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Partially Observed Discrete-Valued Time Series in Fractional Gaussian Noise

(Recommended by Prof. E. Dshalalow)

Stochastic processes of counts have very broad applications in view of the host of integer-valued time series which cannot be satisfactorily handled within the classical framework of Gaussian-like series. In this paper we discuss recursive filters for partially observed discrete-valued time series where the noise in the observations is a fractional Gaussian noise.

Стохастическая обработка отсчетов широко применяется в множестве задач, содержащих целочисленно-оцениваемые временные ряды, которыми нельзя удовлетворительно оперировать в рамках классических Гауссово-подобных рядов. Рассмотрены рекурсивные фильтры для частично наблюдаемых дискретно-оцениваемых рядов, в которых шумы наблюдений являются дробными Гауссовыми шумами.

K e y w o r d s: change of measure, discrete-valued time series, fractional Gaussian noise.

Introduction. The analysis of time series of counts is a rapidly developing area (e.g. [1—7]). It has very broad application in view of the host of integer-valued time series which cannot be satisfactorily handled within the classical framework of Gaussian-like series. Many of the phenomena which occur in practice are by their very nature discrete-valued [4].

To start, we make use of The Binomial thinning operator $\circ$ introduced in [2, 6], namely: for any nonnegative integer-valued random variable X and $\alpha \in \{0,1\}$,

$$\alpha \circ X = \sum_{j=1}^{X} Y_j ,$$

where Y_1, Y_2, … is a sequence of of i.i.d. random variables independent of X, such that $P(Y_j = 1) = 1 - P(Y_j = 0) = \alpha$.

With the operator $\circ$ on hand and x_k standing the realization of an integer-valued process in period k, let $x_{k+1} = \alpha \circ x_{k+1} + v_{k+1}$, where $\{v_k\}$ is some integer-valued stochastic process.

One may think of x_k as referring to the number of patients in a hospital in period k, then the number of patients in period $k+1$ is made up of a portion of those patients who were present in period k ($\alpha \circ x_{k+1}$) and new arriving patients v_{k+1}. Time series models incorporating $\circ$ have been extensively examined in [1—3, 5—7].

Dynamics with fractional Gaussian noise. Let Z denote the set of integers, and Z^+ denote the set of non- negative integers. Following [8] we define a set of functions $\mathcal{L}$ on Z^+ with values in R. We suppose that if $i<0$, then $f(i)=0$. These functions could be considered as infinite sequences: $f(i)=f_i$, $i=0, 1, \ldots$. Then we define: if f^1, f^2 are in $\mathcal{L}$ the convolution product $f^1 * f^2$ is defined by

$$(f^1 * f^2)(n) = \sum_{i=0}^{\infty} f_{n-i}^1 f_i^2 = \sum_{i=0}^{n} f_{n-i}^1 f_i^2 .$$

In this set of functions, consider the function u, which is defined as $u=(u_0, u_1, \ldots)=(1, 1, \ldots)$.

The convolution powers of u are as follows:

$$u^0 = (1, 0, 0, \ldots),$$

$$u^2 = u * u = (1, 2, 3, \ldots),$$

$$u^3 = u^2 * u = (1, 3, 6, \ldots)$$

$$\ldots$$

$$u^k = \left(1, \frac{r}{1!}, \frac{r(r+1)}{2!}, \frac{r(r+1)(r+2)}{3!}, \ldots\right).$$

Note that for any f in $\mathcal{L}$, $f * u^0 = u^0 * f = f$ and for any s, r in R, $u^r * u^s = u^{s+r}$. In particular $u^r * u^{-r} = u^0$.

