_0)(\mu \otimes 1) - 1 : \\ Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C} \otimes Ts\mathcal{C}.$$

It implies

(4.7)
$$\mu^{(3)}\Delta_0 = (\Delta_0 \otimes 1 \otimes 1)(1 \otimes \mu^{(3)}) + (1 \otimes 1 \otimes \Delta_0)(\mu^{(3)} \otimes 1) \\ + (1 \otimes \Delta_0 \otimes 1)(\mu \otimes \mu) - (1 \otimes \mu) - (\mu \otimes 1) : \\ Ts\mathcal{C} \otimes Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C} \otimes Ts\mathcal{C}.$$

Since $\mathbf{i}_0^{\mathcal{C}}\Delta_0 = \mathbf{i}_0^{\mathcal{C}} \otimes \eta + \eta \otimes \mathbf{i}_0^{\mathcal{C}} : \Bbbk \operatorname{Ob}\mathcal{C} \to Ts\mathcal{C} \otimes Ts\mathcal{C}$, it follows that

$$(1 \otimes \mathbf{i}_0^{\mathcal{C}}\Delta_0 \otimes 1)(\mu \otimes \mu) - (1 \otimes (\mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu) - ((1 \otimes \mathbf{i}_0^{\mathcal{C}})\mu \otimes 1) = 0 : \\ Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C} \otimes Ts\mathcal{C}.$$

Equation (4.7) yields

$$(1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(3)}\Delta_0 \\ = (\Delta_0 \otimes 1)(1 \otimes (1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(3)}) + (1 \otimes \Delta_0)((1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(3)} \otimes 1),$$

i.e., $\xi = (1 \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1)\mu^{(3)} : Ts\mathcal{C} \otimes Ts\mathcal{C} \to Ts\mathcal{C}$ is a double $(1,1)$-coderivation. Its the only non-vanishing components are ${}_X\xi_{0,0} = {}_X\mathbf{i}_0^{\mathcal{C}} \in s\mathcal{C}(X,X)$, $X \in \operatorname{Ob}\mathcal{C}$.

Since both ξB_1 and ν are double $(1,1)$-coderivations of degree 0, the equation $\xi B_1 = \nu$ is equivalent to its particular case $\xi B_1 \mathrm{pr}_1 = \nu \mathrm{pr}_1$, i.e., for each $n, m \geqslant 0$

$$\sum_{\substack{0\leqslant p\leqslant n\\0\leqslant q\leqslant m}} (1^{\otimes n-p} \otimes \xi_{p,q} \otimes 1^{\otimes m-q}) b_{n-p+1+m-q} + \sum_{a+k+c=n} (1^{\otimes a} \otimes b_k \otimes 1^{\otimes c+m}) \xi_{a+1+c,m} + \sum_{u+t+v=m} (1^{\otimes n+u} \otimes b_t \otimes 1^{\otimes v}) \xi_{n,u+1+v} = \nu_{n,m} : T^n s\mathcal{C} \otimes T^m s\mathcal{C} \to s\mathcal{C}.$$

It reduces to the the equation

$$(1^{\otimes n} \otimes \mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes m}) b_{n+1+m} = \nu_{n,m} : T^n s\mathcal{C} \otimes T^m s\mathcal{C} \to s\mathcal{C},$$

which holds true, since $\mathbf{i}_0^{\mathcal{C}}$ is a strict unit. □

Note that the maps ν_n, ξ_n obey the following relations:

$$\xi_n = (\xi_{n-1} \otimes 1)\xi, \qquad \nu_n = (1^{\otimes n} \otimes \varepsilon) - (\nu_{n-1} \otimes 1), \qquad n \geqslant 1. \tag{4.8}$$

In particular, $\xi_n \varepsilon = 0 : (Ts\mathcal{C})^{\otimes n+1} \to \Bbbk \mathrm{Ob}\mathcal{C}$, for each $n \geqslant 1$, as $\xi\varepsilon = 0$ by equation (4.3).

4.4. **Lemma.** *The following equations hold true:*

$$\xi_n \Delta_0 = \sum_{i=0}^{n} (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(\xi_i \otimes \xi_{n-i}), \quad n \geqslant 0, \tag{4.9}$$

$$\xi_n b - (-)^n \sum_{i=0}^{n} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \xi_n = \nu_n \xi_{n-1}, \quad n \geqslant 1. \tag{4.10}$$

Proof. Let us prove (4.9). The proof is by induction on n. The case $n = 0$ is trivial. Let $n \geqslant 1$. By (4.8) and Lemma 4.3,

$$\xi_n \Delta_0 = (\xi_{n-1} \otimes 1)\xi\Delta_0 = (\xi_{n-1}\Delta_0 \otimes 1)(1 \otimes \xi) + (\xi_{n-1} \otimes \Delta_0)(\xi \otimes 1).$$

