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NECESSARY AND SUFFICIENT CONDITION FOR SOLVABILITY OF A PARTIAL INTEGRAL EQUATION

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Abstract. Let $T_1 : L_2(\Omega^2) \to L_2(\Omega^2)$ be a partial integral operator [4,7] with the kernel from $C(\Omega^3)$ where $\Omega = [a, b]^\nu$, $\nu \in N$ is fixed. In this paper we investigate solvability of the partial integral equation $f - \varkappa T_1 f = g_0$ in the space $L_2(\Omega^2)$ in the case where $\varkappa$ is a characteristic number. We prove a the theorem that gives a necessary and sufficient condition for solvability of the partial integral equation $f - \varkappa T_1 f = g_0$.

In models of solid state physics [1] and also in the lattice field theory [2], there appear so-called discrete Schrodinger operators, which are lattice analogues of the usual Schrodinger operators in a continuous space. The study of spectra of lattice Hamiltonians (that is, discrete Schrodinger operators) is an important topic in mathematical physics. Nevertheless, when studying spectral properties of discrete Schrodinger operators there appear partial integral equations in a Hilbert space of multi-variable functions [1,3]. Therefore, to investigate spectra of Hamiltonians considered on a lattice, a study of the solvability problem for a partial integral equations in L_2 is essential (and even interesting from the point of view of functional analysis).

A question on existence of a solution of partial integral equation (PIE) for functions of two variables was considered in [4–8] and other works. In the work by the author [9], the PIE $f - \varkappa T_1 f = g_0$ was studied in the space $L_2(\Omega^2)$, where $\Omega = [a, b]^\nu$, for a partial integral operator (PIO) $T_1 : L_2(\Omega^2) \to L_2(\Omega^2)$ with the kernel $k(x, s, y)$ being a continuous function in three variables on Ω^3. The concept of a determinant for the PIE as a continuous function on Ω and the concepts of a regular number, a singular number, a characteristic number, and an essential number for a PIE are given. Theorems on solvability of the PIE are proved in the case where $\varkappa$ is a regular and essential number [9]. In this paper we study solvability of the PIE $f - \varkappa T_1 f = g_0$ when $\varkappa$ is a characteristic number, i.e., the paper continues the work by the author [9].

Let $L^0 = L^0(\Omega)$ be a space of classes of complex-valued measurable functions $b = b(y)$ on Ω. We denote by $L_{2,0}(\Omega^2)$ the totality of classes of complex-valued measurable functions $f(x, y)$ on $\Omega \times \Omega$ satisfying the condition: $\int |f(x, y)|^2 dx$ exists for almost all $y \in \Omega$. It is easy to note that $L_{2,0}(\Omega^2)$ is a linear space over $\mathbb{C}$ and $L_2(\Omega^2) \subset L_{2,0}(\Omega^2)$. For each $b(y) \in L^0$ and $f(x, y) \in L_{2,0}(\Omega^2)$, we define the function $b \circ f$ by the formula $(b \circ f)(x, y) = b(y)f(x, y)$. Then for any $b \in L^0$ we have $b \circ f \in L_{2,0}(\Omega^2)$, where $f \in L_{2,0}(\Omega^2)$. For any $f, g \in L_{2,0}(\Omega^2)$, the integral $\int f(x, t)g(x, t)\, dx$ exists for almost all $t \in \Omega$ and $\varphi(t) = \int f(x, t)g(x, t)\, dx \in L^0$.

Let ∇ be the Boolean algebra of idempotents in L^0. A system $\{f_1, f_2, \dots, f_n\} \subset L_{2,0}(\Omega^2)$ is called ∇-*linearly independent,* if for all $\pi \in \nabla$ and $b_1(y), b_2(y), \dots, b_n(y) \in L^0$ from $\sum_{k=1}^n \pi \circ (b_k \circ f_k) = \theta$ it follows that $\pi \cdot b_1 = \pi \cdot b_2 = \cdots = \pi \cdot b_n = \theta$ [10,11].

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Consider the mapping $\langle\cdot,\cdot\rangle : L_{2,0}(\Omega^2)\times L_{2,0}(\Omega^2)\to L^0$ acting by the rule

$$\langle f,g\rangle = \int f(s,y)\overline{g(s,y)}\,ds,\quad f,g\in L_{2,0}(\Omega^2).$$

For every $b\in L^0$, we have $\langle b\circ f,g\rangle = b\cdot\langle f,g\rangle$, where $f,g\in L_{2,0}(\Omega^2)$, i.e., the mapping $\langle\cdot,\cdot\rangle$ satisfies the condition of L^0-valued internal product [12].

