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SPECTRAL GAPS OF THE ONE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH SINGULAR PERIODIC POTENTIALS

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To the memory of A. Ya. Povzner

Abstract. The behavior of the lengths of spectral gaps $\{\gamma_n(q)\}_{n=1}^{\infty}$ of the Hill-Schrödinger operators

$$S(q)u = -u'' + q(x)u, \qquad u \in \operatorname{Dom}(S(q)),$$

with real-valued 1-periodic distributional potentials $q(x) \in H^{-1}_{1\text{-per}}(\mathbb{R})$ is studied. We show that they exhibit the same behavior as the Fourier coefficients $\{\widehat{q}(n)\}_{n=-\infty}^{\infty}$ of the potentials $q(x)$ with respect to the weighted sequence spaces $h^{s,\varphi}$, $s > -1$, $\varphi \in \mathrm{SV}$. The case $q(x) \in L^2_{1\text{-per}}(\mathbb{R})$, $s \in \mathbb{Z}_+$, $\varphi \equiv 1$, corresponds to the Marchenko-Ostrovskii Theorem.

1. Introduction

The Hill-Schrödinger operators

$$S(q)u := -u'' + q(x)u, \quad u \in \operatorname{Dom}(S(q))$$

with real-valued 1-periodic distributional potentials $q(x) \in H^{-1}_{1\text{-per}}(\mathbb{R})$ are well defined on the Hilbert space $L^2(\mathbb{R})$ in the following *equivalent* basic ways [21]:

- as form-sum operators;
- as quasi-differential operators;
- as limits of operators with smooth 1-periodic potentials in the norm resolvent sense.

The operators $S(q)$ are lower semibounded and self-adjoint on the Hilbert space $L^2(\mathbb{R})$. Their spectra are absolutely continuous and have a band and gap structure as in the classical case of $L^2_{1\text{-per}}(\mathbb{R})$-potentials [9, 13, 4, 21].

The object of our investigation is the behavior of the lengths of spectral gaps. Under the assumption that

$$q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(k) e^{ik2\pi x} \in H^{-1+}_{1\text{-per}}(\mathbb{R},\mathbb{R}), \tag{1}$$

that is,

$$\sum_{k\in\mathbb{Z}} (1+2|k|)^{2s} |\widehat{q}(k)|^2 < \infty \quad \forall s > -1, \quad \text{and} \quad \operatorname{Im} q(x) = 0,$$

we will prove many terms asymptotic estimates for the lengths $\{\gamma_n(q)\}_{n=1}^{\infty}$ and midpoints $\{\tau_n(q)\}_{n=1}^{\infty}$ of spectral gaps of the Hill-Schrödinger operators $S(q)$ (Theorem 1). These estimates enable us to establish a relationship between the rate of *decreasing/increasing* of the lengths of the spectral gaps and the *regularity* of the singular potentials (Theorem 2 and Theorem 3).

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It is well known that if the potentials satisfy

$$q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(k) e^{ik2\pi x} \in L^2_{1\text{-per}}(\mathbb{R},\mathbb{R}), \quad \operatorname{Im} q(x) = 0,$$

i.e., if

$$\sum_{k\in\mathbb{Z}} |\widehat{q}(k)|^2 < \infty \quad \text{and} \quad \widehat{q}(k) = \overline{\widehat{q}(-k)} \quad \forall k \in \mathbb{Z},$$

then the Hill-Schrödinger operators $S(q)$ are lower semibounded and self-adjoint on the Hilbert space $L^2(\mathbb{R})$ with absolutely continuous spectra that have a zone structure [5, 28].

The spectra $\operatorname{spec}(S(q))$ are defined by the location of the endpoints $\{\lambda_0(q), \lambda_n^{\pm}(q)\}_{n=1}^{\infty}$ of the spectral gaps, and the endpoints satisfy the following inequalities:

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \le \lambda_1^+(q) < \lambda_2^-(q) \le \lambda_2^+(q) < \cdots .$$

Moreover, for even/odd numbers $n \in \mathbb{Z}_+$, the endpoints of the spectral gaps are eigenvalues of the periodic/semiperiodic problems on the interval $[0,1]$,

$$S_{\pm}(q)u := -u'' + q(x)u = \lambda u,$$

$$\operatorname{Dom}(S_{\pm}(q)) := \left\{ u \in H^2[0,1] \,\middle|\, u^{(j)}(0) = \pm\, u^{(j)}(1),\, j = 0,1 \right\} \equiv H^2_{\pm}[0,1].$$

The spectral bands (stability or tied zones),

$$\mathcal{B}_0(q) := [\lambda_0(q), \lambda_1^-(q)], \qquad \mathcal{B}_n(q) := [\lambda_n^+(q), \lambda_{n+1}^-(q)], \quad n \in \mathbb{N},$$

are characterized as a locus of those real $\lambda \in \mathbb{R}$ for which all solutions of the equation $S(q)u = \lambda u$ are bounded. On the other hand, the spectral gaps (instability or forbidden zones),

$$\mathcal{G}_0(q) := (-\infty, \lambda_0(q)), \qquad \mathcal{G}_n(q) := (\lambda_n^-(q), \lambda_n^+(q)), \quad n \in \mathbb{N},$$

are a locus of those real $\lambda \in \mathbb{R}$ for which any nontrivial solution of the equation $S(q)u = \lambda u$ is unbounded.