By induction hypothesis,

$$\xi_{n-1}\Delta_0 = \sum_{i=0}^{n-1}(1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-1-i})(\xi_i \otimes \xi_{n-1-i}),$$

therefore

$$\begin{aligned}
\xi_n\Delta_0 = \sum_{i=0}^{n-1}(1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(\xi_i \otimes \xi_{n-1-i} \otimes 1)(1 \otimes \xi) \\
+ (1^{\otimes n} \otimes \Delta_0)((\xi_{n-1} \otimes 1)\xi \otimes 1) \\
= \sum_{i=0}^{n}(1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(\xi_i \otimes \xi_{n-i}),
\end{aligned}$$

since $(\xi_{n-1-i} \otimes 1)\xi = \xi_{n-i}$ if $0 \leqslant i \leqslant n-1$.

Let us prove (4.10). The proof is by induction on n. The case $n = 1$ follows from Lemma 4.3. Let $n \geqslant 2$. By (4.8) and Lemma 4.3,

$$\begin{aligned}
&\xi_n b - (-)^n \sum_{i=0}^{n}(1^{\otimes i} \otimes b \otimes 1^{\otimes n-i})\xi_n \\
&= (\xi_{n-1} \otimes 1)\xi b - (-)^n \sum_{i=0}^{n-1}((1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i})\xi_{n-1} \otimes 1)\xi \\
&\qquad - (-)^n(1^{\otimes n} \otimes b)(\xi_{n-1} \otimes 1)\xi \\
&= -(\xi_{n-1} b \otimes 1)\xi - (\xi_{n-1} \otimes b)\xi + (\xi_{n-1} \otimes 1)\nu \\
&+ (-)^{n-1} \sum_{i=0}^{n-1}((1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i})\xi_{n-1} \otimes 1)\xi + (\xi_{n-1} \otimes b)\xi \\
&= (\xi_{n-1} \otimes 1)\nu \\
&- \left(\left[\xi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1}(1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i})\xi_{n-1}\right] \otimes 1\right)\xi.
\end{aligned}$$

By induction hypothesis

$$\xi_{n-1}b-(-)^{n-1}\sum_{i=0}^{n-1}(1^{\otimes i}\otimes b\otimes 1^{\otimes n-1-i})\xi_{n-1}=\nu_{n-1}\xi_{n-2},$$

therefore

$$\xi_n b-(-)^n\sum_{i=0}^{n}(1^{\otimes i}\otimes b\otimes 1^{\otimes n-i})\xi_n=(\xi_{n-1}\otimes 1)\nu-(\nu_{n-1}\xi_{n-2}\otimes 1)\xi.$$

Since by (4.8),

$$\begin{aligned}(\xi_{n-1}\otimes 1)\nu &- (\nu_{n-1}\xi_{n-2}\otimes 1)\xi \\ &= (\xi_{n-1}\otimes\varepsilon)-(\xi_{n-1}\varepsilon\otimes 1)-(\nu_{n-1}\otimes 1)\xi_{n-1} \\ &= (1^{\otimes n}\otimes\varepsilon)\xi_{n-1}-(\nu_{n-1}\otimes 1)\xi_{n-1}=\nu_n\xi_{n-1},\end{aligned}$$

equation (4.10) is proven. □

5. An augmented differential graded cocategory

Let now $\mathcal{C}=\mathcal{A}^{\mathsf{su}}$, where $\mathcal{A}$ is an A_∞-category. There is an isomorphism of graded $\Bbbk$-quivers, identity on objects:

$$\zeta:\bigoplus_{n\geqslant 0}(Ts\mathcal{A})^{\otimes n+1}[n]\to Ts\mathcal{A}^{\mathsf{su}}.$$

The morphism ζ is the sum of morphisms

$$(5.1)\quad \zeta_n=\Bigl[(Ts\mathcal{A})^{\otimes n+1}[n]\xrightarrow{s^{-n}}(Ts\mathcal{A})^{\otimes n+1} \xhookrightarrow{e^{\otimes n+1}}(Ts\mathcal{A}^{\mathsf{su}})^{\otimes n+1}\xrightarrow{\xi_n}Ts\mathcal{A}^{\mathsf{su}}\Bigr],$$

where $e:\mathcal{A}\hookrightarrow\mathcal{A}^{\mathsf{su}}$ is the natural embedding. The graded quiver

$$\mathcal{E}\overset{\text{def}}{=}\bigoplus_{n\geqslant 0}(Ts\mathcal{A})^{\otimes n+1}[n]$$

