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COMPACT VARIATION, COMPACT SUBDIFFERENTIABILITY AND INDEFINITE BOCHNER INTEGRAL

I. V. ORLOV AND F. S. STONYAKIN

Abstract. The notions of compact convex variation and compact convex subdifferential for the mappings from a segment into a locally convex space (LCS) are studied. In the case of an arbitrary complete LCS, each indefinite Bochner integral has compact variation and each strongly absolutely continuous and compact subdifferentiable a.e. mapping is an indefinite Bochner integral.

0. Introduction and preliminaries

As it is well known, the main difference between properties of a Bochner integral ([1]–[6]) on a segment and the classical Lebesgue integral is that the class of indefinite Bochner integrals $(B)\int_a^x f(t)\,dt$ is essentially smaller than the one of absolutely continuous mappings. Recall ([2], Theorems 3.7.11, 3.8.5, 3.8.6) that, in the case of Banach-valued mappings, the class of indefinite Bochner integrals coincides with the one of absolutely continuous and a.e. differentiable mappings.

This problem was resulted, in particular, in the notion of a Radon-Nikodym property (RNP). Each absolutely continuous mapping taking values in a space with RNP (or Radon-Nikodym space) is an indefinite Bochner integral. This class of spaces plays an important role in the modern theory of Banach spaces and locally convex spaces, especially in connection with probability theory, harmonic analysis and topology ([7]–[9]).

In this paper, a rather different way is chosen. Not restricting, as far as possible, the class of spaces under consideration, a description of the absolute continuous mapping that are indefinite Bochner integrals is given by means of the nonsmooth analysis. To this end, the notion of a (set-valued) convex variation of a mapping from a segment into a LCS is introduced. The use of properties of compact variation (i.1) and compact subdifferential [10, 12] permitted us to prove the main results of the paper, which include representing every (strongly) absolutely continuous and compact subdifferentiable a.e. mapping into an arbitrary LCS as an indefinite Bochner integral (Theorems 3.1), proving that the class of indefinite Bochner integrals coincides with the class of (strongly) absolutely continuous and differentiable a.e. mappings in the case of Frechet spaces (Theorem 3.2), and showing a presence of a compact variation at each indefinite Bochner integral in an arbitrary complete LCS (Theorem 3.3). Note, in particular, that Theorem 3.2 generalizes the above-mentioned result from the case of Banach spaces ([2], Theorems 3.7.11, 3.8.5, 3.8.6) to the case of Frechet spaces. Note also that in the case of an arbitrary (not Frechet) LCS, the indefinite Bochner integral, being absolutely continuous, can nevertheless be nowhere differentiable (see Example 2.2).

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Let us mention a certain similarity of the relevant conditions with the classical conditions of the absolute continuity in measure and bounded variation of a charge in Radon-Nikodym type theorems for Bochner integral ([5], [13]–[17]).

Let us also introduce, in the conclusion of this section, some auxiliary notions. Throughout the paper, for arbitrary sets $A, B \subset E$,

$$A - B = \{x - y \mid x \in A,\ y \in B\}\ ,$$

$$\sup \|A\|_c = \sup\{\|x\|_c \mid x \in A\} \quad \text{for every continuous seminorm } \|\cdot\|_c \text{ on } E\ ,$$

co$A$ denotes the convex hull of A.

Definition 0.1. Given a real vector space (VS) E and a subset $A \subset E$, denote $\triangle A = A - A$, $w(A) = \mathrm{co}(\triangle A)$. In the case of a LCS E, denote by $\overline{w}(A)$ the closure of $w(A)$ and call it *the oscillation of* A.

The following property of the oscillations is directly verified.

Proposition 0.1. *Let E be a real LCS ; $A,\ B \subset E$. Then*

(*i*) $(A \bigcap B \neq \emptyset) \Rightarrow (\overline{w}(A \bigcup B) \subset \overline{w}(A) + \overline{w}(B))$.

