Methods of Functional Analysis and Topology
Vol. 15 (2009), no. 2, pp. 152–167

EXTENSION OF SOME LIONS-MAGENES THEOREMS

ALEKSANDR A. MURACH

Dedicated to the memory of A. Ya. Povzner

Abstract. A general form of the Lions-Magenes theorems on solvability of an elliptic boundary-value problem in the spaces of nonregular distributions is proved. We find a general condition on the space of right-hand sides of the elliptic equation under which the operator of the problem is bounded and has a finite index on the corresponding couple of Hilbert spaces. Extensive classes of the spaces satisfying this condition are constructed. They contain the spaces used by Lions and Magenes and many others spaces.

1. Introduction and statement of the problem

Let Ω be a bounded domain in the Euclidean space $\mathbb{R}^n$, $n \geq 2$, with the boundary Γ which is an infinitely smooth closed manifold of the dimension $n-1$. The domain Ω is situated locally on the same side from Γ.

We consider the nonhomogeneous boundary-value problem in the domain Ω:

$$A\,u = f \ \text{ in } \ \Omega, \quad B_j\,u = g_j \ \text{ on } \ \Gamma \ \text{ for } \ j = 1, \ldots, q. \tag{1}$$

In what follows A is a linear differential expression on $\overline{\Omega}$ of an arbitrary even order $2q \geq 2$, whereas B_j with $j = 1, \ldots, q$ is a boundary linear differential expression on Γ of order $m_j \leq 2q-1$. All coefficients of A and B_j are complex-valued functions infinitely smooth on $\overline{\Omega} := \Omega \cup \Gamma$ and on Γ respectively.

Everywhere in the paper the boundary-value problem (1) is assumed to be *regular elliptic*. This means [1, Ch. 2, Sec. 5.1], [2, Ch. III, § 6, Sec. 5] that the expression A is properly elliptic on $\overline{\Omega}$ and the collection of boundary expressions $B := (B_1, \ldots, B_q)$ is normal and satisfies the complementing condition with respect to A on Γ. It follows from the condition of normality that all orders m_j with $j = 1, \ldots, q$ are mutually distinct.

Along with (1) we consider the boundary-value problem

$$A^{+}\,v = \omega \ \text{ in } \ \Omega, \quad B_j^{+}\,v = h_j \ \text{ on } \ \Gamma, \ \ j = 1, \ldots, q, \tag{2}$$

formally adjoint to the problem (1) with respect to the Green formula

$$(Au, v)_\Omega + \sum_{j=1}^{q} (B_j u,\, C_j^{+} v)_\Gamma = (u, A^{+} v)_\Omega + \sum_{j=1}^{q} (C_j u,\, B_j^{+} v)_\Gamma, \quad u, v \in C^\infty(\,\overline{\Omega}\,).$$

Here, A^{+} is the linear differential expression formally adjoint to A and having the order $2q$ and coefficients from $C^\infty(\,\overline{\Omega}\,)$. In addition, $\{B_j^{+}\}$, $\{C_j\}$, and $\{C_j^{+}\}$ are certain normal systems of linear differential boundary expression with coefficients from $C^\infty(\Gamma)$. They orders satisfy the condition

$$\operatorname{ord} B_j + \operatorname{ord} C_j^{+} = \operatorname{ord} C_j + \operatorname{ord} B_j^{+} = 2q - 1.$$

2000 *Mathematics Subject Classification.* Primary 35J40; Secondary 46E35.

Key words and phrases. Elliptic boundary-value problem, weak solution, Sobolev spaces, Lions-Magenes theorems, multipliers.

In what follows we denote by $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ the inner products in the spaces $L_2(\Omega)$ and $L_2(\Gamma)$ (formed by functions square-integrable over Ω or Γ respectively) and natural extensions by continuity of these inner products.

We set

$$N := \{u \in C^\infty(\overline{\Omega}) : \ Au = 0 \ \text{ in } \ \Omega, \ \ B_j u = 0 \ \text{ on } \ \Gamma \ \ \forall \ j = 1, \dots, q\},$$

$$N^+ := \{v \in C^\infty(\overline{\Omega}) : \ A^+ v = 0 \ \text{ in } \ \Omega, \ \ B_j^+ v = 0 \ \text{ on } \ \Gamma \ \ \forall \ j = 1, \dots, q\}.$$

Since both the problem (1) and (2) are regular elliptic, both the spaces N and N^+ are finite-dimensional [1, Ch. 2, Theorem 5.3], [2, Ch. III, § 6, Sec. 4].

The fundamental property of every elliptic boundary-value problem consists in that the problem generates the bounded and Fredholm operator on appropriate couples of functional spaces. Note that a linear bounded operator $T : E_1 \to E_2$, with E_1 and E_2 being Banach spaces, is called the Fredholm operator if its kernel $\ker T$ and co-kernel $\operatorname{coker} T := E_2 / T(E_1)$ are finite-dimensional. The Fredholm operator T has the closed range in the space E_2 and the finite index $\operatorname{ind} T := \ker T - \operatorname{coker} T$. This operator naturally generates the homeomorphism $T : E_1 / \ker T \leftrightarrow T(E_1)$.

Let us formulate the classical theorem on elliptic boundary-value problems (see, e.g., [2, Ch. 3, Sec. 6], [1, Ch. 2, Sec. 5.4]). In the paper, we restrict ourselves to the Hilbert spaces case, which is the most important for applications.

Theorem 0. *The mapping*

$$u \mapsto (Au, Bu), \quad u \in C^\infty(\overline{\Omega}), \tag{3}$$

can be extended by continuity to the bounded and Fredholm operator

$$(A, B) : \ H^{s+2q}(\Omega) \to H^s(\Omega) \oplus \bigoplus_{j=1}^{q} H^{s+2q-m_j-1/2}(\Gamma) =: \mathcal{H}_s(\Omega, \Gamma) \tag{4}$$

for every real $s \geq 0$. The kernel of this operator coincides with N, whereas the range consists of all vectors $(f, g_1, \dots, g_q) \in \mathcal{H}_s(\Omega, \Gamma)$ satisfying the condition

$$(f, v)_\Omega + \sum_{j=1}^{q} (g_j, C_j^+ v)_\Gamma = 0 \quad \textit{for every} \quad v \in N^+. \tag{5}$$

The index of the operator (4) *is equal to* $\dim N - \dim N^+$ *and independent of s.*

In what follows $H^\sigma(\Omega)$ and $H^\sigma(\Gamma)$ are Hilbert spaces with the index $\sigma \in \mathbb{R}$ consisting of some distributions given in the domain Ω or on the manifold Γ respectively (we will remind their definitions in Sec. 2). In addition, as usual $\mathcal{D}'(\Omega)$ and $\mathcal{D}'(\Gamma)$ stand for the linear topological spaces of all distributions given in Ω or on Γ. We always interpret distributions as antilinear functionals.

Theorem 0 has a *generic* nature because the spaces in which the operator (4) acts are common for all elliptic boundary-value problems of the same order. By this theorem, the operator (A, B) establishes the homeomorphism of the factor space $H^{s+2q}(\Omega)/N$ onto the subspace

$$\{(f, g_1, \dots, g_q) \in \mathcal{H}_s(\Omega, \Gamma) : \ (5) \text{ is true}\}$$

for each $s \geq 0$. Therefore the theorems on operators generated by elliptic boundary-value problems are called Homeomorphisms Theorems.

Generally, Theorem 0 is not true in the case $s < 0$ because the mapping $u \mapsto B_j u$ with $u \in C^\infty(\overline{\Omega})$ can not be extended to the bounded operator $B_j : H^{s+2q}(\Omega) \to \mathcal{D}'(\Gamma)$ if $s + 2q \leq m_j + 1/2$. Therefore we have to use the space narrower than $H^{s+2q}(\Omega)$ as the domain of (A, B), namely

$$D^{s+2q}_{A,X}(\Omega) := \{u \in H^{s+2q}(\Omega) : \ Au \in X^s(\Omega)\}, \tag{6}$$

where $X^s(\Omega)$ is a Hilbert space embedded continuously in $\mathcal{D}'(\Omega)$. In what follows the image Au of $u \in \mathcal{D}'(\Omega)$ is understood in the theory of distributions. We endow the space (6) with the graphics inner product

$$(u_1, u_2)_{D^{s+2q}_{A,X}(\Omega)} := (u_1, u_2)_{H^{s+2q}(\Omega)} + (Au_1, Au_2)_{X^s(\Omega)} \tag{7}$$

and the corresponding norm.

The space $D^{s+2q}_{A,X}(\Omega)$ with the inner product (7) is complete. Indeed, if (u_k) is a Cauchy sequence in $D^{s+2q}_{A,X}(\Omega)$, then by a completeness of $H^{s+2q}(\Omega)$ and $X^s(\Omega)$ there are two limits: $u := \lim u_k$ in $H^{s+2q}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ and $f := \lim Au_k$ in $X^s(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ (embeddings are continuous). Since the differential operator A is continuous in $\mathcal{D}'(\Omega)$, we deduce from the first limit that $Au = \lim Au_k$ in $\mathcal{D}'(\Omega)$. This implies by the second limit the equality $Au = f \in X^s(\Omega)$. Therefore $u \in D^{s+2q}_{A,X}(\Omega)$ and $\lim u_k = u$ in the space $D^{s+2q}_{A,X}(\Omega)$, that is this space is complete.

J.-L. Lions and E. Magenes [3, 4, 5, 1] found the certain important examples of $X^s(\Omega)$ such that the mapping (3) can be extended by continuity to the bounded and Fredholm operator

$$(A, B) : D^{s+2q}_{A,X}(\Omega) \to X^s(\Omega) \oplus \bigoplus_{j=1}^{q} H^{s+2q-m_j-1/2}(\Gamma) =: \mathcal{X}_s(\Omega, \Gamma) \tag{8}$$

if $s < 0$. In contrast to Theorem 0, the domain of the operator (8) depends on coefficients of the elliptic expression A. Therefore the theorems on properties of the operator (8) we naturally shall call the *individual* theorems. Let us formulate two individual theorems proved by Lions and Magenes.