Due to Marchenko and Ostrovskii [14], the endpoints of spectral gaps of the Hill-Schrödinger operators $S(q)$ satisfy the asymptotic estimates

$$\lambda_n^{\pm}(q) = n^2\pi^2 + \widehat{q}(0) \pm |\widehat{q}(n)| + h^1(n), \quad n \to \infty. \tag{2}$$

As a consequence, for the lengths of spectral gaps,

$$\gamma_n(q) := \lambda_n^+ - \lambda_n^-, \quad n \in \mathbb{N},$$

the following asymptotic formulae hold:

$$\gamma_n(q) = 2\,|\widehat{q}(n)| + h^1(n), \quad n \to \infty. \tag{3}$$

Hochstadt [10] ($\Rightarrow$) and Marchenko, Ostrovskii [14], McKean, Trubowitz [15] ($\Leftarrow$) proved that the potential $q(x)$ is an infinitely differentiable function if and only if the lengths of spectral gaps $\{\gamma_n(q)\}_{n=1}^{\infty}$ decrease faster than an arbitrary power of $1/n$,

$$q(x) \in C^{\infty}_{1\text{-per}}(\mathbb{R},\mathbb{R}) \Leftrightarrow \gamma_n(q) = O(n^{-k}), \quad n \to \infty \quad \forall k \in \mathbb{Z}_+.$$

Marchenko and Ostrovskii [14] discovered that

$$q(x) \in H^k_{1\text{-per}}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^k, \quad k \in \mathbb{Z}_+. \tag{4}$$

The relationship (4) was extended by Kappeler, Mityagin [11] ($\Rightarrow$) and Djakov, Mityagin [2] ($\Leftarrow$) (see also the survey [3] and the references therein) to the case of symmetric, monotone, submultiplicative and subexponential weights $\Omega = \{\Omega(n)\}_{n\in\mathbb{Z}}$,

$$q(x) \in H^{\Omega}_{1\text{-per}}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{\Omega}.$$

Pöschel [27] proved the latter statement in a quite different way.

Here and throughout the rest of the paper we use the complex Hilbert spaces $H^{w}_{1\text{-per}}(\mathbb{R})$ (as well as $H^{w}_{\pm}[0,1]$) of 1-periodic functions and distributions defined by means of their Fourier coefficients

$$f(x)=\sum_{k\in\mathbb{Z}}\widehat{f}(k)e^{ik2\pi x}\in H^{w}_{1\text{-per}}(\mathbb{R})\Leftrightarrow\left\{\widehat{f}(k)\right\}_{k\in\mathbb{Z}}\in h^{w},$$

$$h^{w}=\left\{a=\{a(k)\}_{k\in\mathbb{Z}}\ \middle|\ \|a\|_{h^{w}}=\left(\sum_{k\in\mathbb{Z}}w^{2}(k)|a(k)|^{2}\right)^{1/2}<\infty\right\}.$$

Basically we use the power weights

$$w_{s}=\{w_{s}(k)\}_{k\in\mathbb{Z}}:\quad w_{s}(k):=(1+2|k|)^{s},\quad s\in\mathbb{R}.$$

The corresponding spaces we denote by

$$H^{w_{s}}_{1\text{-per}}(\mathbb{R})\equiv H^{s}_{1\text{-per}}(\mathbb{R}),\quad H^{w_{s}}_{\pm}[0,1]\equiv H^{s}_{\pm}[0,1],\quad\text{and}\quad h^{w_{s}}\equiv h^{s},\quad s\in\mathbb{R}.$$

For more details, see Appendix.

2. Main results

As we have already remarked, if assumption (1) is satisfied, the Hill-Schrödinger operators $S(q)$ are lower semibounded and self-adjoint on the Hilbert space $L^{2}(\mathbb{R})$. Their spectra are absolutely continuous and have a classical zone structure [9, 13, 4, 21, 23].