Let (Ω, F, P) be a probability space upon which $\{w_k\}$, $k \in$ N are independent and identically distributed (i.i.d.) Gaussian random variables, having zero means and variances 1 ($N(0, 1)$). Then [8] the fractional Gaussian noise is defined as

$$w_n^r \triangleq (u^r * w)(n) = \sum_{i=0}^{n} u_i^r w_{n-i} .$$

Then w^r is a sequence of Gaussian random variables which have memory and are correlated. Also,

$$E[w_n^r] = 0,$$

$$Var(w_n^r) = \sum_{i=0}^{n} (u_i^r)^2,$$

$$Cov(w_n^r, w_{n-1}^r) = \sum_{i=0}^{n-1} u_{n-i}^r \; u_{n-1-i}^r + 1 .$$

Now consider a system whose state at time k is $x_k \in Z_+$. The time index k of the state evolution will be discrete and identified with $N = \{0, 1, 2, ...\}$. The state of the system satisfies the dynamics

$$x_{k+1} = \alpha_k \circ x_k + v_{k+1}. \tag{1}$$

Here $\{v_k\}$, $k \in N$ are independent and identically distributed random variables such that, for all k, $v_k \in Z_+$ has probability distribution ϕ.

A noisy observation of x_k is to suppose it is given as a linear function of x_k plus a random «noise» term. That is, we suppose that for some real numbers c_k and positive real numbers d_k our observations have the form

$$y_k = c_k x_k + d_k w_k^r.$$

Following [8] let $z_k = (u^{-r} * y)(k)$. Therefore

$$z_k = c_k (u^{-r} * x)(k) + d_k w_k \triangleq c_k h_k (x_0, x_1, ...) + d_k w_k, \tag{2}$$

where w is a sequence of i.i.d. $N(0, 1)$. Write $\{\mathcal{Z}_k\}$, $k \in N$ for the complete filtration generated by $\{z_0, z_1, ..., z_k\}$.

Using measure change techniques we shall derive a recursive expression for the conditional distribution of x_k given $\mathcal{Z}_k$.

Filtering. Initially we suppose all processes are defined on an «ideal» probability space $(\Omega, F, \overline{P})$; then under a new probability measure P, to be defined, the model dynamics (1) and (2) will hold. Suppose that under $\overline{P}$:

1) $\{x_k\}$, $k \in N$ is an i.i.d. sequence with probability distribution $\phi(x)$ with support in Z_+;

2) $\{z_k\}$, $k \in N$ is an i.i.d. $N(0, 1)$ sequence with density function

$$\psi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

Let

$$\overline{\lambda}_0 = \frac{\psi(d_0^{-1}(z_0 - c_0 h_0(x_0)))}{d_0 \psi(z_0)}$$

and for $l = 1, 2, ...$.

$$\overline{\lambda}_l = \frac{\phi(x_l - \alpha_{l-1} \circ x_{l-1}) \psi(d_l^{-1}(z_l - c_l h_l(x_0, ..., x_l)))}{d_l \phi(x_l) \psi(z_l)} .$$

Set

$$\overline{\Lambda}_k = \prod_{l=0}^{k} \overline{\lambda}_l . \tag{3}$$

Let $\mathcal{G}_k$ be the complete σ-field generated by $\{x_0, x_1, ..., x_k, \alpha_0 \circ x_0, ...$ $..., \alpha_k \circ x_k, z_0, z_1, ..., z_k\}$ for $k \in \mathrm{N}$.

Lemma 1. The process $\{\overline{\Lambda}_k\}$, is a $\overline{P}$-martingale with respect to the filtration $\{\mathcal{G}_k\}$, with $k \in \mathrm{N}$.

P r o o f. Since $\overline{\Lambda}_k$ is $\mathcal{G}_k$-measurable $\overline{E}\,[\overline{\Lambda}_{k+1} \mid \mathcal{G}_k] = \overline{\Lambda}_k \overline{E}\,[\overline{\lambda}_{k+1} \mid \mathcal{G}_k]$. Therefore we must show that $\overline{E}\,[\overline{\lambda}_{k+1} \mid \mathcal{G}_k] = 1$:

$$\overline{E}\,[\overline{\lambda}_{k+1} \mid \mathcal{G}_k] =$$

$$= \overline{E}\left[\frac{\phi\,(x_{k+1} - \alpha_k \circ x_k)\,\psi\,(d_{k+1}^{-1}(y_{k+1} - c_{k+1} h_{k+1}(x_0, ..., x_{k+1})))}{d_{k+1} \phi\,(x_{k+1})\,\psi\,(z_{k+1})} \mid \mathcal{G}_k \right] =$$

$$= \overline{E}\left[\frac{\phi\,(x_{k+1} - \alpha_k \circ x_k)}{\phi\,(x_{k+1})}\, \overline{E}\left[\frac{\psi\,(d_{k+1}^{-1}(y_{k+1} - c_{k+1} h_{k+1}(x_0, ..., x_{k+1})))}{d_{k+1} \psi\,(z_{k+1})} \times \right.\right.$$

$$\left.\left. \times \mid \mathcal{G}_k,\, x_{k+1} \right] \mid \mid \mathcal{G}_k \right].$$

Now,

$$\overline{E}\left[\frac{\psi\,(d_{k+1}^{-1}(z_{k+1} - c_{k+1} h_{k+1}(x_0, ..., x_{k+1})))}{d_{k+1} \psi\,(z_{k+1})} \mid \mathcal{G}_k, x_{k+1} \right] =$$

$$= \int_{\mathrm{R}} \frac{\psi\,(d_{k+1}^{-1}(z - c_{k+1} h_{k+1}(x_0, ..., x_{k+1})))}{d_{k+1} \psi\,(z)}\, \psi\,(z)\, dz = 1;$$

$$\overline{E}\left[\frac{\phi\,(x_{k+1} - \alpha_k \circ x_k)}{\phi\,(x_{k+1})} \mid \mathcal{G}_k \right] = \overline{E}\left[\sum_{x \in \mathrm{Z}_+} \frac{\phi\,(x - \alpha_k \circ x_k)}{\phi\,(x)} \phi\,(x) \mid \mathcal{G}_k \right] = \sum_{u \in \mathrm{Z}_+} \phi\,(u) = 1.$$

Define P on (Ω, F) by setting the restriction of the Radon-Nykodim derivative $\frac{dP}{d\overline{P}}$ to $\mathcal{G}_k$ equal to $\overline{\Lambda}_k$.

A key result which relates expectation under P and $\overline{P}$ is given by a Bayes's like formulae ([9, 10])

$$E\,[I(x_k = x \mid \mathcal{G}_k] = \frac{\overline{E}\,[\overline{\Lambda}_k I(x_k = x \mid \mathcal{G}_k]}{\overline{E}\,[\overline{\Lambda}_k \mid \mathcal{G}_k]},$$

where $\overline{E}$ (resp. E) denotes expectations with respect to $\overline{P}$ (resp. P).

The next result shows that the original model can be recovered by some simple transformations.

Lemma 2. The stochastic process $\{v_k\}$, $k \in \mathrm{N}$ is an i.i.d sequence with density function $\phi(x)$ with support in Z_+ and $\{w_k\}$, $k \in \mathrm{N}$ are i.i.d. $N(0, 1)$ sequences of random variables, where

$$v_{k+1} \overset{\Delta}{=} (x_{k+1} - \alpha_k \circ x_k),$$

$$w_k \overset{\Delta}{=} d_k^{-1}(z_k - c_k h_k(x_0, \ldots, x_k)).$$

P r o o f. Suppose $f, g : \mathrm{R} \to \mathrm{R}$ are «test» functions (i.e. measurable functions with compact support). Then with E (resp. $\overline{E}$) denoting expectation under P (resp. $\overline{P}$) and using Bayes' Theorem

$$E[f(v_{k+1})g(w_{k+1}) | \mathcal{G}_k] = \frac{\overline{\Lambda}_k \overline{E}[\overline{\lambda}_{k+1} f(v_{k+1}) g(w_{k+1}) | \mathcal{G}_k]}{\overline{\Lambda}_k \overline{E}[\overline{\lambda}_{k+1} | \mathcal{G}_k]} =$$