Theorem LM$_1$. [3, 4]. *Let $s < 0$ and $X^s(\Omega) := L_2(\Omega)$. Then the mapping* (3) *can be extended by continuity to the bounded and Fredholm operator* (8). *The kernel of this operator coincides with N, whereas the domain consists of all vectors $(f, g_1, \dots, g_q) \in \mathcal{X}_s(\Omega, \Gamma)$ satisfying* (5). *The index of the operator* (8) *is* $\dim N - \dim N^+$ *and independent of s.*

Here, we especially note the case $s = -2q$, which is important in the spectral theory of elliptic operators with general boundary conditions [6–10]; see also the survey [11, Sec. 7.7, 9.6]. In this case the space

$$D^0_{A,L_2}(\Omega) = \{u \in L_2(\Omega) : Au \in L_2(\Omega)\} \tag{9}$$

is the domain of the maximal operator $A_{\max}$ corresponding to the differential expression A [12, Sec. 1.2]. If A is formally self-adjoint, then the minimal differential operator $A_{\min} := A^*_{\max}$ is semi-bounded symmetric operator in the Hilbert space $L_2(\Omega)$. Applying Theorem LM$_1$ and some abstract results, one can describe the class of all self-adjoint extensions $\{\widetilde{A}\}$ of operator $A_{\min}$ by means of boundary conditions. After that one can select among them self-adjoint realizations $\widetilde{A}$ with classical spectral properties (semi-boundedness, discrete spectrum, Weyl's spectral asymptotics etc).

Even when all coefficients of A are constant, the space (9) depends essentially on each of them. One can see it from the following result of L. Hörmander [12, Sec. 3.1, Theorem 3.1].

Let A_1 and A_2 be constant-coefficient linear differential expressions. If $D^0_{A_1,L_2}(\Omega) \subseteq D^0_{A_2,L_2}(\Omega)$, then either $A_2 = \alpha A_1 + \beta$ for some $\alpha, \beta \in \mathbb{C}$, or both A_1 and A_2 are certain polynomials in the derivation operator with respect to a vector e and moreover $\operatorname{ord} A_2 \le \operatorname{ord} A_1$. Note that the second possibility is excluded for elliptic operators.

To formulate the second Lions-Magenes theorem we need the following weighted space

$$\varrho H^s(\Omega) := \{f = \varrho v : v \in H^s(\Omega)\},$$

where $s<0$, and a function $\varrho \in C^\infty(\Omega)$ is positive. We endow this space with the inner product

$$(f_1, f_2)_{\varrho H^s(\Omega)} := (\varrho^{-1} f_1, \varrho^{-1} f_2)_{H^s(\Omega)}$$

and the corresponding norm. The space $\varrho H^s(\Omega)$ is complete and embedded continuously in $\mathcal{D}'(\Omega)$. This follows from that the operator of multiplication by ϱ is continuous in $\mathcal{D}'(\Omega)$ and establishes the homeomorphism from the complete space $H^s(\Omega)$ onto $\varrho H^s(\Omega)$.

We consider a weight function $\varrho := \varrho_1^{-s}$ such that

$$\varrho_1 \in C^\infty(\overline{\Omega}), \quad \varrho_1 > 0 \text{ in } \Omega, \quad \varrho_1(x) = \operatorname{dist}(x, \Gamma) \quad \text{in a neighborhood of } \Gamma. \tag{10}$$

Theorem LM$_2$. [1, Ch. 2, Sec. 7.3]. *Let $s<0$, $s+1/2 \notin \mathbb{Z}$, and $X^s(\Omega) := \varrho_1^{-s} H^s(\Omega)$. Then the assertion of Theorem* LM$_1$ *remains true.*

We note that Lions and Magenes used a certain Hilbert space $\Xi^s(\Omega)$ as $X^s(\Omega)$. This space coincides (up to equivalence of norms) with the weighted space $\varrho_1^{-s} H^s(\Omega)$ for each non half-integer $s<0$ [1, Ch. 2, Sec. 7.1]. Theorem LM$_2$ also holds true for every half-integer $s<0$ if we define the space $X^s(\Omega)$ by means of the complex (holomorphic) interpolation, for instance,

$$X^s(\Omega) := [X^{2s}(\Omega), L_2(\Omega)]_{1/2}. \tag{11}$$

(See the definition and properties of this interpolation, e.g., in [1, Ch. 1, Sec. 14.1]).

In the paper, we find a general enough condition on the space $X^s(\Omega)$ under which the operator (8) is well defined, bounded, and Fredholm if $s<0$. The condition consists in the following.

Condition I$_s$. *The set $X^\infty(\Omega) := X^s(\Omega) \cap C^\infty(\overline{\Omega})$ is dense in $X^s(\Omega)$, and there exists a number $c>0$ such that*

$$\|\mathcal{O} f\|_{H^s(\mathbb{R}^n)} \leq c \, \|f\|_{X^s(\Omega)} \quad \textit{for each} \quad f \in X^\infty(\Omega). \tag{12}$$

Here, $\mathcal{O} f(x) := f(x)$ for $x \in \overline{\Omega}$, and $\mathcal{O} f(x) := 0$ for $x \in \mathbb{R}^n \setminus \overline{\Omega}$.

In (12), we define by $H^s(\mathbb{R}^n)$ the Hilbertian Sobolev space with index s and given over $\mathbb{R}^n$. Note that if s is smaller, then Condition I$_s$ is weaker for the same space $X^s(\Omega)$.

Both of the spaces $X^s(\Omega) := L_2(\Omega)$ and $X^s(\Omega) := \varrho_1^{-s} H^s(\Omega)$ used by Lions and Magenes satisfy Condition I$_s$.

In the paper, we find all the Hilbertian Sobolev spaces $X^s(\Omega) = H^\sigma(\Omega)$ for which Condition I$_s$ is fulfilled. In addition, we describe the class of all weights $\varrho \in C^\infty(\Omega)$ such that the weighted space $X^s(\Omega) := \varrho H^s(\Omega)$ satisfies Condition I$_s$. This class contains the weight $\varrho := \varrho_1^{-s}$ as a special case. Thus, we get some generalizations of the Lions-Magenes theorems mentioned above to more extensive classes of the Hilbertian spaces $X^s(\Omega)$ of right-hand sides of the elliptic equation.

Note that we generalize the Lions-Magenes theorems staying in classes of distributions given in the domain Ω. The different Homeomorphism Theorems for elliptic boundary-value problems were proved by Yu. M. Berezansky, S. G. Krein, Ya. A. Roitberg [13], M. Schechter [14], Roitberg [15–18], Yu. V. Kostarchuk and Roitberg [19] (see also the monograph [2, Ch. III, Sec. 6], survey [11, Sec. 7.9], and textbook [20, Ch. XVI, Sec. 1]). In these theorems, the solution and/or the right-hand side of the elliptic equation are not distributions in Ω. Namely, the right-hand side is assumed to be in the negative space dual to $H^{-s}(\Omega)$, $s<0$, with respect to the inner product in $L_2(\Omega)$. We denote this space by $H^{s,(0)}(\Omega)$; it consists of some distributions in $\mathbb{R}^n$ supported on $\overline{\Omega}$. The solution is considered in $H^{s+2q,(0)}(\Omega)$ [13, 14] or in the special space $H^{s+2q,(2q)}(\Omega)$ introduced by Roitberg [15–18] (see the definition in Sec. 3). Roitberg, Z. G. Sheftel' and their disciples used the space $H^{s+2q,(2q)}(\Omega)$ systematically in the theory of elliptic problems (their results are summed up in Roitberg's monographs [17, 18]).

These Homeomorphism Theorems have also different applications. Among them we mention the theorems on increase in smoothness of the solution up to the boundary, application to the investigation of Green function of an elliptic boundary-value problem, applications to elliptic problems with power singularities, the transmission problem, the Odhnoff problem, and many others (see the monographs [2, 17, 18], survey [11, Sec. 7.9], and references given therein).

We have to note that Roitberg [21, Sec. 2.4] considered a condition on the space $X^s(\Omega)$, which was somewhat stronger than our Condition I_s. He demanded additionally that $C^\infty(\overline{\Omega}) \subset X^s(\Omega)$. Under this stronger condition, Roitberg [21, Sec. 2.4], [17, p. 190] proved the boundedness of the operator (8) for all $s < 0$. Homeomorphism Theorem for this operator was formulated in the survey [11, p. 85] provided that $-2q \leq s \leq 0$ and $N = N^+ = \{0\}$. We also mention Homeomorphism Theorems [19, Theorem 4], [18, Sec. 1.3.8], in which this condition was used, but solutions of the elliptic boundary-value problem were considered in the space $H^{s+2q,(2q)}(\Omega)$. Remark that Roitberg's condition does not cover the important case where $X^s(\Omega) = \{0\}$ as well as some weighted spaces $X^s(\Omega) = \varrho H^s(\Omega)$ which we consider.

We also note that Condition I_s is fulfilled for some classes of the Hilbertian Hörmander spaces [22, Sec. 2.2], [23, Sec. 10.1]. Their applications to elliptic operators and elliptic boundary-value problems were studied by V. A. Mikhailets and the author in [24–33] .

The results of the paper are formulated in Section 2 as Theorems 1, 2, 3, and Corollaries 1, 2. The main result, Theorem 1, is proved in Section 4, all the rest in Section 5. In Section 3, we formulate the auxiliary propositions needed for our proofs. At the end of the paper, we give Appendix, in which a useful proposition on weight functions of the form $\varrho = \varrho_1^\delta$ is established.