(*ii*) $\sup \|\overline{w}(A)\|_c = \sup \|\triangle A\|_c$ *for every continuous seminorm* $\|\cdot\|_c$ *on* E*.*

1. Compact variation mappings and their properties

Definition 1.1. For a fixed segment $I = [a; b] \subset \mathbb{R}$ consider its various partitions $P = \{I_k = [x_{k-1};\ x_k]\}_{k=1}^n$, $a = x_0 < x_1 < \cdots < x_n = b$, and set $\mathcal{P}(I) = \{P\}$. Introduce a partial order in $\mathcal{P}(I)$ by

$$(P_1 \preceq P_2) :\Leftrightarrow (P_2 \text{ is refinement of } P_1)\ . \tag{1.1}$$

Note that the order (1.1) is obviously inductive.

Definition 1.2. Let E be a real LCS, $F : I \to E$. Associate with every partition $P \in \mathcal{P}(I)$ the *partial convex variation (or C-variation)* of F,

$$CV(F, P) = \sum_{k=1}^{n} \overline{w}F(I_k)\ .$$

It is obvious that each set $CV(F, P)$ is closed and absolutely convex.

Proposition 1.1. *The set of the all C-variations $\{CV(F, P) \mid P \in \mathcal{P}(I)\}$ is inductively ordered by the embedding with respect to $P_1 \preceq P_2$.*

Proof. It suffices to consider the case

$$P_1 = \{I_k\}_{k=1}^n, \quad P_2 = \{I_k\}_{k \neq m} \bigcup \{I'_m,\ I''_m\}, \quad I_m = I'_m \bigcup I''_m.$$

Then, in view of Proposition 0.1(i),

$$\begin{aligned} CV(F, P_1) &= \sum_{k \neq m}^{n} \overline{w}F(I_k) + \overline{w}F(I'_m \bigcup I''_m) \\ &\subset \sum_{k \neq m}^{n} \overline{w}F(I_k) + \overline{w}F(I'_m) + \overline{w}F(I''_m) = CV(F, P_2)\ . \quad \square \end{aligned}$$

Definition 1.3. Let us introduce, using the notation of Definitions 1.1–1.2, the (*total*) *convex variation* (or the (*total*) *C-variation*) of F on I by

$$CV_I(F) \equiv CV(F) = \overline{\bigcup_{P \in \mathcal{P}(I)} CV(F, P)}\ . \tag{1.2}$$

In the case of bounded $CV(F)$ we call F the *bounded C-variation mapping* ($F \in BV(I)$), in the case of compact $CV(F)$ we call F the *compact C-variation mapping* or simply *compact variation mapping* ($F \in CV(I)$).

Note that in any case, $CV(F)$ is a closed and absolutely convex set. Let us consider some properties of the C-variation and, first of all, a relationship between the C-variation and the usual variation of F in the Banach case and in the finite-dimensional case.

Proposition 1.2. *Let E be a real Banach space, $F : I \to E$. Denote by*

$$V(F) = \sup_{P \in \mathcal{P}(I)} \sum_{k=1}^{n} \|F(x_k) - F(x_{k-1})\|$$

the usual total variation of F. Then

$$\sup \|CV(F)\| \leqslant V(F) \ . \tag{1.3}$$

Proof. Fix $\varepsilon > 0,\ P \in \mathcal{P}(I)$, and choose $x'_k,\ x''_k \in I_k (k = \overline{1,n})$ such that

$$\|F(x'_k) - F(x''_k)\| > \sup_{x',\ x'' \in I_k} \|F(x') - F(x'')\| - \frac{\varepsilon}{2^k} = \sup \|\overline{w}F(I_k)\| - \frac{\varepsilon}{2^k} \ . \tag{1.4}$$

Summing (1.4) over $k = \overline{1,n}$, we obtain

$$\begin{aligned} V(F) &\geqslant \sum_{k=1}^{n} V_{I_k}(F) \geqslant \sum_{k=1}^{n} \|F(x'_k) - F(x''_k)\| > \sum_{k=1}^{n} \sup \|\overline{w}F(I_k)\| - \sum_{k=1}^{n} \frac{\varepsilon}{2^k} \\ &\geqslant \sup \left\| \sum_{k=1}^{n} \overline{w}F(I_k) \right\| - \sum_{k=1}^{n} \frac{\varepsilon}{2^k} > \sup \|CV(F,P)\| - \varepsilon \ , \end{aligned}$$

whence $V(F) \geqslant \sup \|CV(F,P)\|$ for every $P \in \mathcal{P}(I)$. This immediately implies (1.3) in view of (1.2). □