In the space $\mathcal{H}=L_{2,0}(\Omega^2)$, we consider a partial integral operator (PIO) S defined by

$$Sf=\int_\Omega q(x,s,y)f(s,y)\,ds,\quad f\in\mathcal{H},$$

where $q(x,s,y)\in L_2(\Omega^3)$. The function $q(x,s,y)$ is called *kernel* of the PIO S.

The kernel $\overline{q(s,x,y)}$ corresponds to the adjoint operator S^*, i.e.,

$$S^*f=\int_\Omega \overline{q(s,x,y)}f(s,y)\,ds,\quad f\in\mathcal{H}.$$

Let $\Omega'=\{\alpha\in\Omega:\ \ q(x,s,\alpha)\in L_2(\Omega^2)\}$. Consider a family of compact operators $\{S_\alpha\}_{\alpha\in\Omega'}$ in $L_2(\Omega)$ associated to S by the following formula

$$S_\alpha\varphi=\int_\Omega q(x,s,\alpha)\varphi(s)\,ds,\quad \varphi\in L_2(\Omega)\quad(\alpha\in\Omega'),$$

where $q(x,s,y)$ is the kernel of S.

Further, if no set of integration is indicated, we mean integration over the set Ω.

Now we consider the equation

$$f-\varkappa Sf=g_0 \tag{1}$$

on the space $\mathcal{H}$ where f is an unknown function from $\mathcal{H}$, $g_0\in\mathcal{H}$ is a given function, $\varkappa\in\mathbb{C}$ is a parameter of the equation.

For each $n\in\mathbb{N}$, we define a measurable function

$$\Pi^{(n)}=\Pi^{(n)}(x_1,\dots,x_n,s_1,\dots,s_n,\alpha)$$

on $\Omega^n\times\Omega^n\times\Omega$ by means of the order n determinant,

$$\Pi^{(n)}(x_1,\dots,x_n,s_1,\dots,s_n,\alpha)=\begin{vmatrix} q(x_1,s_1,\alpha) & \dots & q(x_1,s_n,\alpha)\\ \vdots & \vdots & \vdots\\ q(x_n,s_1,\alpha) & \dots & q(x_n,s_n,\alpha)\end{vmatrix}.$$

Now, for every $\varkappa\in\mathbb{C}$ we "formally" define functions $D_1(y)=D_1(y;\varkappa)$ on Ω and $M_1(x,s,y)=M_1(x,s,y;\varkappa)$ on Ω^3 by means of the sum of measurable functional series composed from sequences of measurable functions $d_n(y)$ on Ω and $q_n(x,s,y)$ on Ω^3, respectively, by the following rules

$$D_1(\alpha)=D_1(\alpha;\varkappa)=1+\sum_{n\in\mathbb{N}}\frac{(-\varkappa)^n}{n!}d_n(\alpha),\quad \alpha\in\Omega, \tag{a}$$

and

$$M_1(x,s,\alpha)=M_1(x,s,\alpha;\varkappa)=q(x,s,\alpha)+\sum_{n\in\mathbb{N}}\frac{(-\varkappa)^n}{n!}q_n(x,s,\alpha),\quad (x,s,\alpha)\in\Omega^3, \tag{b}$$

where

$$d_k(\alpha)=\int\dots\int \Pi^{(k)}(\xi_1,\dots,\xi_k,\xi_1,\dots,\xi_k,\alpha)\,d\mu(\xi_1)\dots d\mu(\xi_k),$$

$$q_k(x,s,\alpha)=\int\dots\int \Pi^{(k+1)}(x,\xi_1,\dots,\xi_k,s,\xi_1,\dots,\xi_k,\alpha)\,d\mu(\xi_1)\dots d\mu(\xi_k).$$

Lemma 1. *For each $\varkappa \in \mathbb{C}$ the functions $D_1(y) = D_1(y;\varkappa)$ (a) and $M_1(x,s,y) = M_1(x,s,y;\varkappa)$ (b) are measurable on Ω and Ω^3, respectively. Moreover, for almost all $\alpha \in \Omega$, there exists the integral $\int\int |M_1(x,s,\alpha)|^2 dxds$.*