2. Results

We introduce some necessary function spaces. Let $s \in \mathbb{R}$. Recall that

$$H^s(\mathbb{R}^n) := \bigl\{ w \in \mathcal{S}'(\mathbb{R}^n) : \, \|w\|_{H^s(\mathbb{R}^n)} := \|(1 + |\xi|^2)^{s/2}\, \widehat{w}(\xi)\|_{L_2(\mathbb{R}^n_\xi)} < \infty \bigr\}.$$

Here, $\mathcal{S}'(\mathbb{R}^n)$ is the topological linear space of tempered distributions in $\mathbb{R}^n$, whereas $\widehat{w}$ is the Fourier transform of w. For a closed set $Q \subset \mathbb{R}^n$, we put

$$H^s_Q(\mathbb{R}^n) := \bigl\{ w \in H^s(\mathbb{R}^n) : \, \operatorname{supp} w \subseteq Q \bigr\}.$$

The space $H^s_Q(\mathbb{R}^n)$ is Hilbert with respect to the inner product in $H^s(\mathbb{R}^n)$. We are interested in the cases where $Q \in \{\overline{\Omega}, \widehat{\Omega}, \Gamma\}$ with $\widehat{\Omega} := \mathbb{R}^n \setminus \Omega$.

Following [1, Ch. 1, Sec. 12.1], we will define the Hilbert space $H^s(\Omega)$. For arbitrary $s \geq 0$ we set

$$H^s(\Omega) := H^s(\mathbb{R}^n)/H^s_{\widehat{\Omega}}(\mathbb{R}^n) = \bigl\{ w \restriction \Omega : \, w \in H^s(\mathbb{R}^n) \bigr\}.$$

The space $H^s(\Omega)$ is complete with respect to the Hilbertian norm

$$\|u\|_{H^s(\Omega)} := \inf \bigl\{ \, \|w\|_{H^s(\mathbb{R}^n)} : \, w \in H^s(\mathbb{R}^n), \ w = u \text{ in } \Omega \, \bigr\}.$$

The set $C^\infty(\overline{\Omega})$ is dense in $H^s(\Omega)$, each measurable function u over Ω being identified with the antilinear functional $(u, \cdot)_\Omega$. We denote by $H^s_0(\Omega)$ the closure of the linear manifold

$$C^\infty_0(\Omega) := \{ u \in C^\infty(\overline{\Omega}) : \, \operatorname{supp} u \subset \Omega \}$$

in the topology of $H^s(\Omega)$. The space $H^s_0(\Omega)$ is complete with respect to the inner product in $H^s(\Omega)$.

For arbitrary $s < 0$, we denote by $H^s(\Omega)$ the Hilbert space antidual to the space $H^{-s}_0(\Omega)$ with respect to the inner product in $L_2(\Omega)$. Since antilinear functionals from $H^s(\Omega)$ are defined uniquely by their values on functions from $C^\infty_0(\Omega)$, we can correctly identify these functionals with distributions in Ω. It useful to keep in mind that $H^s(\Omega) =$

$H^s_{\overline{\Omega}}(\mathbb{R}^n)/H^s_\Gamma(\mathbb{R}^n)$ with equality of norms for every $s<0$ [1, Ch. 1, Remark 12.5], and $H^s(\Omega)=H^s(\mathbb{R}^n)/H^s_{\overline{\Omega}}(\mathbb{R}^n)$ with equivalence of norms for all non half-integer $s<0$ [34, Sec. 4.8.2]. It follows from the first equality that $C^\infty_0(\Omega)$ is dense in $H^s(\Omega)$ for every $s<0$ [34, Sec. 4.3.2, Theorem 1(b)].

Thus, the Hilbert space $H^s(\Omega)$ is defined for every $s\in\mathbb{R}$ and embedded continuously in $\mathcal{D}'(\Omega)$. We have the compact dense embedding $H^{s+\delta}(\Omega)\hookrightarrow H^s(\Omega)$ if $\delta>0$.

We denote by $H^s(\Gamma)$ the Hilbertian Sobolev space with the index $s\in\mathbb{R}$ defined over the closed compact manifold Γ [1, Ch. 1, Sec. 7.3]. The space consists of all distributions on Γ belonging to $H^s(\mathbb{R}^{n-1})$ in local coordinates and does not depend on a chose of local charts on Γ up to equivalence of norms.

Let us formulate the main result of the paper.

Theorem 1. *Let $s<0$, and $X^s(\Omega)$ be an arbitrary Hilbert space embedded continuously in $\mathcal{D}'(\Omega)$ and satisfying Condition* I_s. *Then the elliptic boundary-value problem* (1) *possesses the following properties:*

(i) *The set*
$$D^\infty_{A,X}(\Omega):=\{\,u\in C^\infty(\,\overline{\Omega}\,):\,Au\in X^s(\Omega)\,\}$$
is dense in $D^{s+2q}_{A,X}(\Omega)$.

(ii) *The mapping $u\to(Au,Bu)$ with $u\in D^\infty_{A,X}(\Omega)$ can be extended by continuity to the bounded linear operator* (8).

(iii) *The operator* (8) *is Fredholm. Its kernel is N, and its range consists of all vectors $(f,g_1,\dots,g_q)\in\mathcal{X}_s(\Omega,\Gamma)$ satisfying* (5).

(iv) *If the set $\mathcal{O}(X^\infty(\Omega))$ is dense in $H^s_{\overline{\Omega}}(\mathbb{R}^n)$, then the index of* (8) *is* $\dim N-\dim N^+$.

Let us consider some applications of Theorem 1 caused by a particular choice of the space $X^s(\Omega)$. Evidently, the space $X^s(\Omega):=\{0\}$ satisfies Condition I_s. In this case, Theorem 1 describes properties of the semihomogeneous boundary-value problem (1) with $f=0$ and holds true for every $s\in\mathbb{R}$ (see also [25]).

All the Hilbertian Sobolev spaces satisfying Condition I_s are found in the next theorem.

Theorem 2. *Let $s<0$ and $\sigma\in\mathbb{R}$. The space $X^s(\Omega):=H^\sigma(\Omega)$ satisfies Condition I_s if and only if*

$$\sigma\geq\max\{s,-1/2\}. \tag{13}$$

The next result follows from Theorems 1 and 2.

Corollary 1. *Let $s<0$, and* (13) *be valid. Then the mapping $u\mapsto(Au,Bu)$ with $u\in C^\infty(\,\overline{\Omega}\,)$ can be extended by continuity to the bounded and Fredholm operator*

$$(A,B):\,\bigl\{u\in H^{s+2q}(\Omega):Au\in H^\sigma(\Omega)\bigr\}\to H^\sigma(\Omega)\oplus\bigoplus_{j=1}^{q}H^{s+2q-m_j-1/2}(\Gamma), \tag{14}$$

its domain being the Hilbert space with respect to the norm
$$\bigl(\,\|u\|^2_{H^{s+2q}(\Omega)}+\|Au\|^2_{H^\sigma(\Omega)}\bigr)^{1/2}.$$
The index of (14) *is* $\dim N-\dim N^+$ *and independent of s, σ.*

Here, we note the special case where $\sigma=s$. If $-1/2<\sigma=s<0$, then the domain of (14) coincides with the space $H^{s+2q}(\Omega)$ up to equivalence of norms. If $\sigma=s=-1/2$, then the domain is narrower than $H^{2q-1/2}(\Omega)$ but does not depend on A as well (see Sec. 5.3 below).

We always have $X^s(\Omega)\subseteq H^{-1/2}(\Omega)$ in Theorem 2. But we can get a space $X^s(\Omega)$ containing an extensive class of some distributions $f\notin H^{-1/2}(\Omega)$ and satisfying Condition I_s if we use certain weighted spaces $\varrho H^s(\Omega)$.

A function ϱ given in Ω is called a *multiplier* in the space $H^s(\Omega)$ if the operator of multiplication by ϱ maps this space into itself and is bounded on it. A description of the class of all multipliers in $H^s(\Omega)$ with $s \geq 0$ was given in [35, Sec. 6.3.3].

Let $s < -1/2$. We introduce the following condition on the weight function ϱ.

Condition II$_s$. *The function ϱ is a multiplier in the space $H^{-s}(\Omega)$, and*

$$D_\nu^j\,\varrho = 0 \quad \text{on} \quad \Gamma \quad \text{for every} \quad j \in \mathbb{Z}, \quad 0 \leq j < -s - 1/2. \tag{15}$$

Here, D_ν is the derivation operator with respect to the unit vector ν of inner normal to the boundary Γ of Ω. Note that if ϱ is a multiplier in $H^{-s}(\Omega)$, then evidently $\varrho \in H^{-s}(\Omega)$. By the trace theorem [1, Ch. 1, Sec. 9.2], there is a trace $(D_\nu^j \varrho)\!\restriction\!\Gamma \in H^{-s-j-1/2}(\Gamma)$ for every integer $j \geq 0$ such that $-s - j - 1/2 > 0$. Hence, Condition II$_s$ is formulated correctly.

Theorem 3. *Let $s < -1/2$, and a function $\varrho \in C^\infty(\Omega)$ be positive. The space $X^s(\Omega) := \varrho H^s(\Omega)$ satisfies Condition* I$_s$ *if and only if the function ϱ satisfies Condition* II$_s$.

The next result follows from Theorems 1 and 3.

Corollary 2. *Let $s < -1/2$, and a positive function $\varrho \in C^\infty(\Omega)$ satisfy Condition* II$_s$. *Then the mapping $u \to (Au, Bu)$ with $u \in C^\infty(\overline{\Omega})$, $Au \in \varrho H^s(\Omega)$ can be extended by continuity to the bounded and Fredholm operator*

$$(A, B) : \bigl\{u \in H^{s+2q}(\Omega) : Au \in \varrho H^s(\Omega)\bigr\} \to \varrho H^s(\Omega) \oplus \bigoplus_{j=1}^{q} H^{s+2q-m_j-1/2}(\Gamma), \tag{16}$$

its domain being the Hilbert space with respect to the norm

$$\bigl(\,\|u\|^2_{H^{s+2q}(\Omega)} + \|\varrho^{-1}Au\|^2_{H^s(\Omega)}\bigr)^{1/2}.$$

The index of (16) *is* $\dim N - \dim N^+$ *and independent of s, ϱ.*

We give an important example of a function ϱ satisfying Condition II$_s$ for fixed $s < -1/2$ if we set $\varrho := \varrho_1^\delta$, where ϱ_1 meets (10), and the number δ is such that $\delta \geq -s-1/2 \in \mathbb{Z}$ or $\delta > -s - 1/2 \notin \mathbb{Z}$ (we will prove it in Appendix).