Corollary 1.1. *If, with the preceding notation, $V(F) < +\infty$, then $F \in BV(I)$.*

Proposition 1.3. *Let $F : I \to \mathbb{R}$. Then*

$$\sup CV(F) = V(F) \ . \tag{1.5}$$

Proof. Fix $P \in \mathcal{P}(I)$. Then Proposition 0.1(i) implies that

$$\sup \overline{w}F(I_k) = \sup_{x',\ x'' \in I_k} |F(x') - F(x'')| \quad (k = \overline{1,n}) \ .$$

Hence, in view of symmetry of partial C-variations,

$$\begin{aligned} \sup CV(F,P) &= \sup \sum_{k=1}^{n} \overline{w}F(I_k) = \sum_{k=1}^{n} \sup \overline{w}F(I_k) \\ &\geqslant \sum_{k=1}^{n} |F(x_k) - F(x_{k-1})| = V(F,P) \ , \end{aligned}$$

whence

$$\sup CV(F) \geqslant \sup CV(F,P) \geqslant V(F,P) \ ,$$

and hence

$$\sup CV(F) \geqslant \sup_{p \in \mathcal{P}(I)} V(F,P) = V(F) \ . \tag{1.6}$$

Since, by virtue of (1.3),

$$\sup CV(F) = \sup |CV(F)| \leqslant V(F) \ , \tag{1.7}$$

we see that (1.5) follows from (1.6) and (1.7). □

Corollary 1.2. *Let $F : I \to \mathbb{R}^n$. Then $F \in CV(I)$ if and only if $V(F) < \infty$.*

Remark 1.1. 1). Below, in Section 2, we'll show that the case: $F \in CV(I)$ but $V(F) = +\infty$ is possible for $\dim E = \infty$.

2). The inequality (1.3) can easily be generalized to an arbitrary continuous seminorm $\| \cdot \|_c$ in LCS E:

$$\sup \|CV(F)\|_c \leqslant V_c(F) \ .$$

In particular, if $\{\| \cdot \|_i\}_{i \in Y}$ is a defining system of seminorms in E and $V_i(F) < +\infty$ $(i \in Y)$, then $F \in BV(I)$.

Let us continue with a study of properties of the C-variation.

Theorem 1.1. (Additivity). *Let E be a real LCS, $F : I \to E$, $I = I' \bigcup I''$. Then*

$$CV_I(F) = CV_{I'}(F) + CV_{I''}(F) \ .$$

Proof. Let $P' = \{I_k\}_{k=1}^m$, $P'' = \{I_k\}_{k=m+1}^n$ be partitions of I' and I'', respectively, $P = P' \bigcup P''$. Then

$$\begin{aligned} CV(F,P) &= \sum_{k=1}^m \overline{w}F(I_k) + \sum_{k=m+1}^n \overline{w}F(I_k) \\ &= CV(F,P') + CV(F,P'') \subset CV_{I'}(F) + CV_{I''}(F) \ . \end{aligned} \tag{1.8}$$

Next, for an arbitrary $P_1 \in \mathcal{P}(I)$ put $P_2 = P_1 \vee P$ (i.e., P_2 is the mutual refinement of P_1 and P). Then, applying Proposition 1.1 and (1.8) to P_2, we obtain

$$CV(F,P) \subset CV(F,P_2) \subset CV_{I'}(F) + CV_{I''}(F) \ ,$$

whence the inclusion

$$CV_I(F) \subset CV_{I'}(F) + CV_{I''}(F)$$

immediately follows. To check the inverse inclusion, return to (1.8),

$$CV(F,P') + CV(F,P'') = CV(F,P) \subset CV_I(F) \ ,$$

whence we obtain

$$CV_{I'}(F) + CV_{I''}(F) \subset CV_I(F) \ . \quad \square$$

Corollary 1.3. *With the preceding notation,*

(i) $F \in BV(I) \Leftrightarrow F \in BV(I')$ *and* $F \in BV(I'')$ *;*

(ii) $F \in CV(I) \Leftrightarrow F \in CV(I')$ *and* $F \in CV(I'')$ *.*

To formulate the following property, we need to extend the notion of a K-limit, which was introduced by the first author ([10]) for decreasing the net of closed convex sets (the projective case), to the case of an increasing net (inductive case).

