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SCHRÖDINGER OPERATORS WITH PURELY DISCRETE SPECTRUM

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Dedicated to A. Ya. Povzner

Abstract. We prove that $-\Delta + V$ has purely discrete spectrum if $V \geq 0$ and, for all M, $|\{x \mid V(x) < M\}| < \infty$ and various extensions.

1. Introduction

Our main goal in this note is to explore one aspect of the study of Schrödinger operators

$$H = -\Delta + V \tag{1.1}$$

which we will suppose have V's which are nonnegative and in $L^1_{\text{loc}}(\mathbb{R}^\nu)$, in which case (see, e.g., Simon [15]) H can be defined as a form sum. We are interested here in criteria under which H has purely discrete spectrum, that is, $\sigma_{\text{ess}}(H)$ is empty. This is well known to be equivalent to proving $(H+1)^{-1}$ or e^{-sH} for any (and so all) $s > 0$ is compact (see [9, Thm. XIII.16]). One of the most celebrated elementary results on Schrödinger operators is that this is true if

$$\lim_{|x|\to\infty} V(x) = \infty. \tag{1.2}$$

But (1.2) is not necessary. Simple examples where (1.2) fails but H still has compact resolvent were noted first by Rellich [10]—one of the most celebrated examples is in $\nu = 2$, $x = (x_1, x_2)$, and

$$V(x_1, x_2) = x_1^2 x_2^2 \tag{1.3}$$

where (1.2) fails in a neighborhood of the axes. For proof of this and discussions of eigenvalue asymptotics, see [11, 16, 17, 20, 21].

There are known necessary and sufficient conditions on V for discrete spectrum in terms of capacities of certain sets (see, e.g., Maz'ya [6]), but the criteria are not always so easy to check. Thus, I was struck by the following simple and elegant theorem:

Theorem 1. *Define*

$$\Omega_M(V) = \{x \mid 0 \leq V(x) < M\}. \tag{1.4}$$

If (with $|\cdot|$ Lebesgue measure)

$$|\Omega_M(V)| < \infty \tag{1.5}$$

for all M, then H has purely discrete spectrum.

I learned of this result from Wang–Wu [25], but there is much related work. I found an elementary proof of Theorem 1 and decided to write it up as a suitable tribute and

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appreciation of A. Ya. Povzner, whose work on continuum eigenfunction expansions for Schrödinger operators in scattering situation [7] was seminal and inspired me as a graduate student forty years ago!

The proof has a natural abstraction:

Theorem 2. *Let μ be a measure on a locally compact space, X with $L^2(X, d\mu)$ separable. Let L_0 be a selfadjoint operator on $L^2(X, d\mu)$ so that its semigroup is ultracontractive* ([1])*: For some $s > 0$, e^{-sL_0} maps L^2 to $L^\infty(X, d\mu)$. Suppose V is a nonnegative multiplication operator so that*

$$\mu(\{x \mid 0 \leq V(x) < M\}) < \infty \tag{1.6}$$

for all M. Then $L = L_0 + V$ has purely discrete spectrum.

Remark. By $L_0 + V$, we mean the operator obtained by applying the monotone convergence theorem for forms (see, e.g., [13, 14]) to $L_0 + \min(V(x), k)$ as $k \to \infty$.

The reader may have noticed that (1.3) does not obey Theorem 1 (but, e.g.,

$$V(x_1, x_2) = x_1^2 x_2^4 + x_1^4 x_2^2$$

does). But out proof can be modified to a result that does include (1.3). Given a set Ω in $\mathbb{R}^\nu$, define for any x and any $\ell > 0$,

$$\omega_x^\ell(\Omega) = |\Omega \cap \{y \mid |y - x| \leq \ell\}|. \tag{1.7}$$

For example, for (1.3), for $x \in \Omega_M$,

$$\omega_x^\ell(\Omega_M) \leq \frac{C_\ell}{|x| + 1}. \tag{1.8}$$

We will say a set Ω is r-polynomially thin if

$$\int_{x \in \Omega} \omega_x^\ell(\Omega)^r \, d^\nu x < \infty \tag{1.9}$$

for all ℓ. For the example in (1.3), Ω_M is r-polynomially thin for any M and any $r > 0$. We'll prove

Theorem 3. *Let V be a nonnegative potential so that for any M, there is an $r > 0$ so that Ω_M is r-polynomially thin. Then H has purely discrete spectrum.*

As mentioned, this covers the example in (1.3). It is not hard to see that if $P(x)$ is any polynomial in $x_1, \dots, x_\nu$ so that for no $v \in \mathbb{R}^\nu$ is $\vec{v} \cdot \vec{\nabla} P \equiv 0$ (i.e., P isn't a function of fewer than ν linear variables), then $V(x) = P(x)^2$ obeys the hypotheses of Theorem 3.