Let us compare Theorem 1 and its Corollaries 1, 2 with the Lions-Magenes theorems [1, 3, 4, 5] on elliptic boundary-value problems in the spaces of distributions.

We restrict ourselves to the case of Hilbertian spaces, whereas the non-Hilbertian Sobolev spaces were considered in [3, 4, 5] as well.

A proposition similar to Theorem 1 was proved in [5, Sec. 6.10] for non half-integers $s \leq -2q$ and the Dirichlet problem, the space $X^s(\Omega)$ obeying some different conditions depending on the problem. Our Condition I$_s$ does not depend on it.

Theorem LM$_1$ is a special case of Corollary 1 where $\sigma = 0$, i.e. $X^s(\Omega) = L_2(\Omega)$. Note that some spaces $X^s(\Omega)$ containing $L_2(\Omega)$ are permissible in Theorem 2 and Corollary 1. The space $X^s(\Omega) = H^{-1/2}(\Omega)$ is the most extensive among them provided that $s \leq -1/2$. If $-1/2 < \sigma = s < 0$, then Corollary 1 coincides with Theorem 7.5 of Lions and Magenes [1, Ch. 2] proved under the additional assumption that $N = N^+ = \{0\}$.

Theorem LM$_2$ for $s < -1/2$ is a special case of Corollary 2 because the function $\varrho := \varrho_1^{-s}$ satisfies Condition II$_s$. As we have mentioned, in the case where $-1/2 < s < 0$ one can use the more extensive space $H^s(\Omega)$ instead of the weighted space $\varrho H^s(\Omega)$ in (16).

3. Auxiliary propositions

First we note [1, Ch. 1, Theorem 11.5] that for every $s > 1/2$

$$H^s_0(\Omega) := \bigl\{u \in H^s(\Omega) : D_\nu^j\, u = 0 \;\text{ on }\; \Gamma \;\;\forall\;\, j \in \mathbb{Z},\; 0 \leq j < s - 1/2\bigr\}. \tag{17}$$

In addition [1, Ch. 1, Theorem 11.1]

$$H^s_0(\Omega) = H^s(\Omega) \quad \text{for} \quad 0 \le s \le 1/2. \tag{18}$$

Further we will use some results of Roitberg [15, 17] on properties of the problem (1) in the Hilbert scale

$$H^{s,(2q)}(\Omega) := \widetilde{H}^{s,2,(2q)}(\Omega), \quad s \in \mathbb{R}, \tag{19}$$

introduced by him. We also need some properties of this scale.

Let us give the definition of the scale (19). Let $s \in \mathbb{R}$. In the case were $s \ge 0$ we denote by $H^{s,(0)}(\Omega)$ the space $H^s(\Omega)$. In the case were $s < 0$ we denote by $H^{s,(0)}(\Omega)$ the space $H^s_{\overline{\Omega}}(\mathbb{R}^n)$ antidual to $H^{-s}(\Omega)$ with respect to the inner product in $L_2(\Omega)$. The space $H^{s,(0)}(\Omega)$ is Hilbert for every $s \in \mathbb{R}$, with the set $C^\infty(\,\overline{\Omega}\,)$ being dense in it. Here as usual, the function $f \in C^\infty(\,\overline{\Omega}\,)$ is identified with the functional $(f, \cdot\,)_\Omega$.

In view of (18)

$$H^{s,(0)}(\Omega) = H^s(\Omega) \quad \text{with equality of norms} \quad \text{for} \quad s \ge -1/2. \tag{20}$$

If $s < -1/2$, then the spaces $H^{s,(0)}(\Omega)$ and $H^s(\Omega)$ are different.

Let $s \in \mathbb{R}$ and $s \ne j - 1/2$ for all $j = 1, \ldots, 2q$. We denote by $H^{s,(2q)}(\Omega)$ the completion of the linear system $C^\infty(\,\overline{\Omega}\,)$ with respect to the norm

$$\|\, u \,\|_{H^{s,(2q)}(\Omega)} := \Bigl(\|\, u \,\|^2_{H^{s,(0)}(\Omega)} + \sum_{j=1}^{2q} \|(D^{j-1}_\nu u)\restriction\Gamma\|^2_{H^{s-j+1/2}(\Gamma)} \Bigr)^{1/2}.$$

The space $H^{s,(2q)}(\Omega)$ is separable Hilbert.

In the case where $s \in \{j - 1/2 : j = 1, \ldots, 2q\}$ we define the separable Hilbert space $H^{s,(2q)}(\Omega)$ by means of the complex interpolation

$$H^{s,(2q)}(\Omega) := \bigl[\, H^{s-1/2,(2q)}(\Omega), H^{s+1/2,(2q)}(\Omega) \,\bigr]_{1/2}.$$

We note that by the trace theorem [1, Ch. 1, Sec. 9.2]

$$H^{s,(2q)}(\Omega) = H^s(\Omega) \quad \text{with equivalence of norms for} \;\; s > 2q - 1/2. \tag{21}$$

The spaces $H^{s,(2q)}(\Omega)$ and $H^s(\Omega)$ are different if $s \le 2q - 1/2$.

The embeddings $H^{s_2,(0)}(\Omega) \hookrightarrow H^{s_1,(0)}(\Omega)$ and $H^{s_2,(2q)}(\Omega) \hookrightarrow H^{s_1,(2q)}(\Omega)$ are compact and dense for arbitrary $s_1, s_2 \in \mathbb{R}$, $s_1 < s_2$. This follows from the compactness of the embeddings $H^{s_2}(\Omega) \hookrightarrow H^{s_1}(\Omega)$ and $H^{s_2}(\Gamma) \hookrightarrow H^{s_1}(\Gamma)$.

Proposition 1. [[17], Theorems 4.1.1, 5.3.1]. *Let $s \in \mathbb{R}$. The mapping $u \mapsto (Au, Bu)$ with $u \in C^\infty(\,\overline{\Omega}\,)$ can be extended by continuity to the linear bounded operator*

$$(A, B) :\, H^{s+2q,(2q)}(\Omega) \to H^{s,(0)}(\Omega) \oplus \bigoplus_{j=1}^{q} H^{s+2q-m_j-1/2}(\Gamma) =: \mathcal{H}_{s,(0)}(\Omega, \Gamma). \tag{22}$$

This operator is Fredholm. Its kernel coincides with N, whereas its range consists of all vectors $(f, g_1, \ldots, g_q) \in \mathcal{H}_{s,(0)}(\Omega, \Gamma)$ satisfying condition (5). The index of (22) is $\dim N - \dim N^+$.

Proposition 1 is another example of the generic theorem on elliptic boundary-value problems. If $s \ge 0$, then Proposition 1 coincides with Theorem 0 by (21). But if $s < -1/2$, then both the spaces $H^{s+2q,(2q)}(\Omega)$ and $H^{s,(0)}(\Omega) = H^s_{\overline{\Omega}}(\mathbb{R}^n)$ consist of the elements which are not distributions in the domain Ω.

Proposition 2. [[17], Theorem 7.1.1]. *Let $s \in \mathbb{R}$, $\delta > 0$, and $u \in H^{s+2q,(2q)}(\Omega)$. If $(A, B)u \in \mathcal{H}_{s+\delta,(0)}(\Omega, \Gamma)$, then $u \in H^{s+2q+\delta,(2q)}(\Omega)$.*

Proposition 3. [[17], Theorem 6.1.1]. *Let $s \in \mathbb{R}$. The following assertions are true:*

(i) *The norm in the space* $H^{s+2q,(2q)}(\Omega)$ *is equivalent to the norm*

$$\big(\,\|u\|^2_{H^{s+2q,(0)}(\Omega)}+\|Au\|^2_{H^{s,(0)}(\Omega)}\big)^{1/2} \tag{23}$$

on the set of all functions $u\in C^\infty(\,\overline{\Omega}\,)$. *Therefore the space* $H^{s+2q,(2q)}(\Omega)$ *coincides with the completion of the linear system* $C^\infty(\,\overline{\Omega}\,)$ *with respect to the norm* (23).

(ii) *The mapping* $I_A:\,u\mapsto(u,Au)$ *with* $u\in C^\infty(\,\overline{\Omega}\,)$ *can be extended by continuity to the homeomorphism*

$$I_A:\,H^{s+2q,(2q)}(\Omega)\leftrightarrow K_{s+2q,A}(\Omega).$$

Here,

$$\begin{aligned} K_{s+2q,A}(\Omega):=\big\{\,&(u_0,f):\,u_0\in H^{s+2q,(0)}(\Omega),\ f\in H^{s,(0)}(\Omega),\\ &(u_0,A^+v)_\Omega=(f,v)_\Omega\ \ \forall\ v\in H^{2q}_0(\Omega)\cap H^{-s,(0)}(\Omega)\,\big\} \end{aligned} \tag{24}$$

is a closed subspace in $H^{s+2q,(0)}(\Omega)\oplus H^{s,(0)}(\Omega)$.

Proposition 4. [[17], Theorem 6.2.1]. *Let* $s<-2q-1/2$. *For each couple of distributions* $u_0\in H^{s+2q,(0)}(\Omega)$ *and* $f\in H^{s,(0)}(\Omega)$ *satisfying the condition*

$$(u_0,A^+v)_\Omega=(f,v)_\Omega\quad\text{for all}\quad v\in C^\infty_0(\Omega), \tag{25}$$

there exists a unique couple $(u_0^*,f)\in K_{s+2q,A}(\Omega)$ *such that*

$$(u_0,v)_\Omega=(u_0^*,v)_\Omega\quad\text{for all}\quad v\in C^\infty_0(\Omega). \tag{26}$$

Furthermore,

$$\|u_0^*\|_{H^{s+2q,(0)}(\Omega)}\leq c\,\big(\,\|u_0\|^2_{H^{s+2q,(0)}(\Omega)}+\|f\|^2_{H^{s,(0)}(\Omega)}\big)^{1/2}, \tag{27}$$

with number $c>0$ *being independent of* u_0, f, *and* u_0^*.