Using the results of the papers [12, 26], the Isospectral Theorem 5, and [21, Theorem C] we obtain uniform many terms asymptotic estimates for the lengths of spectral gaps, $\{\gamma_{n}(q)\}_{n=1}^{\infty}$, and for their midpoints $\{\tau_{n}(q)\}_{n=1}^{\infty}$,

$$\tau_{n}(q):=\frac{\lambda_{n}^{+}(q)+\lambda_{n}^{-}(q)}{2},\quad n\in\mathbb{N}.$$

Theorem 1. ([18, 25]). *Let $q(x)\in H^{-\alpha}_{1\text{-per}}(\mathbb{R},\mathbb{R})$, $\alpha\in[0,1)$. Then for any $\varepsilon>0$, uniformly on bounded sets of distributions $q(x)$ in the corresponding Sobolev spaces $H^{-\alpha}_{1\text{-per}}(\mathbb{R})$, the lengths $\{\gamma_{n}(q)\}_{n=1}^{\infty}$ and the midpoints $\{\tau_{n}(q)\}_{n=1}^{\infty}$ of spectral gaps of the Hill-Schrödinger operators $S(q)$ for $n\geq n_{0}\left(\|q\|_{H^{-\alpha}_{1\text{-per}}(\mathbb{R})}\right)$ satisfy the following asymptotic formulae:*

$$\gamma_{n}(q)=2\,|\widehat{q}(n)|+h^{1-2\alpha-\varepsilon}(n),\tag{5}$$

$$\tau_{n}(q)=n^{2}\pi^{2}+\widehat{q}(0)+h^{1-2\alpha-\varepsilon}(n).\tag{6}$$

Corollary. ([18, 25]). *Let $q(x)\in H^{-\alpha}_{1\text{-per}}(\mathbb{R},\mathbb{R})$ with $\alpha\in[0,1)$. Then for any $\varepsilon>0$, uniformly in $q(x)$, for the endpoints of spectral gaps of the Hill-Schrödinger operators $S(q)$ the following asymptotic estimates hold:*

$$\lambda_{n}^{\pm}(q)=n^{2}\pi^{2}+\widehat{q}(0)\pm|\widehat{q}(n)|+h^{1-2\alpha-\varepsilon}(n).$$

Now, we can describe a two-way relationship between the rate of decreasing/increasing of the lengths of spectral gaps, $\{\gamma_{n}(q)\}_{n=1}^{\infty}$, and regularity of the potentials $q(x)$ in a refined scale.

Let

$$w_{s,\varphi}=\{w_{s,\varphi}(k)\}_{k\in\mathbb{Z}}:\quad w_{s,\varphi}(k):=(1+2|k|)^{s}\,\varphi(|k|),\quad s\in\mathbb{R},\quad\varphi\in\mathrm{SV},$$

where φ is a function slowly varying at $+\infty$ in the sense of Karamata [30]. This means that it is a function that is positive, measurable on $[a,\infty)$, $a>0$, and obeys the condition

$$\lim_{t\to+\infty}\frac{\varphi(\lambda t)}{\varphi(t)}=1,\quad\lambda>0.$$

For example,

$$\varphi(t) = (\log t)^{r_1} (\log\log t)^{r_2} \dots (\log \dots \log t)^{r_k} \in \mathrm{SV}, \quad \{r_1, \dots, r_k\} \subset \mathbb{R}, \quad k \in \mathbb{N}.$$

The Hörmander spaces

$$H^{w_{s,\varphi}}_{1\text{-per}}(\mathbb{R}) \equiv H^{s,\varphi}_{1\text{-per}}(\mathbb{R}) \simeq H^{s,\varphi}(\mathbb{S}), \quad \mathbb{S} := \mathbb{R}/2\pi\mathbb{Z},$$

and the weighted sequence spaces

$$h^{w_{s,\varphi}} \equiv h^{s,\varphi}$$

form the refined scales:

$$H^{s+\varepsilon}_{1\text{-per}}(\mathbb{R}) \hookrightarrow H^{s,\varphi}_{1\text{-per}}(\mathbb{R}) \hookrightarrow H^{s-\varepsilon}_{1\text{-per}}(\mathbb{R}), \tag{7}$$

$$h^{s+\varepsilon} \hookrightarrow h^{s,\varphi} \hookrightarrow h^{s-\varepsilon}, \qquad s \in \mathbb{R}, \quad \varepsilon > 0, \quad \varphi \in \mathrm{SV}, \tag{8}$$

which, in a general situation, were studied by Mikhailets and Murach [22].

The following statements show that the sequence $\{\gamma_n(q)\}_{n=1}^{\infty}$ has the same behavior as the Fourier coefficients $\{\widehat{q}(n)\}_{n=-\infty}^{\infty}$ with respect to the refined scale $\{h^{s,\varphi}\}_{s\in\mathbb{R},\varphi\in\mathrm{SV}}$.