admits a unique structure of an augmented differential graded cocategory such that ζ becomes an isomorphism of augmented differential graded cocategories. The comultiplication $\widetilde{\Delta} : \mathcal{E} \to \mathcal{E} \otimes \mathcal{E}$ is found from the equation

$$\big[\mathcal{E} \xrightarrow{\zeta} Ts\mathcal{A}^{\mathsf{su}} \xrightarrow{\Delta_0} Ts\mathcal{A}^{\mathsf{su}} \otimes Ts\mathcal{A}^{\mathsf{su}}\big] = \big[\mathcal{E} \xrightarrow{\widetilde{\Delta}} \mathcal{E} \otimes \mathcal{E} \xrightarrow{\zeta\otimes\zeta} Ts\mathcal{A}^{\mathsf{su}} \otimes Ts\mathcal{A}^{\mathsf{su}}\big].$$

Restricting the left hand side of the equation to the summand $(Ts\mathcal{A})^{\otimes n+1}[n]$ of $\mathcal{E}$, we obtain

$$\begin{aligned}\zeta_n\Delta_0 &= s^{-n}e^{\otimes n+1}\xi_n\Delta_0 \\ &= s^{-n}\sum_{i=0}^{n}(e^{\otimes i} \otimes e\Delta_0 \otimes e^{\otimes n-i})(\xi_i \otimes \xi_{n-i}) : \\ &\qquad (Ts\mathcal{A})^{\otimes n+1}[n] \to Ts\mathcal{A}^{\mathsf{su}} \otimes Ts\mathcal{A}^{\mathsf{su}},\end{aligned}$$

by equation (4.9). Since e is a morphism of augmented graded cocategories, it follows that

$$\begin{aligned}\zeta_n\Delta_0 &= s^{-n}\sum_{i=0}^{n}(1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(e^{\otimes i+1}\xi_i \otimes e^{\otimes n-i+1}\xi_{n-i}) \\ &= s^{-n}\sum_{i=0}^{n}(1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(s^i \otimes s^{n-i})(\zeta_i \otimes \zeta_{n-i}) : \\ &\qquad (Ts\mathcal{A})^{\otimes n+1}[n] \to Ts\mathcal{A}^{\mathsf{su}} \otimes Ts\mathcal{A}^{\mathsf{su}}.\end{aligned}$$

This implies the following formula for $\widetilde{\Delta}$:

$$(5.2)\quad \widetilde{\Delta}|_{(Ts\mathcal{A})^{\otimes n+1}[n]} = s^{-n}\sum_{i=0}^{n}(1^{\otimes i}\otimes\Delta_0\otimes 1^{\otimes n-i})(s^i\otimes s^{n-i}) :$$

$$(Ts\mathcal{A})^{\otimes n+1}[n] \to \bigoplus_{i=0}^{n}(Ts\mathcal{A})^{\otimes i+1}[i]\bigotimes(Ts\mathcal{A})^{\otimes n-i+1}[n-i].$$

The counit of $\mathcal{E}$ is $\widetilde{\varepsilon} = [\mathcal{E} \xrightarrow{\mathrm{pr}_0} Ts\mathcal{A} \xrightarrow{\varepsilon} \Bbbk\mathrm{Ob}\mathcal{A} = \Bbbk\mathrm{Ob}\mathcal{E}]$. The augmentation of $\mathcal{E}$ is $\widetilde{\eta} = [\Bbbk\mathrm{Ob}\mathcal{E} = \Bbbk\mathrm{Ob}\mathcal{A} \xrightarrow{\eta} Ts\mathcal{A} \xrightarrow{\mathrm{in}_0} \mathcal{E}]$. The differential $\widetilde{b} : \mathcal{E} \to \mathcal{E}$ is found from the following equation:

$$\big[\mathcal{E} \xrightarrow{\zeta} Ts\mathcal{A}^{\mathsf{su}} \xrightarrow{b} Ts\mathcal{A}^{\mathsf{su}}\big] = \big[\mathcal{E} \xrightarrow{\widetilde{b}} \mathcal{E} \xrightarrow{\zeta} Ts\mathcal{A}^{\mathsf{su}}\big].$$

Let $\widetilde{b}_{n,m} : (Ts\mathcal{A})^{\otimes n+1}[n] \to (Ts\mathcal{A})^{\otimes m+1}[m]$, $n, m \geqslant 0$, denote the matrix coefficients of $\widetilde{b}$. Restricting the left hand side of the above equation to the summand $(Ts\mathcal{A})^{\otimes n+1}[n]$ of $\mathcal{E}$, we obtain

$$\begin{aligned}\zeta_n b &= s^{-n}e^{\otimes n+1}\xi_n b \\ &= s^{-n}e^{\otimes n+1}\nu_n\xi_{n-1} + (-)^n s^{-n}\sum_{i=0}^{n}(e^{\otimes i}\otimes eb\otimes e^{\otimes n-i})\xi_n : \\ &\qquad (Ts\mathcal{A})^{\otimes n+1}[n] \to Ts\mathcal{A}^{\mathsf{su}},\end{aligned}$$

by equation (4.10). Since e preserves the counit, it follows that

$$e^{\otimes n+1}\nu_n = \nu_n e^{\otimes n} : (Ts\mathcal{A})^{\otimes n+1} \to (Ts\mathcal{A}^{\mathsf{su}})^{\otimes n}.$$