Remark 1. [[17], Sec. 6.2]. The conditions (24) and (25) are equivalent for $s\geq-2q-1/2$, but they are not equivalent if $s<-2q-1/2$.

4. Proof of the main result

Now we will prove the main result of the paper, Theorem 1. We assume that its condition be fulfilled; i.e., $s<0$ and the Hilbert space $X^s(\Omega)$ is embedded continuously in $\mathcal{D}'(\Omega)$ and satisfies Condition I_s. It follows that the mapping $f\mapsto\mathcal{O}f$ with $f\in X^\infty(\Omega)$ can be extended by continuity to the bounded linear operator

$$\mathcal{O}:\,X^s(\Omega)\to H^s_{\overline{\Omega}}(\mathbb{R}^n)=H^{s,(0)}(\Omega). \tag{28}$$

This operator is injective. Indeed, let $\mathcal{O}f=0$ for a distribution $f\in X^s(\Omega)$. Chose a sequence $(f_k)\subset X^\infty(\Omega)$ such that $f_k\to f$ in $X^s(\Omega)\hookrightarrow\mathcal{D}'(\Omega)$. Then $\mathcal{O}f_k\to0$ in $H^s_{\overline{\Omega}}(\mathbb{R}^n)\hookrightarrow\mathcal{S}'(\mathbb{R}^n)$ that implies

$$(f,v)_\Omega=\lim(f_k,v)_\Omega=\lim(\mathcal{O}f_k,v)_\Omega=0\quad\text{for every}\quad v\in C^\infty_0(\Omega).$$

Thus, $f=0$ as a distribution belonging to $X^s(\Omega)\hookrightarrow\mathcal{D}'(\Omega)$; i. e., the operator (28) is injective. This operator defines the continuous embedding $X^s(\Omega)\hookrightarrow H^{s,(0)}(\Omega)$.

According to Proposition 1, the element $Au\in H^{s,(0)}(\Omega)$ is correctly defined for an arbitrary $u\in H^{s+2q,(2q)}(\Omega)$ by means of passing to the limit. We set

$$D^{s+2q,(2q)}_{A,X}(\Omega):=\{\,u\in H^{s+2q,(2q)}(\Omega):\,Au\in X^s(\Omega)\,\}.$$

We also endow the space $D^{s+2q,(2q)}_{A,X}(\Omega)$ with the graphics inner product

$$(u_1,u_2)_{D^{s+2q,(2q)}_{A,X}(\Omega)}:=(u_1,u_2)_{H^{s+2q,(2q)}(\Omega)}+(Au_1,Au_2)_{X^s(\Omega)}.$$

The space $D^{s+2q,(2q)}_{A,X}(\Omega)$ is complete with respect to it. Indeed, let (u_k) be a Cauchy sequence in $D^{s+2q,(2q)}_{A,X}(\Omega)$. Since both the spaces $H^{s+2q,(2q)}(\Omega)$ and $X^s(\Omega)$ are complete, the limits $u := \lim u_k$ in $H^{s+2q,(2q)}(\Omega)$ and $f := \lim Au_k$ in $X^s(\Omega)$ exist. The first of them implies that $Au = \lim Au_k$ in $H^{s,(0)}(\Omega)$. We have from this in view of the second limit and the continuity of (28) that $Au = f \in X^s(\Omega)$. Hence, $u \in D^{s+2q,(2q)}_{A,X}(\Omega)$ and $\lim u_k = u$ in $D^{s+2q,(2q)}_{A,X}(\Omega)$; i.e., the space $D^{s+2q,(2q)}_{A,X}(\Omega)$ is complete.

By Proposition 1, the restriction of the operator (22) to $D^{s+2q,(2q)}_{A,X}(\Omega)$ gives the bounded operator

$$(A,B) :\, D^{s+2q,(2q)}_{A,X}(\Omega) \to \mathcal{X}_s(\Omega,\Gamma). \tag{29}$$

The kernel of (29) is N, and the range consists of all vectors $(f, g_1, \dots, g_q) \in \mathcal{X}_s(\Omega,\Gamma)$ satisfying condition (5). Hence, the operator (29) is Fredholm, with its co-kernel being of a dimension $\beta \le \dim N^+$.

Moreover, if $\mathcal{O}(X^\infty(\Omega))$ is dense in $H^s_{\overline{\Omega}}(\mathbb{R}^n)$, then $\beta = \dim N^+$. Indeed, denoting the operator (22) by Λ and the narrower operator (29) by Λ_0, we consider the operators Λ^* and Λ_0^* adjoint to them. Since the embedding $\mathcal{X}_s(\Omega,\Gamma) \hookrightarrow \mathcal{H}_{s,(0)}(\Omega,\Gamma)$ is continuous and dense, we have $\ker \Lambda_0^* \supseteq \ker \Lambda^*$. Hence

$$\beta = \dim \operatorname{coker} \Lambda_0 = \dim \ker \Lambda_0^* \ge \dim \ker \Lambda^* = \dim \operatorname{coker} \Lambda = \dim N^+.$$

Therefore $\beta = \dim N^+$ and the index of (29) is equal to $\dim N - \dim N^+$ if $\mathcal{O}(X^\infty(\Omega))$ is dense in $H^s_{\overline{\Omega}}(\mathbb{R}^n)$.

Let us show that the set $D^\infty_{A,X}(\Omega)$ is dense in $D^{s+2q,(2q)}_{A,X}(\Omega)$. Since $X^\infty(\Omega) \times (C^\infty(\Gamma))^q$ is dense in $\mathcal{X}_s(\Omega,\Gamma)$, we can write by the Gohberg-Krein lemma [36, Lemma 2.1]

$$\mathcal{X}_s(\Omega,\Gamma) = (A,B)\big(D^{s+2q,(2q)}_{A,X}(\Omega)\big) \dotplus \mathcal{Q}(\Omega,\Gamma), \tag{30}$$

where $\mathcal{Q}(\Omega,\Gamma)$ is a finite-dimensional subspace satisfying the condition

$$\mathcal{Q}(\Omega,\Gamma) \subset X^\infty(\Omega) \times (C^\infty(\Gamma))^q. \tag{31}$$

Denote by Π the projector of the space $\mathcal{X}_s(\Omega,\Gamma)$ onto the first term in (30) parallel to the second term.

Let $u \in D^{s+2q,(2q)}_{A,X}(\Omega)$. Approximate $F := (A,B)u$ by a sequence $(F_k) \subset X^\infty(\Omega) \times (C^\infty(\Gamma))^q$ in the the topology of $\mathcal{X}_s(\Omega,\Gamma)$. We have

$$\lim \Pi F_k = \Pi F = F = (A,B)u \quad \text{in} \quad \mathcal{X}_s(\Omega,\Gamma), \tag{32}$$

and by (31)

$$(\Pi F_k) \subset X^\infty(\Omega) \times (C^\infty(\Gamma))^q \tag{33}$$

The Fredholm operator (29) naturally generates the topological isomorphism

$$\Lambda_0 := (A,B) :\, D^{s+2q,(2q)}_{A,X}(\Omega)/N \leftrightarrow \Pi(\mathcal{X}_s(\Omega,\Gamma)).$$

In view of (32),

$$\lim \Lambda_0^{-1}\, \Pi F_k = \{u + w : w \in N\} \quad \text{in} \quad D^{s+2q,(2q)}_{A,X}(\Omega)/N.$$

Hence, there is a sequence of representatives $u_k \in D^{s+2q,(2q)}_{A,X}(\Omega)$ of cosets $\Lambda_0^{-1}\, \Pi F_k$ such that

$$\lim u_k = u \quad \text{in} \quad D^{s+2q,(2q)}_{A,X}(\Omega). \tag{34}$$

In addition, by (33) we have

$$(A,B)u_k = \Pi F_k \in C^\infty(\,\overline{\Omega}\,) \times (C^\infty(\Gamma))^q.$$

It follows in view of Proposition 2, equality (20) and the Sobolev embedding theorem that

$$u_k \in \bigcap_{\delta>0} H^{s+2q+\delta,(2q)}(\Omega) = \bigcap_{\delta>0} H^{s+2q+\delta}(\Omega) = C^\infty(\,\overline{\Omega}\,).$$

Thus, in (34) we have $(u_k) \subset D^\infty_{A,X}(\Omega)$; therefore $D^\infty_{A,X}(\Omega)$ is dense in $D^{s+2q,(2q)}_{A,X}(\Omega)$.

Next, we will consider the cases $-2q-1/2\le s<0$ and $s<-2q-1/2$ separately.

The first case: $-2q-1/2\le s<0$. Then $H^{s+2q,(0)}(\Omega)=H^{s+2q}(\Omega)$ in view of (20). We use Proposition 3 and consider the mapping $I_0: u\mapsto u_0$ in which $u\in D^{s+2q,(2q)}_{A,X}(\Omega)$ and $(u_0,f):=I_A u$. This mapping establishes the homeomorphism

$$I_0:\, D^{s+2q,(2q)}_{A,X}(\Omega)\leftrightarrow D^{s+2q}_{A,X}(\Omega). \tag{35}$$

Indeed, note that (24) $\Leftrightarrow$ (25) for arbitrary $u_0\in H^{s+2q,(0)}(\Omega)=H^{s+2q}(\Omega)$ and $f\in X^s(\Omega)\hookrightarrow H^{s,(0)}(\Omega)$ (see remark 1). Condition (25) means that $Au_0=f$ as distributions in Ω. It follows by Proposition 3 that $I_0(D^{s+2q,(2q)}_{A,X}(\Omega))=D^{s+2q}_{A,X}(\Omega)$. Moreover, we have the equivalence of the norms:

$$\begin{aligned}\|u\|^2_{D^{s+2q,(2q)}_{A,X}(\Omega)}&=\|u\|^2_{H^{s+2q,(2q)}(\Omega)}+\|f\|^2_{X^s(\Omega)}\\&\asymp\|u_0\|^2_{H^{s+2q,(0)}(\Omega)}+\|f\|^2_{H^{s,(0)}(\Omega)}+\|f\|^2_{X^s(\Omega)}\\&\asymp\|u_0\|^2_{H^{s+2q}(\Omega)}+\|f\|^2_{X^s(\Omega)}=\|u_0\|^2_{D^{s+2q}_{A,X}(\Omega)}.\end{aligned}$$

Hence, the mapping I_0 establishes the homeomorphism (35).