Theorem 2. *Let $q(x) \in H^{-1+}_{1\text{-per}}(\mathbb{R},\mathbb{R})$. Then*

$$q(x) \in H^{s,\varphi}_{1\text{-per}}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi}, \quad s \in (-1,0], \quad \varphi \in \mathrm{SV}.$$

Note that the Hörmander spaces $H^{s,\varphi}_{1\text{-per}}(\mathbb{R})$ with $\varphi \equiv 1$ coincide with the Sobolev spaces,

$$H^{s,1}_{1\text{-per}}(\mathbb{R}) \equiv H^{s}_{1\text{-per}}(\mathbb{R}), \quad \text{and} \quad h^{s,1} \equiv h^s, \quad s \in \mathbb{R}.$$

Corollary. ([18, 25]). *Let $q(x) \in H^{-1+}_{1\text{-per}}(\mathbb{R},\mathbb{R})$, then*

$$q(x) \in H^{s}_{1\text{-per}}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s}, \quad s \in (-1,0]. \tag{9}$$

Theorem 2, together with [11, Theorem 1.2], and properties (7) and (8), gives the following extension of the Marchenko-Ostrovskii Theorem (4).

Theorem 3. *Let $q(x) \in H^{-1+}_{1\text{-per}}(\mathbb{R},\mathbb{R})$. Then*

$$q(x) \in H^{s,\varphi}_{1\text{-per}}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi}, \quad s \in (-1,\infty), \quad \varphi \in \mathrm{SV}.$$

In particular,

$$q(x) \in H^{s}_{1\text{-per}}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s}, \quad s \in (-1,\infty).$$

Remark. In the preprint [4], the authors have announced, without a proof, a more general statement,

$$q(x) \in H^{\widehat{\Omega}}_{1\text{-per}}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{\widehat{\Omega}}, \quad \widehat{\Omega} = \left\{ \frac{\Omega(n)}{1+2|n|} \right\}_{n\in\mathbb{Z}},$$

where the weights $\Omega = \{\Omega(n)\}_{n\in\mathbb{Z}}$ are supposed to be symmetric, monotone, submultiplicative and subexponential. This result contains the limiting case

$$q(x) \in H^{-1}_{1\text{-per}}(\mathbb{R},\mathbb{R}) \setminus H^{-1+}_{1\text{-per}}(\mathbb{R},\mathbb{R}).$$

The implication

$$q(x) \in H^{-1}_{1\text{-per}}(\mathbb{R},\mathbb{R}) \Rightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{-1}$$

was proved in the paper [13].

3. Proofs

Spectra of the Hill-Schrödinger operators $S(q)$, $q(x) \in H^{-1}_{1\text{-per}}(\mathbb{R}, \mathbb{R})$ are defined by the endpoints $\{\lambda_0(q), \lambda_n^{\pm}(q)\}_{n=1}^{\infty}$ of spectral gaps. The endpoints as in the case of $L^2_{1\text{-per}}(\mathbb{R})$-potentials satisfy the inequalities

$$-\infty < \lambda_0(q) < \lambda_1^-(q) \le \lambda_1^+(q) < \lambda_2^-(q) \le \lambda_2^+(q) < \cdots .$$

For even/odd numbers $n \in \mathbb{Z}_+$ they are eigenvalues of the periodic/semiperiodic problems on the interval $[0,1]$ [21, Theorem C],

$$S_{\pm}(q)u = \lambda u.$$

The operators

$$S_{\pm}u \equiv S_{\pm}(q)u := D^2_{\pm}u + q(x)u,$$

- $D^2_{\pm} := -d^2/dx^2$, $\operatorname{Dom}(D^2_{\pm}) = H^2_{\pm}[0,1]$;
- $q(x) = \sum_{k\in\mathbb{Z}} \widehat{q}(k)\, e^{i\,k2\pi x} \in H^{-1}_{+}([0,1],\mathbb{R})$;
- $\operatorname{Dom}(S_{\pm}(q)) = \left\{ u \in H^1_{\pm}[0,1] \;\middle|\; D^2_{\pm}u + q(x)u \in L^2(0,1) \right\}$,

are well defined on the Hilbert space $L^2(0,1)$ as lower semibounded, self-adjoint form-sum operators, and they have the pure discrete spectra

$$\operatorname{spec}(S_{\pm}(q)) = \left\{ \lambda_0[S_+(q)],\ \lambda^{\pm}_{2n-1}[S_-(q)],\ \lambda^{\pm}_{2n}[S_+(q)] \right\}_{n=1}^{\infty}.$$

In the papers [18, 25, 19, 20] the authors meticulously investigated the more general periodic/semiperiodic form-sum operators

$$S_{m,\pm}(V) := D^{2m}_{\pm} \dotplus V(x), \quad V(x) \in H^{-m}_{+}[0,1], \quad m \in \mathbb{N},$$

on the Hilbert space $L^2(0,1)$.