$$= \overline{E}[\overline{\lambda}_{k+1} f(v_{k+1}) g(w_{k+1}) | \mathcal{G}_k],$$

where the last equality follows from Lemma 1. Consequently

$$E[f(v_{k+1}) g(w_{k+1}) | \mathcal{G}_k] =$$

$$= \overline{E}\left[\frac{\phi(x_{k+1} - \alpha_k \circ x_k)\psi(d_{k+1}^{-1}(z_{k+1} - c_{k+1}h_{k+1}(x_0, \ldots, x_{k+1})))}{d_{k+1}\phi(x_{k+1})\psi(z_{k+1})} \times\right.$$

$$\left.\times f(x_{k+1} - \alpha_k \circ x_k) g(d_{k+1}^{-1}(z_{k+1} - c_{k+1}h_{k+1}(x_0, \ldots, x_{k+1}))) | \mathcal{G}_k\right] =$$

$$= \overline{E}\left[\frac{\phi(x_{k+1} - \alpha_k \circ x_k)}{\phi(x_{k+1})} f(x_{k+1} - \alpha_k \circ x_k) \times\right.$$

$$\times \overline{E}\left[\frac{\psi(d_{k+1}^{-1}(y_{k+1} - c_{k+1}h_{k+1}(x_0, \ldots, x_{k+1})))}{d_{k+1}\psi(z_{k+1})} \times\right.$$

$$\left.\left.\times g(d_{k+1}^{-1}(z_{k+1} - c_{k+1}h_{k+1}(x_0, \ldots, x_{k+1}))) | \mathcal{G}_k, x_{k+1}\right] | \mathcal{G}_k\right].$$

Now

$$\overline{E}\left[\frac{\psi(d_{k+1}^{-1}(y_{k+1} - c_{k+1}h_{k+1}(x_0, \ldots, x_{k+1})))}{d_{k+1}\psi(z_{k+1})} \times\right.$$

$$\left.\times g(d_{k+1}^{-1}(z_{k+1} - c_{k+1}h_{k+1}(x_0, \ldots, x_{k+1}))) | \mathcal{G}_k, x_{k+1}\right] =$$

$$=\int_{\mathrm{R}} \frac{\psi(d_{k+1}^{-1}(z-c_{k+1}h_{k+1}(x_0,\dots,x_{k+1})))}{d_{k+1}\psi(z)}\psi(z)\times$$

$$\times g(d_{k+1}^{-1}(z-c_{k+1}h_{k+1}(x_0,\dots,x_{k+1})))\,dz=\int_{\mathrm{R}}\psi(u)\,g(u)\,du$$

and

$$\overline{E}\left[\frac{\phi(x_{k+1}-\alpha_k\circ x_k)}{\phi(x_{k+1})}f(x_{k+1}-\alpha_k\circ x_k)\,|\,\mathcal{G}_k\right]=$$

$$=\overline{E}\left[\sum_{x\in \mathrm{Z}_+}\frac{\phi(x-\alpha_k\circ x_k)}{\phi(x)}\phi(x)\,f\,(x-\alpha_k\circ x_k)\,|\,\mathcal{G}_k\right]=\sum_{t\in\mathrm{Z}_+}\phi(t)f(t)\,.$$

Therefore

$$E[f(v_{k+1})\,g(w_{k+1})\,|\,\mathcal{G}_k]=\sum_{t\in\mathrm{Z}_+}\phi(t)\,f(t)\int_{\mathrm{R}}\psi(u)\,g(u)\,du\,.$$

The lemma is proved.

Consider the un-normalized, conditional expectation which is the numerator of (3) and write $\overline{E}\,[\overline{\Lambda}_k I\,(x_k=x)\,|\,\mathcal{Z}_k]=q_k(x)$. If $p_k(\cdot)$ denotes the normalized conditional density, such that $E\,[I\,(x_k=x)\,|\,\mathcal{Z}_k]=p_k(x)$, and from (4) we see that

$$p_k(x)=q_k(x)\left[\sum_{t\in\mathrm{Z}_+}q_k(t)\right]^{-1},\ k\in\mathrm{N}\,.$$

Then we have the following result.