It follows from properties of the operator (29) denoted by Λ_0 and the operator (35) that

$$\Lambda_0 I_0^{-1}:\, D^{s+2q}_{A,X}(\Omega)\to\mathcal{X}_s(\Omega,\Gamma) \tag{36}$$

is a bounded and Fredholm operator, with the range and index being the same as for (29). Since I_0 bijectively maps the set $D^\infty_{A,X}(\Omega)$ onto itself, this set is dense in $D^{s+2q}_{A,X}(\Omega)$, and (36) is an extension by continuity of the mapping $u\to(Au,Bu)$ with $u\in D^\infty_{A,X}(\Omega)$. Theorem 1 is proved in the first case.

The second case: $s<-2q-1/2$. Then $H^{s+2q,(0)}(\Omega)=H^{s+2q}_{\overline{\Omega}}(\mathbb{R}^n)$. In addition

$$H^{s+2q}(\Omega)=\bigl\{w\restriction\Omega:\, w\in H^{s+2q}_{\overline{\Omega}}(\mathbb{R}^n)\bigr\}, \tag{37}$$

$$\|u\|_{H^{s+2q}(\Omega)}=\inf\bigl\{\,\|w\|_{H^{s+2q}(\mathbb{R}^n)}:\, w\in H^{s+2q}_{\overline{\Omega}}(\mathbb{R}^n),\ w=u\text{ in }\Omega\,\bigr\}. \tag{38}$$

This follows immediately from the equality $H^{s+2q}(\Omega)=H^{s+2q}_{\overline{\Omega}}(\mathbb{R}^n)/H^{s+2q}_{\Gamma}(\mathbb{R}^n)$ mentioned in Section 2.

Let us denote $Rw:=w\restriction\Omega$ for $w\in\mathcal{S}'(\mathbb{R}^n)$. We will prove that the mapping $I_0: u\mapsto Ru_0$ with $u\in D^{s+2q,(2q)}_{A,X}(\Omega)$ and $(u_0,f):=I_Au$ establishes the topological isomorphism (35) in the case under consideration. (In the first case, $Ru_0=u_0$.) We use Proposition 3 and note that (24) $\Rightarrow$ (25). For an arbitrary $u\in D^{s+2q,(2q)}_{A,X}(\Omega)$, we have: $Ru_0\in H^{s+2q}(\Omega)$ (see (37)), $f=Au\in X^s(\Omega)$, and

$$(Ru_0,A^+v)_\Omega=(u_0,A^+v)_\Omega=(f,v)_\Omega\quad\text{for every}\quad v\in C^\infty_0(\Omega);$$

i. e., $ARu_0=f$ as distributions in Ω. Therefore $I_0u=Ru_0\in D^{s+2q}_{A,X}(\Omega)$. Moreover, in view of (38) and the definition of $H^{s+2q,(2q)}(\Omega)$ we have:

$$\begin{aligned}\|I_0u\|^2_{D^{s+2q}_{A,X}(\Omega)}&=\|Ru_0\|^2_{H^{s+2q}(\Omega)}+\|f\|^2_{X^s(\Omega)}\\&\le\|u_0\|^2_{H^{s+2q}(\mathbb{R}^n)}+\|f\|^2_{X^s(\Omega)}\le\|u\|^2_{D^{s+2q,(2q)}_{A,X}(\Omega)}.\end{aligned}$$

Hence, the operator $I_0: D^{s+2q,(2q)}_{A,X}(\Omega)\to D^{s+2q}_{A,X}(\Omega)$ is bounded.

Now we will show that this operator is bijective. Let $\omega\in D^{s+2q}_{A,X}(\Omega)$ and $f:=A\omega\in X^s(\Omega)$. Due to (37), there is a distribution $u_0\in H^{s+2q}_{\overline{\Omega}}(\mathbb{R}^n)$ such that $\omega=Ru_0$. Distributions u_0 and f satisfy condition (25) because

$$(u_0,A^+v)_\Omega=(\omega,A^+v)_\Omega=(f,v)_\Omega\quad\text{for every}\quad v\in C^\infty_0(\Omega).$$

According to Proposition 4, for $u_0 \in H^{s+2q,(0)}(\Omega)$ and $f \in X^s(\Omega) \hookrightarrow H^{s,(0)}(\Omega)$ there exists a unique couple $(u_0^*, f) \in K_{s+2q,A}(\Omega)$ such that condition (26) is fulfilled. This implies by Proposition 3 that

$$u^* := I_A^{-1}(u_0^*, f) \in D_{A,X}^{s+2q,(2q)}(\Omega), \quad \text{and} \quad I_0 u^* = R u_0^* = R u_0 = \omega.$$

The element u^* is a unique preimage of ω under the mapping I_0. Indeed, if $I_0 u' = \omega$ for some $u' \in D_{A,X}^{s+2q,(2q)}(\Omega)$, then the couple $(u_0', f') := I_A u' \in K_{s+2q,A}(\Omega)$ satisfies the following conditions:

$$f' = A R u_0' = A\omega = f \quad \text{and} \quad (u_0', v)_\Omega = (\omega, v)_\Omega = (u_0, v)_\Omega \ \ \forall \ v \in C_0^\infty(\Omega).$$

Therefore by Proposition 4, the couples $(u_0', f') = (u_0', f)$ and (u_0^*, f) are equal that implies the equality of their preimages u' and u^* under the mapping I_A.

Thus, the linear bounded operator (35) is bijective in the case examined and therefore is a topological isomorphism by the Banach theorem on inverse operator. Now using the Fredholm property of (29) and reasoning as in the first case, we complete our proof in the second case.

Theorem 1 is proved.

5. Proofs of Theorems 2, 3, and corollaries

5.1. **Proof of Theorem 2.** By the condition, $s < 0$, $\sigma \in \mathbb{R}$, and $X^s(\Omega) := H^\sigma(\Omega)$. Then the set $X^\infty(\Omega) = C^\infty(\overline{\Omega})$ is dense in the space $H^\sigma(\Omega)$.

Sufficiency. Let (13) be fulfilled, i.e. $\sigma \geq \max\{s, -1/2\}$. Then by (20) we have

$$H^\sigma(\Omega) = H^{\sigma,(0)}(\Omega) \hookrightarrow H^{s,(0)}(\Omega),$$

with the embedding being continuous. We remind that each function $f \in C^\infty(\overline{\Omega})$ is identified with the functional $(f, \cdot)_\Omega$, the last being identified with the function $\mathcal{O}f$ from the space $H^s_{\overline{\Omega}}(\mathbb{R}^n) = H^{s,(0)}(\Omega)$. Hence

$$\|\mathcal{O}f\|_{H^s(\mathbb{R}^n)} \leq c\, \|f\|_{H^\sigma(\Omega)} \quad \text{for every} \quad f \in C^\infty(\overline{\Omega}),$$

with number $c > 0$ being independent of f. Sufficiency is proved.

Necessity. Let $X^s(\Omega) := H^\sigma(\Omega)$ satisfy Condition I_s. We assume that $\sigma < 0$. (If $\sigma \geq 0$, then (13) holds true). The operator (28) establishes the continuous dense embedding $H^\sigma(\Omega) \hookrightarrow H^{s,(0)}(\Omega)$. This implies that

$$H^{-s}(\Omega) = (H^{s,(0)}(\Omega))' \subseteq (H^\sigma(\Omega))' = H_0^{-\sigma}(\Omega),$$

we denoting by H' the space antidual to H with respect to the inner product in $L_2(\Omega)$. Hence, $-s \geq -\sigma$. Moreover $-\sigma \leq 1/2$, because if $-\sigma > 1/2$, then the function $f \equiv 1 \in H^{-s}(\Omega)$ would not belong to $H_0^{-\sigma}(\Omega)$ by virtue of (17). Thus, σ satisfies (13). Necessity is proved.

5.2. **Proof of Corollary 1.** Let numbers $s < 0$ and σ satisfy inequality (13). The boundedness and the Fredholm property of (14) follow immediately from Theorems 1 and 2 in which $X^s(\Omega) := H^\sigma(\Omega)$ and $D^\infty_{A,X}(\Omega) = C^\infty(\overline{\Omega})$. Moreover, since the set $\mathcal{O}(C^\infty(\overline{\Omega}))$ identified with $C^\infty(\overline{\Omega})$ is dense in $H^s_{\overline{\Omega}}(\mathbb{R}^n) = H^{s,(0)}(\Omega)$, the index of (14) is equal to $\dim N - \dim N^+$ by Theorem 1(iv) and therefore independent of s, f.

5.3. **Remark to Corollary 1.** Here we consider the special case of Corollary 1 where $-1/2 \leq \sigma = s < 0$. In this case the domain of the operator (14) does not depend on A. Indeed, if $-1/2 < \sigma = s < 0$, then we have the bounded operator $A : H^{s+2q}(\Omega) \to H^s(\Omega)$ because s is not half-integer [1, Ch. 1, Proposition 12.1]. It follows that the domain of (14) coincides with $H^{s+2q}(\Omega)$ and therefore does not depend on A.

If $s = \sigma = -1/2$, we cannot reason as above because the space $H^{-1/2}(\Omega)$ is narrower than $A(H^{2q-1/2}(\Omega))$. However, in view of Theorem 1(i), equality (20), and Proposition 3 we draw a conclusion that the domain of (14) is the completion of the set of all $u \in C^\infty(\overline{\Omega})$ with respect to the norm

$$\|u\|^2_{H^{2q-1/2}(\Omega)} + \|Au\|^2_{H^{-1/2}(\Omega)} = \|u\|^2_{H^{2q-1/2,(0)}(\Omega)} + \|Au\|^2_{H^{-1/2,(0)}(\Omega)} \asymp \|u\|_{H^{2q-1/2,(2q)}(\Omega)}.$$

Hence, the domain coincides with the space $H^{2q-1/2,(2q)}(\Omega)$ independent of A.