Proof. Let $\varkappa \in \mathbb{C}$ be an arbitrary fixed number. We respectively denote by $\Delta_\alpha^{(1)}(\varkappa)$ and $M_\alpha^{(1)}(x,s;\varkappa)$ the Fredholm determinant and the Fredholm minor of the operator $I - \varkappa S_\alpha$ for $\alpha \in \Omega'$, where I is the identity operator in $L_2(\Omega)$. Let $\varphi_n(y)$ and $\psi_n(x,s,y)$ be the partial sums of the functional series (a) and (b), respectively. We have the sequences of measurable functions $\varphi_n(y)$ on Ω and $\psi_n(x,s,y)$ on Ω^3 such that $\lim_{n\to\infty} \varphi_n(y) = \Delta_y^{(1)}(\varkappa) = D_1(y;\varkappa)$ for almost all $y \in \Omega$ and $\lim_{n\to\infty} \psi_n(x,s,y) = M_y^{(1)}(x,s;\varkappa) = M_1(x,s,y;\varkappa)$ for almost all $(x,s,y) \in \Omega^3$. Therefore, the function $D_1(y) = D_1(y;\varkappa)$ and the function $M_1(x,s,y) = M_1(x,s,y;\varkappa)$ are measurable on Ω and Ω^3, respectively. It is known that if the kernel $h(x,s)$ of the integral operator $A\varphi = \int h(x,s)\varphi(s)\,ds$, $\varphi \in L_2(\Omega)$, is an element of the space $L_2(\Omega^2)$, then the minor $M(x,s;\varkappa)$ of the operator $I - \varkappa A$ is also an element of the space $L_2(\Omega^2)$. Hence we have

$$\int\int |M_1(x,s,\alpha)|^2 dxds < \infty \quad \text{for almost all} \quad \alpha \in \Omega. \quad \square$$

The measurable functions $D_1(y) = D_1(y;\varkappa)$ and $M_1(x,s,y) = M_1(x,s,y;\varkappa)$ are, respectively, called the determinant and the minor of the operator $E - \varkappa S$, $\varkappa \in \mathbb{C}$, where E is the identity operator in $L_{2,0}(\Omega^2)$.

Lemma 2. *Let $S : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ be a PIO with a kernel $q \in L_2(\Omega^3)$. If the homogeneous equation $\varphi - \varkappa S_\alpha \varphi = \theta$, $\varkappa \in \mathbb{C}$, has only the trivial solution in $L_2(\Omega)$ for almost all $\alpha \in \Omega'$, then PIE (1) is solvable in the space $L_{2,0}(\Omega^2)$ for every $g_0 \in L_{2,0}(\Omega^2)$.*

Proof. Let $\varkappa \in \mathbb{C}$, $g_0(x,y)$ be an arbitrary function from the space $L_{2,0}(\Omega^2)$. Let the homogeneous equation $\varphi - \varkappa S_\alpha \varphi = \theta$ have only the trivial solution in the space $L_2(\Omega)$ for almost all $\alpha \in \Omega'$. Then $D_1(\alpha) = D_1(\alpha;\varkappa) \neq 0$ for almost all $\alpha \in \Omega$ and the equation $\varphi(x) - \varkappa(S_\alpha\varphi)(x) = h_\alpha(x)$ has a solution $\varphi_\alpha(x) \in L_2(\Omega)$ for almost all $\alpha \in \Omega'$ where $h_\alpha(x) = g_0(x,\alpha) \in L_2(\Omega)$. Moreover, the solution $\varphi_\alpha(x)$ has the form [13]

$$\varphi_\alpha(x) = h_\alpha(x) + \varkappa \int \frac{M_1(x,s,\alpha;\varkappa)}{D_1(\alpha;\varkappa)} h_\alpha(s)\,ds.$$

We have

$$\int\int \left| \frac{M_1(x,s,\alpha;\varkappa)}{D_1(\alpha;\varkappa)} \right|^2 dxds < \infty \quad \text{for almost all} \quad \alpha \in \Omega.$$

This means that we can define a PIO $W = W(\varkappa) : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ with the kernel [14]

$$\frac{M_1(x,s,\alpha;\varkappa)}{D_1(\alpha;\varkappa)}.$$

Therefore we have $f_0(x,y) = g_0(x,y) + \varkappa(Wg_0)(x,y) \in L_{2,0}(\Omega^2)$ and $\varphi_\alpha(x) = f_0(x,\alpha)$ for almost all $\alpha \in \Omega$. So the function $f_0(x,y)$ is a solution of the equation (1). $\square$

The following two propositions are proved analogously to Propositions 1 and 2 from [9].