So, we need to find precise asymptotic estimates for eigenvalues of the operators $S_{\pm}(q)$. It is quite a difficult problem for the form-sum operators $S_{\pm}(q)$ are not convenient for studying. We also cannot apply the approach developed by Savchuk and Shkalikov (see the survey [29] and the references therein) considering the operators $S_{\pm}(q)$ as quasi-differential, since the periodic/semiperiodic boundary conditions are not strongly regular in the sense of Birkhoff. Therefore, we propose an alternative approach which is based on isospectral transformation of the problem.

Kappeler and Möhr [12, 26] investigated the second order differential operators $L_{\pm}(q)$, $q(x) \in H^{-1}_{+}([0,1],\mathbb{R})$ (in general, with complex-valued potentials) defined on the *negative* Sobolev spaces $H^{-1}_{\pm}[0,1]$,

$$L_{\pm} \equiv L_{\pm}(q) := D^2_{\pm} + q(x), \quad \operatorname{Dom}(L_{\pm}(q)) = H^1_{\pm}[0,1].$$

They established that the operators $L_{\pm}(q)$ with $q(x) \in H^{-\alpha}_{+}([0,1],\mathbb{R})$, $\alpha \in [0,1)$, have the real-valued discrete spectra

$$\operatorname{spec}(L_{\pm}(q)) = \left\{ \lambda_0[L_+(q)],\ \lambda^{\pm}_{2n-1}[L_-(q)],\ \lambda^{\pm}_{2n}[L_+(q)] \right\}_{n=1}^{\infty}$$

such that

$$\left|\lambda^{\pm}_{n}[L_{\pm}(q)] - n^2\pi^2 - \widehat{q}(0)\right| \le C n^{\alpha}, \quad n \ge n_0\left(\|q\|_{H^{-\alpha}_{+}[0,1]}\right).$$

More precisely, for the values

$$\gamma_n[L_{\pm}(q)] := \lambda^{+}_{n}[L_{\pm}(q)] - \lambda^{-}_{n}[L_{\pm}(q)], \quad n \in \mathbb{N},$$

$$\tau_n[L_{\pm}(q)] := \frac{\lambda^{+}_{n}[L_{\pm}(q)] + \lambda^{-}_{n}[L_{\pm}(q)]}{2}, \quad n \in \mathbb{N},$$

they proved the following result.

Proposition 4. (Kappeler, Möhr [12, 26]). *Let $q(x) \in H_+^{-\alpha}([0,1],\mathbb{R})$, and $\alpha \in [0,1)$. Then for any $\varepsilon > 0$, uniformly on bounded sets of distributions $q(x)$ in the Sobolev spaces $H_+^{-\alpha}[0,1]$, the values $\{\gamma_n[L_\pm(q)]\}_{n=1}^\infty$ and $\{\tau_n[L_\pm(q)]\}_{n=1}^\infty$, $n \geq n_0\left(\|q\|_{H_+^{-\alpha}[0,1]}\right)$, for the operators $L_\pm(q)$ satisfy the following asymptotic estimates:*

i) $\left\{\min_{\pm}\left|\gamma_n[L_\pm(q)] \pm 2\sqrt{(\widehat{q}+\omega)(-n)(\widehat{q}+\omega)(n)}\right|\right\}_{n\in\mathbb{N}} \in h^{1-2\alpha-\varepsilon}$,

ii) $\tau_n[L_\pm(q)] = n^2\pi^2 + \widehat{q}(0) + h^{1-2\alpha-\varepsilon}(n)$,

where

$$\{\omega(n)\}_{n\in\mathbb{Z}} \equiv \left\{\frac{1}{\pi^2}\sum_{k\in\mathbb{Z}\setminus\{\pm n\}}\frac{\widehat{q}(n-k)\widehat{q}(n+k)}{n^2-k^2}\right\}_{n\in\mathbb{Z}} \in \begin{cases} h^{1-\alpha}, & \alpha\in[0,1/2),\\ h^{3/2-2\alpha-\delta}, & \alpha\in[1/2,1)\end{cases}$$

with any $\delta > 0$ (see the Convolution Lemma [12, 26]).

Remark. In the papers [24, 16, 17, 25], the more general operators

$$L_{m,\pm}(V) := D_\pm^{2m} + V(x), \quad V(x) \in H_+^{-m}[0,1], \quad m \in \mathbb{N},$$

on the spaces $H_\pm^{-m}[0,1]$ were studied. In particular, an analogue of Proposition 4 was proved.

The following statement is an essential point of our approach.