Furthermore, e commutes with the differential b, therefore

$$\zeta_n b = s^{-n}\nu_n s^{n-1}(s^{-(n-1)}e^{\otimes n}\xi_{n-1}) + (-)^n s^{-n}\sum_{i=0}^{n}(1^{\otimes i}\otimes b\otimes 1^{\otimes n-i})s^n(s^{-n}e^{\otimes n+1}\xi_n)$$
$$= s^{-n}\nu_n s^{n-1}\zeta_{n-1} + (-)^n s^{-n}\sum_{i=0}^{n}(1^{\otimes i}\otimes b\otimes 1^{\otimes n-i})s^n\zeta_n : (Ts\mathcal{A})^{\otimes n+1}[n] \to Ts\mathcal{A}^{\mathsf{su}}.$$

We conclude that

$$\widetilde{b}_{n,n} = (-)^n s^{-n}\sum_{i=0}^{n}(1^{\otimes i}\otimes b\otimes 1^{\otimes n-i})s^n : (Ts\mathcal{A})^{\otimes n+1}[n] \to (Ts\mathcal{A})^{\otimes n+1}[n], \tag{5.3}$$

for $n \geqslant 0$, and

$$\widetilde{b}_{n,n-1} = s^{-n}\nu_n s^{n-1} : (Ts\mathcal{A})^{\otimes n+1}[n] \to (Ts\mathcal{A})^{\otimes n}[n-1], \tag{5.4}$$

for $n \geqslant 1$, are the only non-vanishing matrix coefficients of $\widetilde{b}$.

Let $g : \mathcal{E} \to Ts\mathcal{B}$ be a morphism of augmented differential graded cocategories, and let $g_n : (Ts\mathcal{A})^{\otimes n+1}[n] \to Ts\mathcal{B}$ be its components. By formula (5.2), the equation $g\Delta_0 = \widetilde{\Delta}(g\otimes g)$ is equivalent to the system of equations

$$g_n\Delta_0 = s^{-n}\sum_{i=0}^{n}(1^{\otimes i}\otimes\Delta_0\otimes 1^{\otimes n-i})(s^i g_i\otimes s^{n-i}g_{n-i}) : (Ts\mathcal{A})^{\otimes n+1}[n] \to Ts\mathcal{B}\otimes Ts\mathcal{B}, \quad n \geqslant 0.$$

The equation $g\varepsilon = \widetilde{\varepsilon}(\Bbbk\mathrm{Ob}g)$ is equivalent to the equations $g_0\varepsilon = \varepsilon(\Bbbk\mathrm{Ob}g_0)$, $g_n\varepsilon = 0$, $n \geqslant 1$. The equation $\widetilde{\eta}g = (\Bbbk\mathrm{Ob}g)\eta$ is equivalent to the equation $\eta g_0 = (\Bbbk\mathrm{Ob}g_0)\eta$. By formulas (5.3) and (5.4), the equation $gb = \widetilde{b}g$ is equivalent to

$g_0 b = b g_0 : Ts\mathcal{A} \to Ts\mathcal{B}$ and

$$g_n b = (-)^n s^{-n} \sum_{i=0}^{n} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) s^n g_n + s^{-n} \nu_n s^{n-1} g_{n-1} : \\ (Ts\mathcal{A})^{\otimes n+1}[n] \to Ts\mathcal{B}, \quad n \geqslant 1.$$

Introduce $\Bbbk$-linear maps $\phi_n = s^n g_n : (Ts\mathcal{A})^{\otimes n+1}(X,Y) \to Ts\mathcal{B}(Xg, Yg)$ of degree $-n$, $X, Y \in \operatorname{Ob}\mathcal{A}$, $n \geqslant 0$. The above equations take the following form:

$$\phi_n \Delta_0 = \sum_{i=0}^{n} (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(\phi_i \otimes \phi_{n-i}) : \\ (Ts\mathcal{A})^{\otimes n+1} \to Ts\mathcal{B} \otimes Ts\mathcal{B}, \tag{5.5}$$

for $n \geqslant 1$;