Proposition 1. *Let $S : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ be a PIO with the kernel $q \in L_2(\Omega^3)$. Then the following two conditions are equivalent:*

(i) *a number $\lambda \in \mathbb{C}$ is an eigenvalue of the operator S;*
(ii) *a number $\lambda \in \mathbb{C}$ is an eigenvalue of operators $\{S_\alpha\}_{\alpha\in\Omega_0}$, where Ω_0 is a subset of Ω such that $\mu(\Omega_0) > 0$.*

Proposition 2. *If $\lambda \in \mathbb{C}$ is an eigenvalue of a PIO $S : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ with a kernel $q(x,s,y) \in L_2(\Omega^3)$, then the number $\overline{\lambda}$ is an eigenvalue of the operator S^*.*

Theorem 1. *Let $S : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ be a PIO with a kernel $q(x,s,y) \in L_2(\Omega^3)$. Then every eigenvalue of the PIO S corresponds only to a finite number of ∇-linearly independent eigenfunctions.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of the PIO S and

$$f_1,\ f_2,\ \ldots,\ f_m \tag{2}$$

be some ∇-linearly independent eigenfunctions, i.e.,

$$\lambda f_j(x,y) = (Sf_j)(x,y), \quad j = 1,2,\ldots,m. \tag{3}$$

Since any linear combination of the eigenfunctions (2) of the operator S with coefficients from L^0 is also an eigenfunction, we can apply to the functions (2) the process of L^0-orthogonalization [12]. Thus, we can assume that the functions (2) are mutually orthogonal and normed in the sense of L^0-valued internal products, i.e.,

$$\langle f_i, f_j\rangle = 0, \quad i \neq j \quad \text{and} \quad \langle f_i, f_i\rangle = 1.$$

Therefore we can rewrite (3) in the following form:

$$\overline{\lambda} \cdot \overline{f_j(x,y)} = \int \overline{q(x,s,y)} \cdot \overline{f_j(s,y)}\, ds.$$

From here, it is easy to see that for almost all $x \in \Omega$ the left hand-side of this equality is an L^0-valued Fourier coefficient of the function $\overline{q(x,s,y)}$ and it is a function of (s,y) with respect to the orthogonal normed system (2). By the Bessel inequality [12], one can write

$$|\lambda|^2 \sum_{j=1}^{m} |f_j(x,y)|^2 \leq \int |q(s,x,y)|^2 ds \quad \text{for almost all} \quad x \in \Omega.$$

If we integrate both parts of this inequality with respect to x and y, we obtain

$$m \leq |\lambda|^{-2} \int\int\int |q(x,s,y)|^2 dx ds dy < \infty.$$

Hence, the number of ∇-linearly independent functions corresponding to the eigenvalue λ is finite. □

Let S be a PIO with a kernel $q(x,s,y) \in L_2(\Omega^2)$. A number $\varkappa_0 \in \mathbb{C}$ is called *a characteristic value* of the PIE $f - \varkappa_0 Sf = g_0$ if the homogeneous equation $f - \varkappa_0 Sf = 0$ has a non-trivial solution. From here, it is clear that any characteristic value $\varkappa_0$ of the PIE $f - \varkappa Sf = g_0$ is non-zero.

Corollary 1. *Let $S : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ be a PIO with a kernel $q(x,s,y) \in L_2(\Omega^3)$. Then any characteristic value of the PIE $f - \varkappa Sf = g_0$ corresponds only to a finite number of ∇-linearly independent eigenfunctions.*

Theorem 2. *Let $\varkappa$ be a characteristic number of the PIE (1). Then the homogeneous PIE*

$$f - \varkappa Sf = 0 \tag{4}$$

and the adjoint homogeneous PIE

$$f - \overline{\varkappa} S^* f = 0 \tag{4'}$$

have the same number of ∇-linearly independent solutions.

Proof. Let $f_1,\ldots,f_m$ and $g_1,\ldots,g_n$ be ∇-linearly independent solutions of the the homogeneous equations (4) and (4′), respectively. Assume that $m < n$. We can suppose that $f_1,\ldots,f_m$ and $g_1,\ldots,g_n$ are orthonormal systems in the sense of L^0-valued internal product.