Definition 1.4. Let $\{B_t\}_{t \in T}$ be a system of closed convex subsets of a real LCS E, inductively ordered by inclusion. We say that

$$B = K - \varinjlim_{t \in T} B_t \ ,$$

if for each zero neighborhood $U \subset E$ there exists $t_U \in T$ such that

$$(t \succeq t_U) \Rightarrow (B_t \subset B \subset B_t + U) \ . \tag{1.9}$$

Remark 1.2. It obviously follows from (1.9) that

$$\bigcup_{t \in T} B_t \subset B \subset \bigcup_{t \in T} (B_t + U) = \bigcup_{t \in T} B_t + U$$

for every $U \subset E$, whence

$$\bigcup_{t \in T} B_t \subset B \subset \overline{\bigcup_{t \in T} B_t} \ .$$

In particular, B is convex set.

Theorem 1.2. *Let E be a real LCS, $F : I \to E$. If F has compact variation on I, then*

$$CV(F) = K - \lim_{\overrightarrow{P \in \mathcal{P}(I)}} CV(F, P) \ .$$

Proof. It follows from Definition 1.3 and ([18], I.1.1) that for each fixed neighborhood $\widetilde{U} \subset E$ of zero,

$$CV(F) = \overline{\bigcup_{P \in \mathcal{P}} CV(F, P)} = \bigcap_{U \subset E} \Big[\bigcup_{P \in \mathcal{P}} CV(F, P) + U \Big] \subset \bigcup_{P \in \mathcal{P}} (CV(F, P) + \widetilde{U}) \ .$$

Thus, the sets $\{CV(F, P) + \widetilde{U} \mid P \in \mathcal{P}(I)\}$ forms an open covering of the compact set $CV(F)$. Choose a finite subcovering

$$\{CV(F, P_j) + \widetilde{U}\}_{j=1}^{n}$$

and put $P_{\widetilde{U}} = P_1 \vee ... \vee P_n$. Then, by Proposition 1.1, $CV(F, P_j) \subset CV(F, P_{\widetilde{U}}),\ j = \overline{1, n}$, whence

$$CV(F, P_j) + \widetilde{U} \subset CV(F, P_{\widetilde{U}}) + \widetilde{U}. \quad (j = \overline{1, n})$$

Hence,

$$CV(F, P_{\widetilde{U}}) \subset CV(F) \subset \bigcup_{j=1}^{n} (CV(F, P_j) + \widetilde{U}) \subset CV(F, P_{\widetilde{U}}) + \widetilde{U} \ .$$

Replacing $P_{\widetilde{U}}$ with an arbitrary refinement $P \succeq P_{\widetilde{U}}$, the last inclusion remains valid:

$$CV(F, P) \subset CV(F) \subset CV(F, P) + \widetilde{U} \ ,$$

which corresponds to (1.9). □

Remark 1.3. As it is well known, even in the scalar case for discontinuous F, in general, $V(F, P) \nrightarrow V(F)$ (and hence $CV(F, P) \nrightarrow CV(F)$) as the step of the partition, $\lambda(P)$, tends to 0. Let us check now that $CV(F, P) \to CV(F))$ as $\lambda(P) \to 0$ for a continuous mapping $F : I \to E$.