$$\phi_n b = (-)^n \sum_{i=0}^{n} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \phi_n + \nu_n \phi_{n-1} : \\ (Ts\mathcal{A})^{\otimes n+1} \to Ts\mathcal{B}, \tag{5.6}$$

for $n \geqslant 1$;

$$\phi_0 \Delta_0 = \Delta_0 (\phi_0 \otimes \phi_0), \quad \phi_0 \varepsilon = \varepsilon, \quad \phi_0 b = b \phi_0, \tag{5.7}$$

$$\phi_n \varepsilon = 0, \quad n \geqslant 1. \tag{5.8}$$

Summing up, we conclude that morphisms of augmented differential graded cocategories $\mathcal{E} \to Ts\mathcal{B}$ are in bijection with collections consisting of a morphism of augmented differential graded cocategories $\phi_0 : Ts\mathcal{A} \to Ts\mathcal{B}$ and of $\Bbbk$-linear maps $\phi_n : (Ts\mathcal{A})^{\otimes n+1}(X,Y) \to Ts\mathcal{B}(X\phi_0, Y\phi_0)$ of degree $-n$, $X, Y \in \operatorname{Ob}\mathcal{A}$, $n \geqslant 1$, such that equations (5.5), (5.6), and (5.8) hold true.

In particular, A_∞-functors $f : \mathcal{A}^{\mathsf{su}} \to \mathcal{B}$, which are augmented differential graded cocategory morphisms $Ts\mathcal{A}^{\mathsf{su}} \to$

$Ts\mathcal{B}$, are in bijection with morphisms $g = \zeta f : \mathcal{E} \to Ts\mathcal{B}$ of augmented differential graded cocategories. With the above notation, we may say that to give an A_∞-functor $f : \mathcal{A}^{\mathsf{su}} \to \mathcal{B}$ is the same as to give an A_∞-functor $\phi_0 : \mathcal{A} \to \mathcal{B}$ and a system of $\Bbbk$-linear maps $\phi_n : (Ts\mathcal{A})^{\otimes n+1}(X,Y) \to Ts\mathcal{B}(X\phi_0, Y\phi_0)$ of degree $-n$, $X, Y \in \operatorname{Ob}\mathcal{A}$, $n \geqslant 1$, such that equations (5.5), (5.6) and (5.8) hold true.

5.1. **Proposition.** *The following conditions are equivalent.*

(a) *There exists an A_∞-functor $U : \mathcal{A}^{\mathsf{su}} \to \mathcal{A}$ such that*

$$\big[\mathcal{A} \xrightarrow{e} \mathcal{A}^{\mathsf{su}} \xrightarrow{U} \mathcal{A}\big] = \mathrm{id}_{\mathcal{A}}.$$

(b) *There exists a double $(1,1)$-coderivation $\phi : Ts\mathcal{A} \otimes Ts\mathcal{A} \to Ts\mathcal{A}$ of degree -1 such that $\phi B_1 = \nu$.*

Proof. (a)⇒(b) Let $U : \mathcal{A}^{\mathsf{su}} \to \mathcal{A}$ be an A_∞-functor such that $eU = \mathrm{id}_{\mathcal{A}}$, in particular $\operatorname{Ob}U = \mathrm{id} : \operatorname{Ob}\mathcal{A}^{\mathsf{su}} = \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{A}$. It gives rise to the family of $\Bbbk$-linear maps $\phi_n = s^n\zeta_n U : (Ts\mathcal{A})^{\otimes n+1}(X,Y) \to Ts\mathcal{B}(X,Y)$ of degree $-n$, $X, Y \in \operatorname{Ob}\mathcal{A}$, $n \geqslant 0$, that satisfy equations (5.5), (5.6) and (5.8). In particular, $\phi_0 = eU = \mathrm{id}_{\mathcal{A}}$. Equations (5.5) and (5.6) for $n = 1$ read as follows:

$$\begin{aligned} \phi_1\Delta_0 = (\Delta_0 \otimes 1)(\phi_0 \otimes \phi_1) + (1 \otimes \Delta_0)(\phi_1 \otimes \phi_0) \\ = (\Delta_0 \otimes 1)(1 \otimes \phi_1) + (1 \otimes \Delta_0)(\phi_1 \otimes 1), \\ \phi_1 b = (1 \otimes b + b \otimes 1)\phi_1 + \nu_1\phi_0 = (1 \otimes b + b \otimes 1)\phi_1 + \nu. \end{aligned}$$

In other words, ϕ_1 is a double $(1,1)$-coderivation of degree -1 and $\phi_1 B_1 = \nu$.