In Section 2, we'll present a simple compactness criterion on which all theorems rely. In Section 3, we'll prove Theorems 1 and 2. In Section 4, we'll prove Theorem 3.

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2. Segal's Lemma

Segal [12] proved the following result, sometimes called Segal's lemma:

Proposition 2.1. *For A, B positive selfadjoint operators,*

$$\|e^{-(A+B)}\| \leq \|e^{-A} e^{-B}\|. \tag{2.1}$$

Remarks. 1. $A + B$ can always be defined as a closed quadratic form on $Q(A) \cap Q(B)$. That defines $e^{-(A+B)}$ on $\overline{Q(A) \cap Q(B)}$ and we set it to 0 on the orthogonal complement. Since the Trotter product formula is known in this generality (see Kato [5]), (2.1) holds in that generality.

2. Since $\|C^*C\| = \|C\|^2$, $\|e^{-A/2}e^{-B/2}\|^2 = \|e^{-B/2}e^{-A}e^{-B/2}\|$, and since

$$\|e^{-(A+B)/2}\|^2 = \|e^{-(A+B)}\|,$$

(2.1) is equivalent to

$$\|e^{-A+B}\| \le \|e^{-B/2}e^{-A}e^{-B/2}\| \tag{2.2}$$

which is the way Segal [12] stated it.

3. Somewhat earlier, Golden [4] and Thompson [22] proved

$$\mathrm{Tr}(e^{-(A+B)}) \le \mathrm{Tr}(e^{-A}e^{-B}) \tag{2.3}$$

and Thompson [23] later extended this to any symmetrically normed operator ideal.

Proof. There are many; see, for example, Simon [18, 19]. Here is the simplest, due to Deift [2, 3]: If σ is the spectrum of an operator

$$\sigma(CD) \setminus \{0\} = \sigma(DC) \setminus \{0\} \tag{2.4}$$

so with σ_r the spectral radius,

$$\sigma_r(CD) = \sigma_r(DC) \le \|DC\|. \tag{2.5}$$

If CD is selfadjoint, $\sigma_r(CD) = \|CD\|$, so

$$CD \text{ selfadjoint} \Rightarrow \|CD\| \le \|DC\|. \tag{2.6}$$

Thus,

$$\|e^{-A/2}e^{-B/2}\|^2 = \|e^{-B/2}e^{-A}e^{-B/2}\| \le \|e^{-A}e^{-B}\|. \tag{2.7}$$

By induction,

$$\|(e^{-A/2^n}e^{-B/2^n})^{2^n}\| \le \|e^{-A/2^n}e^{-B/2^n}\|^{2n} \le \|e^{-A}e^{-B}\|. \tag{2.8}$$

Take $n \to \infty$ and use the Trotter product formula to get (2.1). □

In [18], I noted that this implies for any symmetrically normed trace ideal, $\mathcal{I}_\Phi$, that

$$e^{-A/2}e^{-B}e^{-A/2} \in \mathcal{I}_\Phi \Rightarrow e^{-(A+B)} \in \mathcal{I}_\Phi. \tag{2.9}$$

I explicitly excluded the case $\mathcal{I}_\Phi = \mathcal{I}_\infty$ (the compact operators) because the argument there doesn't show that, but it is true—and the key to this paper!

Since

$$C \in \mathcal{I}_\infty \Leftrightarrow C^*C \in \mathcal{I}_\infty$$

and $e^{-(A+B)} \in \mathcal{I}_\infty$ if and only if $e^{-\frac{1}{2}(A+B)} \in \mathcal{I}_\infty$, it doesn't matter if we use the symmetric form (2.2) or the following asymmetric form which is more convenient in applications.