Theorem 1.

$$q_{k+1}(x)=\frac{\psi(d_0^{-1}(z_0-c_0h_0(x_0)))}{d_0\dots d_{k+1}\,\psi(z_0)\dots\psi(z_{k+1})}\times$$

$$\times\sum_{x_1\in\mathrm{Z}_+}\dots\sum_{x_k\in\mathrm{Z}_+}\psi(d_1^{-1}(z_1-c_1h_1(x_0,x_1)))\times$$

$$\times\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(x_0,\dots,x_k,x)))\times$$

$$\times\sum_{m_k=0}^{x_k}\phi(x-m_k)\binom{x_k}{m_k}\alpha_k^{m_k}(1-\alpha_k)^{x_k-m_k}\sum_{m_0=0}^{x_0}\phi(x_1-m_0)\binom{x_0}{m_0}\alpha_0^{m_0}(1-\alpha_0)^{x_0-m_0}.$$

P r o o f. In view of (3)

$$\overline{E}\,[\overline{\Lambda}_k\overline{\lambda}_{k+1}I\,(x_{k+1}=x)\,|\,\mathcal{Z}_{k+1}]=$$

$$=\overline{E}\left[\overline{\Lambda}_k \frac{\phi(x-\alpha_k\circ x_k)\,\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(x_0,\dots,x_k,x)))}{d_{k+1}\phi(x)\psi(z_{k+1})}\phi(x)\,|\,Z_{k+1}\right]=$$

$$=\overline{E}\left[\overline{\Lambda}_{k-1}\sum_{x_k\in Z_+}\frac{\psi(d_k^{-1}(z_k-c_kh_k(x_0,\dots,x_k)))}{d_k\phi(x_k)\psi(z_k)}\phi(x_k-\alpha_{k-1}\circ x_{k-1})\phi(x_k)\times\right.$$

$$\times\sum_{m_k=0}^{x_k}\phi(x-m_k)\binom{x_k}{m_k}\alpha_k^{m_k}(1-\alpha_k)^{x_k-m_k}\times$$

$$\left.\times\frac{\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(x_0,\dots,x_k,x)))}{d_{k+1}\,\psi(z_{k+1})}\,|\,Z_{k+1}\right]\dots$$

$$\dots\frac{\psi(d_0^{-1}(z_0-c_0h_0(x_0)))}{d_0\dots d_{k+1}\,\psi(z_0)\dots\psi(z_{k+1})}\times$$

$$\times\sum_{x_1\in Z_+}\dots\sum_{x_k\in Z_+}\psi(d_1^{-1}(z_1-c_1h_1(x_0,x_1)))\dots$$

$$\dots\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(x_0,\dots,x_k,x)))\times$$

$$\times\sum_{m_k=0}^{x_k}\phi(x-m_k)\binom{x_k}{m_k}\alpha_k^{m_k}(1-\alpha_k)^{x_k-m_k}\ \dots\ \sum_{m_0=0}^{x_0}\phi(x_1-m_0)\binom{x_0}{m_0}\alpha_0^{m_0}(1-\alpha_0)^{x_0-m_0}.$$

Approximate recursion. In this section, we give recursive approximate estimates of the hidden states. Assume that x_0 is known and let

$$x_0\triangleq\widetilde{x}_0,\ \ \widetilde{q}_0(x)=\frac{\psi(d_0^{-1}(z_0-c_0h_0(\widetilde{x}_0))}{d_0\psi(z_0)}\delta_{x_0}(x);$$

$$\widetilde{x}_k=\sum_{t\in Z_+}t\,\widetilde{p}_k(t),$$

where

$$\widetilde{p}_k(x)=\widetilde{q}_k(x)=\left[\sum_{t\in Z_+}\widetilde{q}_k(t)\right]^{-1}.$$