(b)⇒(a) Let $\phi : Ts\mathcal{A} \otimes Ts\mathcal{A} \to Ts\mathcal{A}$ be a double $(1,1)$-coderivation of degree -1 such that $\phi B_1 = \nu$. Define $\Bbbk$-linear maps

$$\phi_n : (Ts\mathcal{A})^{\otimes n+1}(X,Y) \to Ts\mathcal{A}(X,Y), \quad X, Y \in \operatorname{Ob}\mathcal{A},$$

of degree $-n$, $n \geqslant 0$, recursively via $\phi_0 = \mathrm{id}_{\mathcal{A}}$ and $\phi_n = (\phi_{n-1} \otimes 1)\phi$, $n \geqslant 1$. Let us show that ϕ_n satisfy equations (5.5), (5.6) and (5.8). Equation (5.8) is obvious: $\phi_n\varepsilon = (\phi_{n-1} \otimes 1)\phi\varepsilon = 0$ as $\phi\varepsilon = 0$ by (4.3). Let us prove equation (5.5) by induction. It holds for $n = 1$ by assumption, since $\phi_1 = \phi$ is a double $(1,1)$-coderivation. Let $n \geqslant 2$. We have:

$$\begin{aligned}\phi_n\Delta_0 &= (\phi_{n-1} \otimes 1)\phi_1\Delta_0 \\ &= (\phi_{n-1} \otimes 1)((\Delta_0 \otimes 1)(1 \otimes \phi_1) + (1 \otimes \Delta_0)(\phi_1 \otimes 1)) \\ &= (\phi_{n-1}\Delta_0 \otimes 1)(1 \otimes \phi_1) \\ &\qquad + (1^{\otimes n} \otimes \Delta_0)((\phi_{n-1} \otimes 1)\phi_1 \otimes 1).\end{aligned}$$

By induction hypothesis,

$$\phi_{n-1}\Delta_0 = \sum_{i=0}^{n-1} (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-1-i})(\phi_i \otimes \phi_{n-1-i}),$$

so that

$$\begin{aligned}\phi_n\Delta_0 &= \sum_{i=0}^{n-1} (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(\phi_i \otimes \phi_{n-1-i} \otimes 1)(1 \otimes \phi_1) \\ &\qquad + (1^{\otimes n} \otimes \Delta_0)((\phi_{n-1} \otimes 1)\phi_1 \otimes 1) \\ &= \sum_{i=0}^{n} (1^{\otimes i} \otimes \Delta_0 \otimes 1^{\otimes n-i})(\phi_i \otimes \phi_{n-i}),\end{aligned}$$

since $(\phi_{n-1-i} \otimes 1)\phi_1 = \phi_{n-i}$, $0 \leqslant i \leqslant n-1$.

Let us prove equation (5.6) by induction. For $n = 1$ it is equivalent to the equation $\phi B_1 = \nu$, which holds by assumption. Let $n \geqslant 2$. We have:

$$\phi_n b - (-)^n \sum_{i=0}^{n} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \phi_n$$

$$= (\phi_{n-1} \otimes 1)\phi b - (-)^n \sum_{i=0}^{n-1} ((1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i})\phi_{n-1} \otimes 1)\phi$$

$$- (-)^n (1^{\otimes n} \otimes b)(\phi_{n-1} \otimes 1)\phi$$

$$= -(\phi_{n-1} b \otimes 1)\phi - (\phi_{n-1} \otimes b)\phi + (\phi_{n-1} \otimes 1)\nu$$

$$+ (-)^{n-1} \sum_{i=0}^{n-1} ((1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i})\phi_{n-1} \otimes 1)\phi + (\phi_{n-1} \otimes b)\phi$$

$$= (\phi_{n-1} \otimes 1)\nu$$

$$- \left(\left[\phi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i})\phi_{n-1} \right] \otimes 1 \right) \phi.$$

By induction hypothesis,

$$\phi_{n-1} b - (-)^{n-1} \sum_{i=0}^{n-1} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-1-i})\phi_{n-1} = \nu_{n-1}\phi_{n-2},$$

therefore

$$\phi_n b - (-)^n \sum_{i=0}^{n} (1^{\otimes i} \otimes b \otimes 1^{\otimes n-i}) \phi_n$$

$$= (\phi_{n-1} \otimes 1)\nu - (\nu_{n-1}\phi_{n-2} \otimes 1)\phi.$$

Since by (4.8)

$$
\begin{aligned}
(\phi_{n-1} \otimes 1)\nu - (\nu_{n-1}\phi_{n-2} \otimes 1)\phi \\
&= (\phi_{n-1} \otimes \varepsilon) - (\phi_{n-1}\varepsilon \otimes 1) - (\nu_{n-1} \otimes 1)\phi_{n-1} \\
&= (1^{\otimes n} \otimes \varepsilon)\phi_{n-1} - (\nu_{n-1} \otimes 1)\phi_{n-1} = \nu_n \phi_{n-1},
\end{aligned}
$$

and equation (5.6) is proven.