Theorem 2.2. *Let $\mathcal{I}_\infty$ be the ideal of compact operators on some Hilbert space, $\mathcal{H}$. Let A, B be nonnegative selfadjoint operators. Then*

$$e^{-A}e^{-B} \in \mathcal{I}_\infty \Rightarrow e^{-(A+B)} \in \mathcal{I}_\infty. \tag{2.10}$$

Proof. For any bounded operator, C, define $\mu_n(C)$ by

$$\mu_n(C) = \min_{\psi_1 \dots \psi_{n-1}} \sup_{\substack{\|\varphi\|=1 \\ \varphi \perp \psi_1, \dots, \psi_{n-1}}} \|C\varphi\|. \tag{2.11}$$

By the min-max principle (see [9, Sect. XIII.1]),

$$\lim_{n\to\infty} \mu_n(C) = \sup(\sigma_{\mathrm{ess}}(|C|)) \tag{2.12}$$

and $\mu_n(C)$ are the singular values if $C \in \mathcal{I}_\infty$. In particular,

$$C \in \mathcal{I}_\infty \Leftrightarrow \lim_{n\to\infty} \mu_n(C) = 0. \tag{2.13}$$

Let $\wedge^\ell(\mathcal{H})$ be the antisymmetric tensor product (see [8, Sects. II.4, VIII.10], [9, Sect. XIII.17], and [18, Sect. 1.5]). As usual (see [18, eqn. (1.14)]),

$$\|\wedge^m(C)\| = \prod_{j=1}^m \mu_j(C). \tag{2.14}$$

Since $\mu_1 \geq \mu_2 \geq \cdots \geq 0$, we have

$$\lim_{n\to\infty} \mu_n(C) = \lim_{n\to\infty} (\mu_1(C)\dots\mu_n(C))^{1/n}. \tag{2.15}$$

(2.13)–(2.15) imply

$$C \in \mathcal{I}_\infty \Leftrightarrow \lim_{n\to\infty} \|\wedge^n(C)\|^{1/n} = 0. \tag{2.16}$$

As usual, there is a selfadjoint operator, $d\wedge^n(A)$ on $\wedge^n(\mathcal{H})$ so

$$\wedge^n(e^{-tA}) = e^{-t\,d\wedge^n(A)} \tag{2.17}$$

so Segal's lemma implies that

$$\|\wedge^n(e^{-(A+B)})\| \leq \|\wedge^n(e^{-A})\wedge^n(e^{-B})\| = \|\wedge^n(e^{-A}e^{-B})\|. \tag{2.18}$$

Thus,

$$\lim_{n\to\infty} \|\wedge^n(e^{-(A+B)})\|^{1/n} \leq \lim_{n\to\infty} \|\wedge^n(e^{-A}e^{-B})\|^{1/n}. \tag{2.19}$$

By (2.16), we obtain (2.10). □

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. By Theorem 2.2, we need only show $C = e^{\Delta}e^{-V}$ is compact. Write

$$C = C_m + D_m \tag{3.1}$$

where

$$C_m = C\chi_{\Omega_m}, \quad D_m = C\chi_{\Omega_m^c} \tag{3.2}$$

with χ_S the operator of multiplication by the characteristic function of a set $S \subset \mathbb{R}^\nu$.

$$\|e^{-V}\chi_{\Omega_m^c}\|_\infty \leq e^{-m}$$

and $\|e^{\Delta}\| = 1$, so

$$\|D_m\| \leq e^{-m} \tag{3.3}$$

and thus,

$$\lim_{m\to\infty} \|C - C_m\| = 0. \tag{3.4}$$

If we show each C_m is compact, we are done. We know e^{Δ} has integral kernel $f(x-y)$ with f a Gaussian, so in L^2. Clearly, since V is positive, C_m has an integral kernel $C_m(x,y)$ dominated by

$$|C_m(x,y)| \leq f(x-y)\chi_{\Omega_m}(y). \tag{3.5}$$

Thus,

$$\int |C_m(x,y)|^2\, d^\nu x\, d^\nu y \leq \|f\|^2_{L^2(\mathbb{R}^\nu)} \|\chi_{\Omega_m}\|_{L^2(\mathbb{R}_\nu)} < \infty$$

since $|\Omega_m| < \infty$. Thus, C_m is Hilbert–Schmidt, so compact. □

Proof of Theorem 2. We can follow the proof of Theorem 1. It suffices to prove that $e^{-sL_0}e^{-sV}$ is compact, and so, that $e^{-sL_0}\chi_{\Omega_m}$ is Hilbert–Schmidt.