Define the function

$$p(x,s,y) = q(x,s,y) - \sum_{j=1}^{m} \overline{f_j(s,y)} g_j(x,y).$$

We have $p(x,s,y) \in L_2(\Omega^3)$ since $f_j, g_k \in L_{2,0}(\Omega^2)$. Consider two homogeneous PIE,

$$f - \varkappa W f = 0 \tag{5}$$

and

$$f - \overline{\varkappa} W^* f = 0, \tag{5'}$$

where W is the PIO with the kernel $p(x,s,y)$.

Let $h(x,y)$ be a solution of the equation (5). Then we have

$$\langle h, g_j \rangle = \langle \varkappa W h, g_j \rangle = \langle h, \overline{\varkappa} S^* g_j \rangle - \varkappa \langle h, f_j \rangle = \langle h, g_j \rangle - \varkappa \langle h, f_j \rangle, \quad j = 1, 2, \ldots, m.$$

Hence, since $\varkappa \neq 0$,

$$\langle h, f_j \rangle = 0, \quad j = 1, 2, \ldots, m. \tag{6}$$

Thus, any solution of the equation (5) satisfies the conditions (6). But by virtue of this conditions, one can rewrite the equation (5) in the form $f - \varkappa S f = 0$, i.e., any solution of the equation (5) satisfies the equation (4), too. We obtain that a solution $h(x,y)$ of the equation (5) is in the form

$$h(x,y) = \sum_{j=1}^{m} (b_j \circ f_j)(x,y), \quad b_j \in L^0, \quad j = 1, 2, \ldots, m.$$

But we have $0 = \langle h, f_k \rangle = \sum_{j=1}^{m} \langle b_j \circ f_j, f_k \rangle = \sum_{j=1}^{m} b_j \cdot \langle f_j, f_k \rangle = b_k$, $k = 1, 2, \ldots, m$. Thus, we have $h(x,y) = \theta$, i.e., the homogeneous PIE (5) has only the trivial solution. We show that the adjoint equation (5′) has non-trivial solutions. If we substitute $g(x,y) = g_k(x,y)$, where $k > m$, in the equation (5′) then we obtain $g_k = \varkappa^* W^* g_k$. Thus, we obtain the contradiction to Proposition 2: the equation (5) has only the trivial solution, but the adjoint equation (5′) has a non-trivial solution. Hence the case $m < n$ is impossible. One can prove similarly that the case $m > n$ is also impossible and we obtain that $m = n$. ☐

Theorem 3. *Let $\varkappa_0$ be a characteristic number of the PIE (1). Then*

a) *the homogeneous equation $f - \varkappa_0 S f = 0$ has a non-trivial solution, moreover, the set of all solutions of the homogeneous equation is an infinite dimensional subspace of $\mathcal{H}$;*

b) *PIE (1) is solvable if and only if the given function g_0 satisfies the condition*

$$\langle g_0, g \rangle = 0, \tag{I}$$

where $g \in \mathcal{H}$ is an arbitrary solution of the adjoint homogeneous equation $f - \overline{\varkappa}_0 S^* f = 0$.

Proof. The proof of the property a) follows immediately from Proposition 1 and Proposition 3 from [9]. We prove the property b).

i) ("if-part") Let $\varkappa_0$ be a characteristic number of the PIE (1) and $f_0 \in \mathcal{H}$ be a solution of the PIE (1) and $g \in \mathcal{H}$ be an arbitrary solution of the adjoint homogeneous equation $f - \overline{\varkappa}_0 S^* f = 0$. Then

$$\begin{aligned}\langle f_0, g \rangle &= \langle g_0 + \varkappa_0 S f_0, g \rangle = \langle g_0, g \rangle + \langle \varkappa_0 S f_0, g \rangle = \langle g_0, g \rangle + \langle f_0, \overline{\varkappa}_0 S^* g \rangle \\ &= \langle g_0, g \rangle + \langle f_0, g \rangle.\end{aligned}$$

Therefore we have $\langle g_0, g \rangle = 0$.

ii) ("only if"-part) Let $\varkappa_0$ be a characteristic number of the PIE (1). Suppose that g_0 satisfies the condition (I), i.e., $\langle g_0, g\rangle = 0$ for every solution $g \in \mathcal{H}$ of the equation $f - \overline{\varkappa}_0 S^* f = 0$.