Theorem 2. The un-normalized density $q_{k+1}(x)$ is approximately computed by the recursion

$$\widetilde{q}_{k+1}(x)=\frac{\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(\widetilde{x}_0,\dots,\widetilde{x}_k,x)))}{d_{k+1}\,\psi(z_{k+1})}\times$$

$$\times \sum_{t \in Z_+} \sum_{r=0}^{t} \phi(x-r) \binom{t}{r} \alpha_k^r (1-\alpha_k)^{t-r} \widetilde{q}_k(t). \tag{4}$$

P r o o f.

$$\overline{E}\,[\overline{\Lambda}_k \overline{\lambda}_{k+1} I(x_{k+1}=x) \,|\, \mathcal{Z}_{k+1}] =$$

$$= \overline{E}\left[\overline{\Lambda}_k \frac{\phi(x-\alpha_k \circ x_k)\,\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(x_0,\dots,x_k,x)))}{d_{k+1}\psi(z_{k+1})} \,|\, \mathcal{Z}_{k+1}\right].$$

Now we replace $x_0,\dots,x_k$ with $\widetilde{x}_0,\dots,\widetilde{x}_k$ which, of course, are $\mathcal{Z}_{k+1}$ measurable:

$$\overline{E}\,[\overline{\Lambda}_k \overline{\lambda}_{k+1} I(x_{k+1}=x) \,|\, \mathcal{Z}_{k+1}] =$$

$$= \overline{E}\left[\overline{\Lambda}_k \frac{\phi(x-\alpha_k \circ x_k)\,\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(\widetilde{x}_0,\dots,\widetilde{x}_k,x)))}{d_{k+1}\psi(z_{k+1})} \,|\, \mathcal{Z}_{k+1}\right] =$$

$$= \frac{\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(\widetilde{x}_0,\dots,\widetilde{x}_k,x)))}{d_{k+1}\,\psi(z_{k+1})}\,\overline{E}\,[\overline{\Lambda}_k \phi(x-\alpha_k \circ x_k) \,|\, \mathcal{Z}_{k+1}] =$$

$$= \frac{\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(\widetilde{x}_0,\dots,\widetilde{x}_k,x)))}{d_{k+1}\,\psi(z_{k+1})};$$

$$\overline{E}\left[\overline{\Lambda}_k \sum_{t \in Z_+} \sum_{r=0}^{t} \phi(x-r) \binom{z}{r} \alpha_k^r (1-\alpha_k)^{t-r}\, I(x_k=t) \,|\, \mathcal{Z}_k\right] =$$

$$= \frac{\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(\widetilde{x}_0,\dots,\widetilde{x}_k,x)))}{d_{k+1}\psi(z_{k+1})} \times$$

$$\times \sum_{t \in Z_+} \sum_{r=0}^{t} \phi(x-r) \binom{z}{r} \alpha_k^r (1-\alpha_k)^{t-r}\, \overline{E}\,[\overline{\Lambda}_k I(x_k=t) \,|\, \mathcal{Z}_k] =$$

$$= \frac{\psi(d_{k+1}^{-1}(z_{k+1}-c_{k+1}h_{k+1}(\widetilde{x}_0,\dots,\widetilde{x}_k,x)))}{d_{k+1}\psi(z_{k+1})} \times$$

$$\times \sum_{t \in Z_+} \sum_{r=0}^{t} \phi(x-r) \binom{z}{r} \alpha_k^r (1-\alpha_k)^{t-r} q_k(t).$$

Taking $\widetilde{q}_k(t)$ as an approximation of $q_k(t)$ the result follows.

In this paper an integer-valued process state space model is proposed. The state space model is not directly observed. The observed process is corrupted with a fractional Gaussian noise. Filters for the partially observed dynamics are derived.

Стохастична обробка відліків широко застосовується у багатьох задачах з цілочисло-оцінюваними часовими рядами, котрими не можна задовільно оперувати в рамках класичних Гаусово-подібних рядів. Розглянуто рекурсивні фільтри для частково спостережуваних дискретно-оцінюваних рядів, в котрих шуми спостережень є дробовими Гаусовими шумами.

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