Theorem 1.3. *Let E be a real LCS, $F : I \to E$ be a continuous mapping. If $F \in CV(I)$ then for every neighborhood $U \subset E$ of zero there exists $\delta_U > 0$ such that for each $P \in \mathcal{P}(I)$,*

$$(\lambda(P) < \delta_U) \Rightarrow (CV(F, P) \subset CV(F) \subset CV(F) + U) \ .$$

We will write the last condition in the form

$$CV(F) = K_\lambda - \lim_{\lambda \to 0} CV(F, P) \ .$$

Proof. Fix $U \subset E$ and choose $W \subset E$ such that $W + W \subset U$. Then, using Theorem 1.2, choose $P_W = \{I_k\}_{k=1}^{N} \in \mathcal{P}(I)$ such that

$$CV(F, P_W) \subset CV(F) \subset CV(F, P_W) + W \ .$$

Next, using uniform continuity of F, choose $\delta > 0$ and $W' \subset E$ such that

$$(|x' - x''| < \delta) \Rightarrow (F(x') - F(x'') \in W') \ . \tag{1.10}$$

Finally, choose a partition $P_\delta = \{I^j\}_{j=1}^{n}$ with $\lambda(P_\delta) < \delta$ and set $\widetilde{P} = P_W \vee P_\delta$.

Then, by Proposition 1.1,

$$CV(F, P_W) \subset CV(F, \widetilde{P}) \subset CV(F, P_\delta) + \sum_{k=1}^{N} \overline{w} F(I_{\pm}^{j(k)}) \ ,$$

where the intervals $I_{\pm}^{j(k)}$ contain inside right or left ends of the interval I_k, respectively, $k = \overline{1, N}$.

Since, by virtue of (1.10), $\overline{w}F(I_{\pm}^{j(k)}) \subset W'$, we finally have

$$\begin{aligned} CV(F,P_\delta) &\subset CV(F) \subset CV(F,P_W) + W \subset [CV(F,P_\delta) + \overbrace{W' + \cdots + W'}^{2N} + W \\ &\subset CV(F,P_\delta) + (W+W) \subset CV(F,P_\delta) + U \ . \quad \square \end{aligned}$$

At the end of this section, let us introduce the notion of a *compact-Lipschitz* (or *K-Lipschitz*) mapping and explain its relation with that of a compact variation.

Definition 1.5. Let E be a real LCS, $F : I \to E$. We say that a mapping F is compact-Lipschitz on I (or $F \in \mathrm{Lip}_K(I)$) if, for some compact absolutely convex set $C \subset E$, the inclusion

$$F(x_2) - F(x_1) \in C \cdot (x_2 - x_1)$$

holds for arbitrary $x_1,\ x_2 \in I$.

It is obvious that, in the case of a Banach space E, each compact-Lipschitz mapping is also Lipschitz. It is also easy to see that each compact-Lipschitz mapping is absolutely continuous. Let us check now that each K-Lipschitz mapping is also of compact variation.

Proposition 1.4. *Let E be a real LCS, $F : I \to E$. If $F \in \mathrm{Lip}_K(I)$ then $F \in CV(I)$.*

Proof. Obviously, the inclusion $F(x_2) - F(x_1) \in C \cdot (x_2 - x_1)$ implies the inclusion $\overline{w}F([x; x+h]) \subset C \cdot h$ for each $[x; x+h] \subset I$. Whence the inclusion $CV(F,P) \subset C \cdot (b-a)$ for each $P \in \mathcal{P}(I)$ and, hence, the inclusion $CV(F) \subset C \cdot (b-a)$ easily follow. $\square$

2. K-subdifferentiability. Examples of K-subdifferentiable and nowhere K-subdifferentiable mappings

First, let us introduce the notion of a K-limit for a decreasing system of closed convex sets, which has been already mentioned earlier in connection with Definition 1.4.

Definition 2.1. Let $\{B_t\}_{t \in T}$ be a system of the closed convex subsets of a real LCS E, inductively ordered oppositely to inclusion. We say that

$$B = K - \varprojlim_{t \in T} B_t \ ,$$

if B is closed and for each neighborhood $U \subset E$ of zero there exists $t_U \in T$ such that

$$(t \succeq t_U) \Rightarrow (B \subset B_t \subset B + U) \ . \tag{2.1}$$

Remark 2.1. It obviously follows from (2.1) that

$$B \subset \bigcap_{t \in T} B_t \subset \bigcap_{U \subset E} (B + U) = \overline{B} \ ,$$

whence, since $\bigcap_{t \in T} B_t$ is closed,

$$B = \bigcap_{t \in T} B_t \ .$$

In particular, B is a closed convex set.