5.4. **Proof of Theorem 3.** By the condition, $s < -1/2$ whereas $\varrho \in C^\infty(\Omega)$ is positive. Let us denote by M_ϱ and $M_{\varrho^{-1}}$ the operators of multiplication by ϱ and ϱ^{-1} respectively. We have the isometric isomorphism $M_\varrho : H^s(\Omega) \leftrightarrow \varrho H^s(\Omega)$. It follows from this and from the density of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ that the set $C_0^\infty(\Omega)$ is dense in $X^s(\Omega) := \varrho H^s(\Omega)$. Hence, the more extensive set $X^\infty(\Omega)$ is dense in $X^s(\Omega)$.

We need the following lemma.

Lemma 1. *Let $s < -1/2$. The multiplication by $\varrho \in C^\infty(\Omega)$ is a bounded operator*

$$M_\varrho : H^{-s}(\Omega) \to H_0^{-s}(\Omega) \tag{39}$$

if and only if ϱ satisfies Condition II_s.

Proof. **Necessity.** If the multiplication by ϱ defines the bounded operator (39), then ϱ is a multiplier in $H^{-s}(\Omega)$ and belongs to $H_0^{-s}(\Omega)$. Therefore ϱ satisfies Condition II_s in view of (17).

Sufficiency. Let ϱ satisfy Condition II_s. We only need to prove that $\varrho u \in H_0^{-s}(\Omega)$ for every $u \in H^{-s}(\Omega)$. Condition II_s implies that $\varrho \in H_0^{-s}(\Omega)$ in view of (17). We chose sequences $(u_k) \subset C^\infty(\overline{\Omega})$ and $(\varrho_j) \subset C_0^\infty(\Omega)$ such that $u_k \to u$ and $\varrho_j \to \varrho$ in $H^{-s}(\Omega)$. Since both the functions ϱ and u_k are multipliers in the space $H^{-s}(\Omega)$, we have therein

$$\lim_{k\to\infty} (\varrho u_k) = \varrho u \quad \text{and} \quad \lim_{j\to\infty} (\varrho_j u_k) = \varrho u_k \quad \text{for every } k.$$

This in view of $\varrho_j u_k \in C_0^\infty(\Omega)$ implies that $\varrho u \in H_0^{-s}(\Omega)$. Sufficiency is proved. □

Now let us define the following space with inner product:

$$\varrho^{-1} H_0^{-s}(\Omega) := \{ f = \varrho^{-1} v : v \in H_0^{-s}(\Omega) \},$$
$$(f_1, f_2)_{\varrho^{-1} H_0^{-s}(\Omega)} := (\varrho f_1, \varrho f_2)_{H^{-s}(\Omega)}.$$

We have the isometric isomorphism

$$M_{\varrho^{-1}} : H_0^{-s}(\Omega) \leftrightarrow \varrho^{-1} H_0^{-s}(\Omega). \tag{40}$$

Hence, the space $\varrho^{-1} H_0^{-s}(\Omega)$ is complete, with $C_0^\infty(\Omega)$ being dense in it.

Note that

$$(\varrho^{-1} H_0^{-s}(\Omega))' = \varrho H^s(\Omega) \quad \text{with equality of norms.} \tag{41}$$

Indeed, passing in (40) to adjoint operator, we get the isometric isomorphism

$$M_{\varrho^{-1}} : (\varrho^{-1} H_0^{-s}(\Omega))' \leftrightarrow (H_0^{-s}(\Omega))' = H^s(\Omega).$$

This by the definition of $\varrho H^s(\Omega)$ implies the isometric isomorphism

$$I = M_\varrho M_{\varrho^{-1}} : (\varrho^{-1} H_0^{-s}(\Omega))' \leftrightarrow \varrho H^s(\Omega),$$

where I is the identity operator. Thus, (41) is proved.

Now we can complete the proof of Theorem 3 in the following way. According to Lemma 1, Condition II_s is equivalent to the boundedness of the operator (39) that by (40) is equivalent to the continuous embedding $H^{-s}(\Omega) \hookrightarrow \varrho^{-1} H_0^{-s}(\Omega)$. This embedding is dense and by (41) is equivalent to the continuous dense embedding

$$\varrho H^s(\Omega) = (\varrho^{-1} H_0^{-s}(\Omega))' \hookrightarrow (H^{-s}(\Omega))' = H^s_{\overline{\Omega}}(\mathbb{R}^n).$$

Finally, the embedding $\varrho H^s(\Omega) \hookrightarrow H^s_{\overline{\Omega}}(\mathbb{R}^n)$ is equivalent to Condition I_s. Note that the last embedding is dense because $C_0^\infty(\Omega)$ is dense in $H^s_{\overline{\Omega}}(\mathbb{R}^n)$ [34, Sec. 4.3.2, Theorem 1(b)]. Thus, Conditions II_s and I_s are equivalent for $X^s(\Omega) = \varrho H^s(\Omega)$.

Theorem 3 is proved.

5.5. **Proof of Corollary 2.** Let $s < -1/2$, and a positive function $\varrho \in C^\infty(\Omega)$ satisfy Condition II_s. The boundedness and Fredholm property of the operator (16) follow from Theorems 1 and 3 with $X^s(\Omega) := \varrho H^s(\Omega)$. In addition, since the set $\mathcal{O}(X^s(\Omega))$ is dense in $H^s_{\overline{\Omega}}(\mathbb{R}^n)$, Theorem 1(iv) implies that the index of (16) is $\dim N - \dim N^+$ and independent of s, ϱ.

Acknowledgments. The author would like to thank Yu. M. Berezansky and V. A. Mikhailets for their valuable remarks and interest in this work.

Appendix

A.1**.** In Appendix we will prove the following proposition, which gives an important example of the function satisfying Condition II_s.

Proposition A. *Let a number $s < -1/2$ and a function ϱ_1 satisfying condition* (10) *be given. Assume that $\delta \geq -s - 1/2 \in \mathbb{Z}$ or $\delta > -s - 1/2 \notin \mathbb{Z}$. Then the function $\varrho := \varrho_1^\delta$ satisfies Condition* II_s*.*

A.2. **Proof of Proposition A.** Condition (15) is fulfilled for the function $\varrho = \varrho_1^\delta$ because $\varrho_1 = 0$ on Γ, and $\delta \geq -s - 1/2$. Therefore we only need to prove that ϱ_1^δ is a multiplier in the space $H^{-s}(\Omega)$. If the positive number δ is integer, then the function ϱ_1^δ belongs to $C^\infty(\overline{\Omega})$ and therefore is a multiplier in $H^{-s}(\Omega)$. Further we assume that $\delta \notin \mathbb{Z}$. Then by the condition, $\delta > -s - 1/2$.

It is not difficult to verify that the function $\eta_\delta(t) := t^\delta$, $0 < t < 1$, belongs to $H^{-s}((0,1))$ (we will do it in the next paragraph). Hence, this function has an extension from the interval $(0,1)$ to $\mathbb{R}$ pertaining to $H^{-s}(\mathbb{R})$. Let us retain the notation η_δ for the extension. By the Strichartz theorem [37], [35, Sec. 2.2.9], every function from the space $H^{-s}(\mathbb{R})$ is a multiplier therein if $-s > 1/2$. Hence, η_δ is a multiplier in $H^{-s}(\mathbb{R})$. Then [35, Sec. 2.4, Proposition 5] the function $\eta_{\delta,n}(t', t_n) := \eta_\delta(t_n)$ of arguments $t' \in \mathbb{R}^{n-1}$, $t_n \in \mathbb{R}$ is a multiplier in $H^{-s}(\mathbb{R}^n)$. This function coincides with ϱ_1^δ in the special local coordinates (x', t_n) near the boundary Γ. Here, x' is a coordinate of a point on Γ, and t_n is the distance from Γ. It follows by [35, Sec. 6.4.1, Lemma 3] that ϱ_1^δ is a multiplier in each space $H^{-s}(\Omega \cap V_j)$, where $\{V_j : j = 1, \ldots, r\}$ is a finite collection of balls in $\mathbb{R}^n$ with a sufficiently small radius ε, and the collection covers the boundary Γ. By supplementing this collection with the set $V_0 := \{x \in \Omega : \operatorname{dist}(x, \Gamma) > \varepsilon/2\}$, we get the finite open covering of the closed domain $\overline{\Omega}$. Let certain functions $\chi_j \in C_0^\infty(V_j)$, $j = 0, 1, \ldots, r$, form the partition of unity on $\overline{\Omega}$ corresponding to this covering. Since the multiplication by a function from $C_0^\infty(V_j)$ is a bounded operator in the space $H^{-s}(\Omega \cap V_j)$, the function $\chi_j \varrho_1^\delta$ has to be a multiplier in this space and therefore in $H^{-s}(\Omega)$. Hence, $\varrho_1^\delta = \sum_{j=0}^r \chi_j \varrho_1^\delta$ is a multiplier in $H^{-s}(\Omega)$.

It remains to proof that $\eta_\delta \in H^{-s}((0,1))$. We use the inner description of the space $H^{-s}((0,1))$. If $-s \in \mathbb{Z}$, the inclusion $\eta_\delta \in H^{-s}((0,1))$ is equivalent to that $\eta_\delta \in L_2((0,1))$ and $\eta_\delta^{(-s)} \in L_2((0,1))$. The last two inclusions are fulfilled because $\delta > -s - 1/2$. Hence, $\eta_\delta \in H^{-s}((0,1))$ in the case examined. If $-s \notin \mathbb{Z}$, then by [38, p. 214, Sec. 7.48] the inclusion $\eta_\delta \in H^{-s}((0,1))$ is equivalent to that $\eta_\delta \in H^{[-s]}((0,1))$ and

$$\int_0^1 \int_0^1 \frac{|D_t^{[-s]} t^\delta - D_\tau^{[-s]} \tau^\delta|^2}{|t - \tau|^{1 + 2\{-s\}}} \, dt \, d\tau < \infty. \tag{42}$$

Here as usual, $[-s]$ and $\{-s\}$ are the integral and fractional parts of $-s$ respectively. Since $\delta > [-s] - 1/2$, we have $\eta_\delta \in H^{[-s]}((0,1))$, that was proved above. In addition, inequality (42) holds true by virtue of the following elementary lemma, which we will prove in Subsection A.3.