Theorem 5. (Isospectral Theorem [18, 25]). *The operators $S_\pm(q)$ and $L_\pm(q)$ are isospectral,*

$$\operatorname{spec}(S_\pm(q)) = \operatorname{spec}(L_\pm(q)).$$

Proof. The inclusions

$$\operatorname{spec}(S_\pm(q)) \subset \operatorname{spec}(L_\pm(q))$$

are obvious, since

$$S_\pm(q) \subset L_\pm(q).$$

Let us prove the inverse inclusions,

$$\operatorname{spec}(L_\pm(q)) \subset \operatorname{spec}(S_\pm(q)).$$

Let $\lambda \in \operatorname{spec}(L_\pm(q))$, and f be a correspondent eigenvector or a rootvector. Therefore

$$(L_\pm(q) - \lambda Id)\, f = g, \quad f, g \in \operatorname{Dom}(L_\pm(q)) = H_\pm^1[0,1],$$

where f is an eigenfunction if $g = 0$, and a rootvector if $g \neq 0$.

So, we get

$$L_\pm(q)f = \lambda Idf + g \in H_\pm^1[0,1],$$

i.e.,

$$L_\pm(q)f = D_\pm^2 f + q(x)f \in L^2(0,1).$$

Thus we have proved that $f \in \operatorname{Dom}(S_\pm(q))$. In the case when f is a rootvector ($g \neq 0$) in a similar fashion we show that $g \in \operatorname{Dom}(S_\pm(q))$, too. Continuing this process as necessary (note that it is finite, since the eigenvalue λ has finite algebraic multiplicity) we obtain that all eigenvectors and rootvectors corresponding to λ belong to the domains $\operatorname{Dom}(S_\pm(q))$ of the operators $S_\pm(q)$. Consequently, we can conclude that

$$\lambda \in \operatorname{spec}(S_\pm(q)),$$

hence we obtain the needed inclusions,

$$\operatorname{spec}(L_\pm(q)) \subset \operatorname{spec}(S_\pm(q)).$$

The proof is complete. □

Now, Theorem 1 follows from Proposition 4, the Isospectral Theorem 5, and [21, Theorem C], since

$$\widehat{q}(n) = \overline{\widehat{q}(-n)}, \quad n \in \mathbb{Z},$$
$$\omega(n) = \overline{\omega(-n)}, \quad n \in \mathbb{Z},$$

and, as a consequence,

$$\min_{\pm} \left| \gamma_n(q) \pm 2\sqrt{(\widehat{q}+\omega)(-n)(\widehat{q}+\omega)(n)} \right| = \left| \gamma_n(q) - 2\left| (\widehat{q}+\omega)(n) \right| \right|.$$

The proof of Theorem 1 is complete.

To prove Theorem 2 we firstly prove its Corollary. The formula (9) follows from [12, Corollary 0.2 (2.6)], the Isospectral Theorem 5 and [21, Theorem C]. Also it can be proved directly as well similarly to [12, Corollary 0.2 (2.6)] using estimates (5).

Now, to prove Theorem 2 it is sufficient to apply the asymptotic estimates (5), properties (7) and (8) of the refined scales, and formula (9),

- $$q \in H^{s,\varphi}_{\text{1-per}}(\mathbb{R},\mathbb{R}) \overset{(7)}{\Longrightarrow} q \in H^{s-\delta}_{\text{1-per}}(\mathbb{R},\mathbb{R}),\ \delta > 0 \overset{(5)}{\Longrightarrow} \gamma_n = 2\left|\widehat{q}(n)\right| + h^{1+2(s-\delta)-\varepsilon}(n) \overset{(8)}{\Longrightarrow} \gamma_n = 2\left|\widehat{q}(n)\right| + h^{s,\varphi}(n) \Longrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi};$$
- $$\{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi} \overset{(8)}{\Longrightarrow} \{\gamma_n\}_{n=1}^{\infty} \in h^{s-\delta},\ \delta > 0 \overset{(9)}{\Longrightarrow} q \in H^{s-\delta}_{\text{1-per}}(\mathbb{R},\mathbb{R}) \overset{(5)}{\Longrightarrow} \gamma_n = 2\left|\widehat{q}(n)\right| + h^{1+2(s-\delta)-\varepsilon}(n) \overset{(8)}{\Longrightarrow} \gamma_n = 2\left|\widehat{q}(n)\right| + h^{s,\varphi}(n) \Longrightarrow \{\widehat{q}(n)\}_{n\in\mathbb{Z}} \in h^{s,\varphi}(n).$$

Note that, since $\delta > 0$ and $\varepsilon > 0$ were chosen arbitrarily, we can take them to be such that

$$1 + s - 2\delta - \varepsilon > 0.$$

The proof of Theorem 2 is complete.

Now, we are ready to prove Theorem 3.

At first, note that from [11, Theorem 1.2] we get the following asymptotic formulae for the lengths of spectral gaps:

$$\gamma_n(q) = 2\left|\widehat{q}(n)\right| + h^{1+s}(n) \quad \text{for} \quad q(x) \in H^{s}_{1\text{-}per}(\mathbb{R},\mathbb{R}),\ s \in [0,\infty), \tag{10}$$

which, for integer numbers $s \in \mathbb{Z}_+$, were proved by Marchenko and Ostrovskii [14].