Consider the function $p(x,s,y) \in L_2(\Omega^3)$ given by the equality

$$p(x,s,y) = q(x,s,y) - \sum_{j=1}^{m} \overline{f_j(s,y)} g_j(x,y),$$

where $f_1, f_2, \dots, f_m$ and $g_1, g_2, \dots, g_m$ are orthonormal systems of solutions of the equations (4) and (4′), respectively, in the sense of L^0-valued internal product. Then for almost all $\alpha \in \Omega$ the homogeneous Fredholm equation $\varphi - \varkappa_0 W_\alpha \varphi = 0$ has in $L_2(\Omega)$ only the trivial solution [13], where W_α is an integral operator in $L_2(\Omega)$ with the kernel $p(x,s,\alpha)$. Hence, by Lemma 2, the PIE $f - \varkappa_0 W f = g_0$ has a solution $f_0 \in \mathcal{H}$ of the form

$$f_0 = g_0(x,y) + \varkappa_0 S f_0(x,y) - \varkappa_0 \sum_{j=1}^{m} \langle f_0, f_j\rangle \cdot g_j(x,y).$$

Therefore, we obtain that

$$\begin{aligned}\langle f_0, g_k\rangle &= \langle g_0, g_k\rangle + \langle \varkappa_0 S f_0, g_k\rangle - \sum_{j=1}^{m} \langle f_0, f_j\rangle \cdot \langle \varkappa_0 g_j, g_k\rangle \\ &= \langle f_0, \overline{\varkappa}_0 S^* g_k\rangle - \varkappa_0 \langle f_0, f_k\rangle = \langle f_0, g_k\rangle - \varkappa_0\langle f_0, f_k\rangle,\end{aligned}$$

i.e., $\langle f_0, f_k\rangle = 0$, since $\varkappa_0 \neq 0$. Thus, the solution f_0 of the equation $f - \varkappa_0 W f = g_0$ has the form $f_0 = g_0 + \varkappa_0 S f_0$ and, hence, the function f_0 is also a solution of the PIE (1) at $\varkappa = \varkappa_0$. □

If there exists a number C such that

$$(II) \qquad |b(t)| \leq C \quad \text{for almost all} \quad t \in \Omega,$$

then the PIO S is a bounded operator on the space $L_2(\Omega^2)$, i.e., $Sf \in L_2(\Omega^2)$, $\forall f \in L_2(\Omega^2) \subset L_{2,0}(\Omega^2)$ and $\|Sf\|_{L_2(\Omega^2)} \leq C_0 \|f\|_{L_2(\Omega^2)}$ for all $f \in L_2(\Omega^2)$, where C_0 is a positive number,

$$b(t) = \int\int |q(x,s,t)|^2 dx ds.$$

Let $k(x,s,y) \in C(\Omega^3)$. Then the subspace $L_2(\Omega^2)$ is invariant for the PIO T_1 : $(T_1 f)(x,y) = \int k(x,s,y) f(s,y) ds$. Therefore it is possible to study solvability for the PIE

$$(7) \qquad f - \varkappa T_1 f = g_0$$

in the space $L_2(\Omega^2)$ where f is an unknown function from $L_2(\Omega^2)$, $g_o \in L_2(\Omega^2)$ is a given (known) function, $\varkappa \in \mathbb{C}$ is a parameter of the equation.

Let χ_{T_1} be a set of characteristic numbers for the PIE (7) (see [9]). the definition of a characteristic number [9] and the obtained Theorem imply the following.

Theorem 4. *Let $\varkappa_0 \in \chi_{T_1}$. Then*

a) *the homogeneous equation $f - \varkappa T_1 f = \theta$ has a non-trivial solution, moreover, the set of all solutions of the homogeneous equation is an infinite dimensional subspace of $L_2(\Omega^2)$;*

b) *PIE (7) is solvable if and only if the given function g_0 satisfies the condition*

$$(III) \qquad \int g_0(s,t)\overline{g(s,t)}\, ds = 0 \quad \textit{for almost all} \quad t \in \Omega,$$

where $g \in L_2(\Omega^2)$ is an arbitrary solution of the adjoint homogeneous equation $f - \overline{\varkappa_0} T_1^* f = \theta$.

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