Lemma A. *Let $\alpha, \beta, \gamma \in \mathbb{R}$, and in addition, $\alpha \neq 0$, $\gamma > 0$. Then*

$$I(\alpha,\beta,\gamma) := \int_0^1 \int_0^1 \frac{|t^\alpha - \tau^\alpha|^\gamma}{|t-\tau|^\beta}\, dt\, d\tau < \infty \tag{43}$$

if and only if the following inequalities are fulfilled:

$$\alpha\gamma - \beta > -2, \quad \gamma - \beta > -1, \quad \alpha\gamma > -1. \tag{44}$$

Indeed, the double integral in (42) is equal to $c\, I(\alpha,\beta,\gamma)$, where c is a positive number, whereas $\alpha = \delta - [-s]$, $\beta = 1 + 2\{-s\}$, and $\gamma = 2$. Equalities (44) are fulfilled for these numbers α, β, and γ, because

$$\alpha\gamma - \beta = 2(\delta + s) - 1 > -2, \quad \gamma - \beta = 1 - 2\{-s\} > -1, \quad \alpha\gamma = 2(\delta - [-s]) > -1.$$

We have used the condition $\delta > -s - 1/2$ in the first and the third inequalities. Thus, the inclusion $\eta_\delta \in H^{-s}((0,1))$ is also valid in the case of non-integer $s < -1/2$.

Proposition A is proved.

A.3. **Proof of Lemma A.** Changing the variable $\lambda := \tau/t$ in the inner integral, we can write the following in view of evident transformations:

$$I(\alpha,\beta,\gamma) = 2\int_0^1 dt \int_0^t \frac{|t^\alpha - \tau^\alpha|^\gamma}{|t-\tau|^\beta}\, d\tau = 2\int_0^1 t^{\alpha\gamma-\beta+1} dt \int_0^1 \frac{|1-\lambda^\alpha|^\gamma}{|1-\lambda|^\beta}\, d\lambda.$$

Here, the integral in variable t is finite if and only if $\alpha\gamma - \beta > -2$, whereas the integral in τ is finite if and only if both the inequalities $\alpha\gamma > -1$ and $\gamma - \beta > -1$ hold. Hence (43) $\Leftrightarrow$ (44), which is what had to prove.

References

1. J.-L. Lions, E. Magenes, *Problèmes aux Limites non Homogènes et Applications*, Vol. 1, Dunod, Paris, 1968. (Russian translation: Mir, Moscow, 1971; English translation: Springer-Verlag, New York, 1972.)
2. Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Amer. Math. Soc., Providence, RI, 1968. (Russian edition: Naukova Dumka, Kiev, 1965)
3. J.-L. Lions, E. Magenes, *Problémes aux limites non homogénes,* V, Ann. Sci. Norm. Sup. Pisa **16** (1962), 1–44.
4. J.-L. Lions, E. Magenes, *Problémes aux limites non homogénes,* VI, J. d'Analyse Math. **11** (1963), 165–188.
5. E. Magenes, *Spazi di interpolazione ed equazioni a derivate parziali*, Atti VII Congr. Un. Mat. Italiana, Genoa, 1963, Edizioni Cremonese, Rome, 1965, pp. 134–197.
6. G. Grubb, *A characterization of non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa **22** (1968), no. 3, 425–513.
7. G. Grubb, *On coerciveness and semiboundedness of general boundary problems*, Isr. J. Math. **10** (1971), 32–95.
8. V. A. Mikhailets, *Asymptotics of the spectrum of elliptic operators and boundary conditions*, Soviet. Math. Dokl. **26** (1982), no. 5, 464–468.
9. V. A. Mikhailets, *A precise estimate of the remainder in the spectral asymptotics of general elliptic boundary problems*, Funct. Anal. Appl. **23** (1989), no. 2, 137–139.
10. V. A. Mikhailets, *To the theory of general boundary problems for elliptic equations*, Nonlinear Boundary Value Problems, vol. 2, Naukova Dumka, Kiev, 1990, pp. 66–70. (Russian)
11. M. S. Agranovich, *Elliptic boundary problems*, Encycl. Math. Sci., vol. 79, Partial Differential Equations, IX, Springer-Verlag, Berlin, 1997, pp. 1–144.
12. L. Hörmander, *On the theory of general partial differential equations*, Acta Math. **94** (1955), no. 1, 161–248. (Russian translation, Izdat. Inostran. Lit., Moscow, 1959)
13. Yu. M. Berezansky, S. G. Krein, Ya. A. Roitberg, *Theorem on homeomorphisms and local increase in smoothness of solutions of elliptic equations up to the boundary*, Soviet Math. Dokl. **4** (1963), 152–155.
14. M. Schechter, *On L_p estimates and regularity,* I, Amer. J. Math. **85** (1963), no. 1, 1–13.
15. Ya. A. Roitberg, *Elliptic problems with nonhomogeneous boundary conditions and local increase in smoothness up to the boundary for generalized solutions*, Soviet Math. Dokl. **5** (1964), 1034–1038.

16. Ya. A. Roitberg, *Homeomorphism theorems defined by elliptic operators*, Soviet Math. Dokl. **9** (1968), 656–660.
17. Ya. A. Roitberg, *Elliptic Boundary Value Problems in the Spaces of Distributions*, Mathematics and its Applications, Vol. 384, Kluwer Academic Publishers, Dordrecht, 1996. (Russian edition in four preprints, Chernigovskii Gosudarstvennyi Pedagogicheskii Institut, Chernigov, 1990, 1991)
18. Ya. A. Roitberg, *Boundary Value Problems in the Spaces of Distributions*, Mathematics and its Applications, Vol. 498, Kluwer Academic Publishers, Dordrecht, 1999.
19. Yu. V. Kostarchuk, Ya. A. Roitberg, *Isomorphism theorems for elliptic boundary value problems with boundary conditions that are not normal*, Ukrainian Math. J. **25** (1973), no. 2, 222–226.
20. Yu. M. Berezansky, Z. G. Sheftel, G. F. Us, *Functional Analysis,* Vols. 1, 2, Birkhäuser Verlag, Basel—Boston—Berlin, 1996. (Russian edition: Vyshcha Shkola, Kiev, 1990)
21. Ya. A. Roitberg, *On boundary values of generalized solutions of elliptic equations*, Math. USSR–Sb. **15** (1971), no. 2, 241–260.
22. L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963. (Russian translation: Mir, Moscow, 1965)
23. L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. 2, Springer-Verlag, Berlin, 1983. (Russian translation: Mir, Moscow, 1986)
24. V. A. Mikhailets, A. A. Murach, *Improved scales of spaces and elliptic boundary-value problems.* II, Ukrainian. Math. J. **58** (2006), no. 3, 398–417.
25. V. A. Mikhailets, A. A. Murach, *Regular elliptic boundary-value problem for homogeneous equation in two-sided refined scale of spaces*, Ukrainian Math. J. **58** (2006), no. 11, 1748–1767.
26. V. A. Mikhailets, A. A. Murach, *Elliptic operator with homogeneous regular boundary conditions in two-sided refined scale of spaces*, Ukrainian Math. Bull. **3** (2006), no. 4, 529–560.
27. A. A. Murach, *Elliptic pseudo-differential operators in a refined scale of spaces on a closed manifold*, Ukrainian Math. J. **59** (2007), no. 6, 874–893.
28. A. A. Murach, *Douglis-Nirenberg elliptic systems in the refined scale of spaces on a closed manifold*, Methods Funct. Anal. Topology **14** (2008), no. 2, 142–158.
29. V. A. Mikhailets, A. A. Murach, *Elliptic boundary-value problem in a two-sided improved scale of spaces*, Ukrainian Math. J. **60** (2008), no. 4, 574–597.
30. V. A. Mikhailets, A. A. Murach, *Elliptic systems of pseudodifferential equations in a refined scale on a closed manifold*, Bull. Pol. Acad. Sci. Math. **56** (2008), no. 3–4, 213–224.
31. A. A. Murach, *On elliptic systems in the Hörmander spaces*, Ukrain. Mat. Zh. **61** (2009), no. 3, 391–399. (Russian)
32. V. A. Mikhailets, A. A. Murach, *On elliptic operators on a closed compact manifold*, Dopov. Nats. Acad. Nauk Ukr., Mat. Pryr. Tehn. Nauky (2009), no. 3, 29–35. (Russian)
33. V. A. Mikhailets, A. A. Murach, *Elliptic problems and Hörmander spaces*, Operator Theory: Advances and Applications, vol. 191, Birkhäuser Verlag, Basel, 2009, pp. 447–470.
34. H. Triebel, *Interpolation. Function Spaces. Differential Operators*, North-Holland, Amsterdam, 1978. (Russian translation: Mir, Moscow, 1980)
35. V. G. Maz'ya, T. O. Shaposhnikova, *Theory of Multipliers in Spaces of Differentiable Functions*, Pitman, Boston, MA, 1985. (Russian edition: Izdatel'stvo Leningradskogo Universiteta, Leningrad, 1986)
36. I. C. Gohberg, M. G. Krein, *The basic propositions on defect numbers, root numbers, and indices of linear operators*, Amer. Math. Soc. Transl. Ser. 2, **13** (1960), 185–264.
37. R. S. Strichartz, *Multipliers on fractional Sobolev spaces*, J. Math. Mech. **16** (1967), no. 9, 1031–1060.
38. R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine; Chernigiv State Technological University, 95 Shevchenka, Chernigiv, 14027, Ukraine

E-mail address: murach@imath.kiev.ua

Received 29/10/2008; Revised 16/03/2009