The system of maps ϕ_n, $n \geqslant 0$, corresponds to an A_∞-functor $U : \mathcal{A}^{\mathsf{su}} \to \mathcal{A}$ such that $\phi_n = s^n \zeta_n U$, $n \geqslant 0$. In particular, $eU = \phi_0 = \mathrm{id}_\mathcal{A}$. □

5.2. **Proposition.** *Let $\mathcal{A}$ be a unital A_∞-category. There exists a double $(1,1)$-coderivation $h : Ts\mathcal{A} \otimes Ts\mathcal{A} \to Ts\mathcal{A}$ of degree -1 such that $hB_1 = \nu$.*

Proof. Let $\mathcal{A}$ be a unital A_∞-category. By [9, Corollary A.12], there exist a differential graded category $\mathcal{D}$ and an A_∞-equivalence $f : \mathcal{A} \to \mathcal{D}$. The functor f is unital by [8, Corollary 8.9]. This means that, for every object X of $\mathcal{A}$, there exists a $\Bbbk$-linear map ${}_X v_0 : \Bbbk \to (s\mathcal{D})^{-2}(Xf, Xf)$ such that ${}_X\mathbf{i}_0^\mathcal{A} f_1 = {}_{Xf}\mathbf{i}_0^\mathcal{D} + {}_X v_0 b_1$. Here ${}_{Xf}\mathbf{i}_0^\mathcal{D}$ denotes the strict unit of the differential graded category $\mathcal{D}$.

By Lemma 4.3, $\xi = (1 \otimes \mathbf{i}_0^\mathcal{D} \otimes 1)\mu^{(3)} : Ts\mathcal{D} \otimes Ts\mathcal{D} \to Ts\mathcal{D}$ is a $(1,1)$-coderivation of degree -1. Let ι denote the double (f,f)-coderivation $(f \otimes f)\xi$ of degree -1. By Lemma 4.3,

$$\iota B_1 = (f \otimes f)(\xi B_1) = (f \otimes f)\nu = \nu f.$$

By Lemma 4.2, the equation $\nu B_1 = 0$ holds true. We conclude that the double coderivations $\nu \in \mathscr{D}(\mathcal{A},\mathcal{A})(\mathrm{id}_\mathcal{A}, \mathrm{id}_\mathcal{A})^0$ and $\iota \in \mathscr{D}(\mathcal{A},\mathcal{D})(f,f)^{-1}$ satisfy the following equations:

$$\nu B_1 = 0, \tag{5.9}$$

$$\iota B_1 - \nu f = 0. \tag{5.10}$$

We are going to prove that there exist double coderivations $h \in \mathscr{D}(\mathcal{A},\mathcal{A})(\mathrm{id}_\mathcal{A},\mathrm{id}_\mathcal{A})^{-1}$ and $k \in \mathscr{D}(\mathcal{A},\mathcal{D})(f,f)^{-2}$ such that the following equations hold true:

$$hB_1 = \nu,$$
$$hf = \iota + kB_1.$$

Let us put ${}_Xh_{0,0} = {}_X\mathbf{i}_0^\mathcal{A}$, ${}_Xk_{0,0} = {}_Xv_0$, and construct the other components of h and k by induction. Given an integer $t \geqslant 0$, assume that we have already found components $h_{p,q}$, $k_{p,q}$ of the sought h, k, for all pairs (p,q) with $p+q < t$, such that the equations

$$(5.11)\quad (hB_1 - \nu)_{p,q} = 0 : \\ s\mathcal{A}(X_0,X_1) \otimes \cdots \otimes s\mathcal{A}(X_{p+q-1},X_{p+q}) \to s\mathcal{A}(X_0,X_{p+q}),$$

$$(5.12)\quad (kB_1 + \iota - hf)_{p,q} = 0 : \\ s\mathcal{A}(X_0,X_1) \otimes \cdots \otimes s\mathcal{A}(X_{p+q-1},X_{p+q}) \to s\mathcal{D}(X_0f,X_{p+q}f)$$

are satisfied for all pairs (p,q) with $p+q < t$. Introduce double coderivations $\widetilde{h} \in \mathscr{D}(\mathcal{A},\mathcal{A})(\mathrm{id}_\mathcal{A},\mathrm{id}_\mathcal{A})$ and $\widetilde{k} \in \mathscr{D}(\mathcal{A},\mathcal{D})(f,f)$ of degree -1 resp. -2 by their components: $\widetilde{h}_{p,q} = h_{p,q}$, $\widetilde{k}_{p,q} = k_{p,q}$ for $p+q < t$, all the other components vanish. Define a double $(1,1)$-coderivation $\lambda = \widetilde{h}B_1 - \nu$ of degree 0 and a double (f,f)-coderivation $\kappa = \widetilde{k}B_1 + \iota - \widetilde{h}f$ of degree -1. Then $\lambda_{p,q} = 0$, $\kappa_{p,q} = 0$ for all $p+q < t$. Let non-negative integers n, m satisfy $n+m = t$. The identity $\lambda B_1 = 0$ implies that

$$\lambda_{n,m}b_1 - \sum_{l=1}^{n+m}(1^{\otimes l-1} \otimes b_1 \otimes 1^{\otimes n+m-l})\lambda_{n,m} = 0.$$