Let us pass to the notion of a *K-subdifferential* (or *compact subdifferential*). The main properties of K-subdifferential, including various forms of the mean value theorem, were investigated by the authors in [11, 12].

Definition 2.2. Let E be a real LCS, $F : I \to E,\ x \in I, \delta > 0$. The set

$$\partial_K F(x,\delta) = \overline{\mathrm{co}} \left\{ \frac{F(x+h) - F(x)}{h} \ \middle|\ 0 < |h| < \delta \right\}$$

is said to be a *partial K-subdifferential* of F at x, corresponding to δ. A set $\partial_K F(x) \subset E$ is said to be a *K-subdifferential* of F at x, if

(*i*) $\partial_K F(x) = K - \lim_{\overleftarrow{\delta\to 0}} \partial_K F(x,\delta)$;
(*ii*) $\partial_K F(x)$ is a compact set.

In an analogous way, the right and left K-subdifferentials, $\partial_K^+ F(x)$ and $\partial_K^- F(x)$, respectively, can be defined.

Remark 2.2. It is shown in [12] that if F is differentiable at x then F is K-subdifferentiable at x and $\partial_K F(x) = \{F'(x)\}$, but the inverse is not true.

Proposition 2.1. *Each K-Lipschitz mapping is everywhere K-subdifferentiable.*

Proof. Obviously, the inclusion $F(x_2) - F(x_1) \in C(x_2 - x_1)$ implies the inclusion

$$\partial_K F(x,\delta) \subset C$$

for each $x \in I$, $\delta > 0$, whence $\partial_K F(x,\delta)$ is compact. Because convergence of a decreasing sequence of compact sets to its intersection is necessarily topological, both conditions of Definition 2.2 are fulfilled. □

Let us pass to examples. First, consider an example of differentiable (and, hence, K-subdifferentiable) a.e. compact variation mapping, not having the usual (strong) bounded variation.

Example 2.1. Let E be a real infinite-dimensional separable Hilbert space, $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in E. Define the mapping $F : [0;1] \to E$ by

$$\begin{cases} F(0) = 0\ ; \quad F(\frac{n}{n+1}) = \sum_{k=1}^{n} \frac{e_k}{k} \quad (n \in \mathbb{N})\ ; \quad F(1) = \sum_{k=1}^{\infty} \frac{e_k}{k}\ ; \\ F \text{ is linear on the segments } \left[\frac{n-1}{n};\ \frac{n}{n+1}\right] \quad (n \in \mathbb{N})\ . \end{cases}$$

Thus, F is continuous and countably piecewise linear on $[0;1]$. For each partition $P_n :\ 0 < \frac{1}{2} < \cdots < \frac{n}{n+1} < 1$ we obtain

$$\begin{aligned} V(F,P_n) &= \Big(\sum_{k=1}^{n} \Big\|F\Big(\frac{k}{k+1}\Big) - F\Big(\frac{k-1}{k}\Big)\Big\|\Big) + \Big\|F(1) - F\Big(\frac{n}{n+1}\Big)\Big\| = \sum_{k=1}^{n} \Big\|\frac{e_k}{k}\Big\| \\ &+ \Big\|\sum_{k=n+1}^{\infty} \frac{e_k}{k}\Big\| = \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=n+1}^{\infty} \frac{1}{k^2} \to +\infty \quad \text{as} \quad n \to \infty\ , \end{aligned}$$

whence $V(F) = +\,\infty$ follows.

However, let us show now that $F \in CV(I)$. Using piecewise linearity of F and additivity of C-variation (Theorem 1.1), we obtain

$$CV_I(F) \supset CV_{[0;\ \frac{n}{n+1}]}(F) = \sum_{k=1}^{n} CV_{[\frac{k-1}{k};\ \frac{k}{k+1}]}(F) = \sum_{k=1}^{n} \Big[-\frac{e_k}{k}; \frac{e_k}{k}\Big]\ . \tag{2.2}$$