Using (9), (10) and (4) it is easy to prove that

$$q(x) \in H^{s}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s}, \quad s \in (-1,\infty). \tag{11}$$

Sufficiency in Theorem 3. Let $q(x) \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R})$. If $s \in (-1,0]$, then using Theorem 2 we obtain that $\{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi}$. If $s > 0$, then

$$q(x) \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \overset{(7)}{\hookrightarrow} H^{s-\delta}_{1\text{-}per}(\mathbb{R},\mathbb{R}),\ \delta > 0 \overset{(10)}{\Longrightarrow} \gamma_n(q) = 2\left|\widehat{q}(n)\right| + h^{1+s-\delta}(n) \overset{(8)}{\Longrightarrow} \gamma_n(q) = 2\left|\widehat{q}(n)\right| + h^{s,\varphi}(n) \Longrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi}.$$

Sufficiency is proved.

Necessity in Theorem 3. Let us assume that $\{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi}$. If $s \in (-1,0]$ then from Theorem 2 it follows that $q(x) \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R})$. If $s > 0$, then

$$\{\gamma_n(q)\}_{n=1}^{\infty} \in h^{s,\varphi} \overset{(8)}{\hookrightarrow} h^{s-\delta},\ \delta > 0 \overset{(11)}{\Longrightarrow} q(x) \in H^{s-\delta}_{1\text{-}per}(\mathbb{R},\mathbb{R}) \overset{(10)}{\Longrightarrow} \gamma_n(q) = 2\left|\widehat{q}(n)\right| + h^{1+s-\delta}(n) \overset{(8)}{\Longrightarrow} \gamma_n(q) = 2\left|\widehat{q}(n)\right| + h^{s,\varphi}(n) \Longrightarrow q(x) \in H^{s,\varphi}_{1\text{-}per}(\mathbb{R},\mathbb{R}).$$

Necessity is proved.

The proof of Theorem 3 is complete.

4. Concluding remarks

In fact, we can prove the following result: if $q(x) \in H^{-1+}_{1\text{-per}}(\mathbb{R},\mathbb{R})$ and

$$(1+2|k|)^s \ll w(k) \ll (1+2|k|)^{1+2s}, \quad s \in (-1,0],$$
$$(1+2|k|)^s \ll w(k) \ll (1+2|k|)^{1+s}, \quad s \in [0,\infty),$$

then

$$q(x) \in H^{w}_{1\text{-per}}(\mathbb{R},\mathbb{R}) \Leftrightarrow \{\gamma_n(q)\}_{n=1}^{\infty} \in h^w.$$

This result is not covered by the theorems in the preprint [4], because it does not require the weight function to be monotone and submultiplicative.

Appendix

The complex Sobolev spaces $H^s_{1\text{-per}}(\mathbb{R})$, $s \in \mathbb{R}$, of 1-periodic functions and distributions on the real axis $\mathbb{R}$ are defined by means of their Fourier coefficients,

$$H^s_{1\text{-per}}(\mathbb{R}) := \left\{ f = \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{ik2\pi x} \,\Big|\, \| f \|_{H^s_{1\text{-per}}(\mathbb{R})} < \infty \right\},$$
$$\| f \|_{H^s_{1\text{-per}}(\mathbb{R})} := \left(\sum_{k\in\mathbb{Z}} \langle 2k\rangle^{2s} |\widehat{f}(k)|^2 \right)^{1/2}, \quad \langle k\rangle := 1+|k|,$$
$$\widehat{f}(k) := \langle f, e^{ik2\pi x}\rangle_{L^2_{1\text{-per}}(\mathbb{R})}, \quad k \in \mathbb{Z}.$$

By $\langle\cdot,\cdot\rangle_{L^2_{1\text{-per}}(\mathbb{R})}$ we denote the sesqui-linear form that gives the pairing between the dual spaces $H^s_{1\text{-per}}(\mathbb{R})$ and $H^{-s}_{1\text{-per}}(\mathbb{R})$ with respect to $L^2_{1\text{-per}}(\mathbb{R})$, and which is an extension by continuity of the $L^2_{1\text{-per}}(\mathbb{R})$-inner product [1, 8],

$$\langle f, g\rangle_{L^2_{1\text{-per}}(\mathbb{R})} := \int_0^1 f(x)\overline{g(x)}\, dx = \sum_{k\in\mathbb{Z}} \widehat{f}(k)\overline{\widehat{g}(k)} \quad \forall f,g \in L^2_{1\text{-per}}(\mathbb{R}).$$

It is useful to notice that

$$H^0_{1\text{-per}}(\mathbb{R}) \equiv L^2_{1\text{-per}}(\mathbb{R}).$$

By $H^{s+}_{1\text{-per}}(\mathbb{R})$, $s \in \mathbb{R}$, we denote the inductive limit of the Sobolev spaces $H^t_{1\text{-per}}(\mathbb{R})$ with $t > s$,

$$H^{s+}_{1\text{-per}}(\mathbb{R}) := \bigcup_{\varepsilon>0} H^{s+\varepsilon}_{1\text{-per}}(\mathbb{R}).$$

It is a topological space with the inductive topology.