The (n,m)-component of the identity $\kappa B_1 + \lambda f = 0$ gives

$$\kappa_{n,m} b_1 + \sum_{l=1}^{n+m} (1^{\otimes l-1} \otimes b_1 \otimes 1^{\otimes n+m-l}) \kappa_{n,m} + \lambda_{n,m} f_1 = 0.$$

The chain map $f_1 : \mathcal{A}(X_0, X_{n+m}) \to s\mathcal{D}(X_0 f, X_{n+m} f)$ is homotopy invertible as f is an A_∞-equivalence. Hence, the chain map Φ given by

$$\underline{\mathsf{C}}_{\Bbbk}^{\bullet}(N, s\mathcal{A}(X_0, X_{n+m})) \to \underline{\mathsf{C}}_{\Bbbk}^{\bullet}(N, s\mathcal{D}(X_0 f, X_{n+m} f)),$$
$$\lambda \mapsto \lambda f_1,$$

is homotopy invertible for each complex of $\Bbbk$-modules N, in particular, for $N = s\mathcal{A}(X_0, X_1) \otimes \cdots \otimes s\mathcal{A}(X_{n+m-1}, X_{n+m})$. Therefore, the complex $\operatorname{Cone}(\Phi)$ is contractible, e.g. by [8, Lemma B.1]. Consider the element $(\lambda_{n,m}, \kappa_{n,m})$ of

$$\underline{\mathsf{C}}_{\Bbbk}^{0}(N, s\mathcal{A}(X_0, X_{n+m})) \oplus \underline{\mathsf{C}}_{\Bbbk}^{-1}(N, \mathcal{D}(X_0 f, X_{n+m} f)).$$

The above direct sum coincides with $\operatorname{Cone}^{-1}(\Phi)$. The equations $-\lambda_{n,m} d = 0$, $\kappa_{n,m} d + \lambda_{n,m} \Phi = 0$ imply that $(\lambda_{n,m}, \kappa_{n,m})$ is a cycle in the complex $\operatorname{Cone}(\Phi)$. Due to acyclicity of $\operatorname{Cone}(\Phi)$, $(\lambda_{n,m}, \kappa_{n,m})$ is a boundary of some element $(h_{n,m}, -k_{n,m})$ of $\operatorname{Cone}^{-2}(\Phi)$, i.e., of

$$\underline{\mathsf{C}}_{\Bbbk}^{-1}(N, s\mathcal{A}(X_0, X_{n+m})) \oplus \underline{\mathsf{C}}_{\Bbbk}^{-2}(N, \mathcal{D}(X_0 f, X_{n+m} f)).$$

Thus, $-k_{n,m}d + h_{n,m}f_1 = \kappa_{n,m}$, $-h_{n,m}d = \lambda_{n,m}$. These equations can be written as follows:

$$- h_{n,m}b_1 - \sum_{u+1+v=n+m} (1^{\otimes u} \otimes b_1 \otimes 1^{\otimes v})h_{n,m} = (\widetilde{h}B_1 - \nu)_{n,m},$$

$$- k_{n,m}b_1 + \sum_{u+1+v=n+m} (1^{\otimes u} \otimes b_1 \otimes 1^{\otimes v})k_{n,m} + h_{n,m}f_1 = (\widetilde{k}B_1 + \iota - \widetilde{h}f)_{n,m}.$$

Thus, if we introduce double coderivations $\overline{h}$ and $\overline{k}$ by their components: $\overline{h}_{p,q} = h_{p,q}$, $\overline{k}_{p,q} = k_{p,q}$ for $p + q \leqslant t$ (using just found maps if $p + q = t$) and 0 otherwise, then these coderivations satisfy equations (5.11) and (5.12) for each p, q such that $p+q \leqslant t$. Induction on t proves the proposition. □

5.3. **Theorem.** *Every unital A_∞-category admits a weak unit.*

Proof. The proof follows from Propositions 5.1 and 5.2. □

6. Summary

We have proved that the definitions of unital A_∞-category given by Lyubashenko, by Kontsevich and Soibelman, and by Fukaya are equivalent.

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