That e^{-sL_0} maps L^2 to L^∞ implies, according to the Dunford–Pettis theorem (see [24, Thm. 46.1]), that there is, for each $x \in X$, a function $f_x(\,\cdot\,) \in L^2(X, d\mu)$ with

$$(e^{-sL_0}g)(x) = \langle f_x, g\rangle \tag{3.6}$$

and

$$\sup_x \|f_x\|_{L^2} = \|e^{-sL_0}\|_{L^2\to L^\infty} \equiv C < \infty. \tag{3.7}$$

Thus, e^{-sL_0} has an integral kernel $K(x,y)$ with

$$\sup_x \int |K(x,y)|^2 \, d\mu(y) = C < \infty \tag{3.8}$$

(for $K(x,y) = f_x(y)$). But e^{-sL_0} is selfadjoint, so its kernel is complex symmetric, so

$$\sup_y \int |K(x,y)|^2 \, d\mu(x) = C < \infty. \tag{3.9}$$

Thus,

$$\int |K(x,y)\chi_{\Omega_m}(y)|^2 \, d\mu(x) d\mu(y) \le C\mu(\Omega_m) < \infty \tag{3.10}$$

and $e^{-sL_0}\chi_{\Omega_m}$ is Hilbert–Schmidt. □

4. Proof of Theorem 3

As with the proof of Theorem 1, it suffices to prove that for each M, $e^{\Delta}\chi_{\Omega_M}$ is compact. e^{Δ} is convolution with an L^1 function, f. Let Q_R be the characteristic function of $\{x \mid |x| < R\}$. Let F_R be convolution with fQ_R. Then

$$\|e^{\Delta} - F_R\| \le \|f(1-Q_R)\|_1 \to 0 \tag{4.1}$$

as $R \to \infty$, so

$$\|e^{\Delta}\chi_{\Omega_M} - F_R\chi_{\Omega_M}\| \to 0 \tag{4.2}$$

and it suffices to prove for each R, M,

$$C_{M,R} = F_R\chi_{\Omega_M} \tag{4.3}$$

is compact. Clearly, this works if we show for some k, $(C^*_{M,R}C_{M,R})^k$ is Hilbert–Schmidt.

Let D be the operator with integral kernel

$$D(x,y) = \chi_{\Omega_M}(x)Q_{2R}(x-y)\chi_{\Omega_M}(y). \tag{4.4}$$

Since f is bounded, it is easy to see that

$$(C^*_{M,R}C_{M,R})(x,y) \le cD(x,y) \tag{4.5}$$

for some constant c, so it suffices to show D^k is Hilbert–Schmidt.

D^k has integral kernel

$$D^k(x,y) = \int D(x,x_1)D(x_1,x_2)\dots D(x_{k-1},y)\, dx_1 \dots dx_{k-1}. \tag{4.6}$$

Fix y. This integral is zero unless $|x-x_1| < 2R, \dots, |x_{k-1}-y| < 2R$, so, in particular, unless $|x-y| \le 2kR$. Moreover, the integrand can certainly be restricted to the regions $|x_j - y| \le 2kR$. Thus,

$$D^k(x,y) \le Q_{2kR}(x-y)\biggl(\int_{|x_j-y|\le 2kR} \prod_{j=1}^{k-1} \chi_{\Omega_M}(x_j)\, dx_1 \dots dx_{k-1}\biggr)\chi_{\Omega_m}(y) \tag{4.7}$$

$$= Q_{2kR}(x-y)(\omega_y^{2kR}(\Omega_M)^{k-1})\chi_{\Omega_M}(y) \tag{4.8}$$

by the definition of ω_x^ℓ in (1.7).

Thus,

$$\int |D^k(x,y)|^2 \, d^\nu x \, d^\nu y \le C(kR)^\nu \int_{x\in\Omega} [\omega_x^{2kR}(\Omega_M)]^{2k-2} \, d^\nu x$$

so if $2k-2 > r$ and (1.9) holds, D^k is Hilbert–Schmidt. □

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