In a similar fashion the Sobolev spaces $H^s_{\pm}[0,1]$, $s \in \mathbb{R}$, of 1-periodic (1-semiperiodic) functions and distributions over the interval $[0,1]$ are defined by

$$H^s_{\pm}[0,1] := \left\{ f = \sum_{k\in\Gamma_\pm} \widehat{f}\left(\frac{k}{2}\right) e^{ik\pi x} \,\Big|\, \| f \|_{H^s_{\pm}[0,1]} < \infty \right\},$$
$$\| f \|_{H^s_{\pm}[0,1]} := \left(\sum_{k\in\Gamma_\pm} \langle k\rangle^{2s} \left|\widehat{f}\left(\frac{k}{2}\right)\right|^2 \right)^{1/2}, \quad \langle k\rangle = 1+|k|,$$
$$\widehat{f}\left(\frac{k}{2}\right) := \langle f(x), e^{ik\pi x}\rangle_{\pm}, \quad k \in \Gamma_\pm.$$

Here

$$\Gamma_+ \equiv 2\mathbb{Z} := \{k \in \mathbb{Z} \mid k \equiv 0 \pmod 2\},$$
$$\Gamma_- \equiv 2\mathbb{Z}+1 := \{k \in \mathbb{Z} \mid k \equiv 1 \pmod 2\},$$

and $\langle\cdot,\cdot\rangle_\pm$ are sesqui-linear forms that define the pairing between the dual spaces $H^s_\pm[0,1]$ and $H^{-s}_\pm[0,1]$ with respect to $L^2(0,1)$; the sesqui-linear forms $\langle\cdot,\cdot\rangle_\pm$ are extensions by continuity of the $L^2(0,1)$-inner product [1, 8],

$$\langle f,g\rangle_\pm := \int_0^1 f(x)\overline{g(x)}\,dx = \sum_{k\in\Gamma_\pm} \widehat{f}\left(\frac{k}{2}\right)\overline{\widehat{g}\left(\frac{k}{2}\right)} \quad \forall f,g \in L^2(0,1).$$

It is obvious that

$$H^0_+[0,1] \equiv H^0_-[0,1] \equiv L^2(0,1).$$

We say that a 1-periodic function or a distribution $f(x)$ is *real-valued* if $\operatorname{Im} f(x) = 0$. Let us recall that

$$\operatorname{Re} f(x) := \frac{1}{2}(f(x)+\overline{f(x)}), \quad \operatorname{Im} f(x) := \frac{1}{2i}(f(x)-\overline{f(x)}),$$

(see, for an example, [31]). In terms of the Fourier coefficients, we have

$$\operatorname{Im} f(x) = 0 \Leftrightarrow \widehat{f}(k) = \overline{\widehat{f}(-k)}, \quad k \in \mathbb{Z}.$$

Set

$$H^s_{1\text{-per}}(\mathbb{R},\mathbb{R}) := \left\{f(x) \in H^s_{1\text{-per}}(\mathbb{R}) \mid \operatorname{Im} f(x) = 0\right\},$$
$$H^{s+}_{1\text{-per}}(\mathbb{R},\mathbb{R}) := \left\{f(x) \in H^{s+}_{1\text{-per}}(\mathbb{R}) \mid \operatorname{Im} f(x) = 0\right\},$$
$$H^s_\pm([0,1],\mathbb{R}) := \left\{f(x) \in H^s_\pm[0,1] \mid \operatorname{Im} f(x) = 0\right\}.$$

Also we will need the Hilbert spaces

$$h^s \equiv h^s(\mathbb{Z},\mathbb{C}), \quad s \in \mathbb{R},$$

of (two-sided) weighted sequences,

$$h^s := \left\{a = \{a(k)\}_{k\in\mathbb{Z}} \,\middle|\, \|a\|_{h^s} := \left(\sum_{k\in\mathbb{Z}} \langle k\rangle^{2s}|a(k)|^2\right)^{1/2} < \infty\right\}, \quad \langle k\rangle = 1+|k|.$$

Note that

$$h^0 \equiv l^2(\mathbb{Z},\mathbb{C}),$$

and

$$a = \{a(k)\}_{k\in\mathbb{Z}} \in h^s, \quad s \in \mathbb{R}, \qquad \Rightarrow \qquad a(k) = o(|k|^{-s}), \quad k \to \pm\infty.$$

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