Let $P = \{I_j\}_{j=1}^{m}$ be an arbitrary partition of I. Choose n such that $x_{m-1} < \frac{n}{n+1} < x_m$ and set

$$P' = P \vee \Big\{\Big[\frac{k-1}{k}; \frac{k}{k+1}\Big]\Big\}_{k=1}^{n}\ .$$

Then, taking into account Proposition 1.1, (2.2) implies

$$\begin{aligned} CV(F,P) &\subset CV(F,P') \subset CV_{[0;\frac{n}{n+1}]}(F) + \overline{w}F\Big(\Big[\frac{n}{n+1};1\Big]\Big) \\ &= \sum_{k=1}^{n} \Big[-\frac{e_k}{k}; \frac{e_k}{k}\Big] + \overline{w}\Big\{\sum_{k=1}^{n+p} \frac{e_k}{k} \quad (p \in \mathbb{N})\ ; \quad \sum_{k=1}^{\infty} \frac{e_k}{k}\Big\}\ , \end{aligned} \tag{2.3}$$

and it remains to estimate the last term in the right-hand side of (2.3).

Denote it by $\overline{w}(A)$. Since

$$\triangle A = \Big\{ \pm \sum_{k=n+p+1}^{n+p+q} \frac{e_k}{k} \ ; \quad 0 \ ; \quad \pm \sum_{k=n+p+1}^{\infty} \frac{e_k}{k} \quad (p \in \mathbb{N}_0, q \in \mathbb{N}) \Big\} \ ,$$

we have

$$\sup \|\overline{w}(A)\| = \sup \|\triangle A\| \leqslant \sum_{k=n+1}^{\infty} \frac{1}{k^2} =: r_n \to 0 \quad \text{as} \quad n \to \infty \ . \tag{2.4}$$

Thus, $\overline{w}(A)$ is contained in the closed ball $B_{r_n}(0)$. From here and (2.3) it follows that

$$CV(F,P) \subset \sum_{k=1}^{n} \left[-\frac{e_k}{k}; \frac{e_k}{k} \right] + B_{r_n}(0) \subset \sum_{k=1}^{\infty} \left[-\frac{e_k}{k}; \frac{e_k}{k} \right] + B_{r_n}(0) \ ,$$

whence for each $n \in \mathbb{N}$,

$$CV(F) \subset \sum_{k=1}^{\infty} \left[-\frac{e_k}{k}; \frac{e_k}{k} \right] + B_{r_n}(0) \ ,$$

and, finally,

$$CV(F) \subset \sum_{k=1}^{\infty} \left[-\frac{e_k}{k}; \frac{e_k}{k} \right] \ . \tag{2.5}$$

(It is easy to check the exact equality in (2.5)). Since the set in the right-hand side of (2.5) is compact (the Hilbert cube), $CV(F)$ is compact, too. So, $F \in CV(I)$. Note that, in view of countable piecewise linearity, F is differentiable a.e.

Secondly, consider an example of a mapping that is nowhere differentiable K-Lipschitz (and, hence, of compact variation) and everywhere K-subdifferentiable.

Example 2.2. Let E_I be the space of the all real functions $\xi = \xi(\theta)$ on $I = (0;1)$ with the defining system of seminorms $\{\|\cdot\|_t\}_{t\in I}$ that corresponds to the topology of pointwise convergence. Then E_I is a separable and complete LCS. Define a mapping $y(\cdot) : I \to E_I$ by

$$y(s)(\theta) = s \quad \text{for} \quad 0 < s \leqslant \theta < 1 \ , \qquad y(s)(\theta) = \theta \quad \text{for} \quad 0 < \theta \leqslant s < 1 \ .$$

1). For each $s \in I$ and corresponding $\triangle s > 0$ small enough (such that both cases $s \leqslant \theta \leqslant s + \triangle s$ and $s - \triangle s \leqslant \theta \leqslant s$ are impossible) it follows that

$$\frac{y(s+\triangle s) - y(s)}{\triangle s}(\theta) = \begin{cases} 1, & \text{for } 0 < s < s + \triangle s \leqslant \theta < 1 \ , \\ 0, & \text{for } 0 < \theta \leqslant s < s + \triangle s < 1 \ , \end{cases}$$

$$\frac{y(s-\triangle s) - y(s)}{-\triangle s}(\theta) = \begin{cases} 1, & \text{for } 0 < s - \triangle s < s \leqslant \theta < 1 \ , \\ 0, & \text{for } 0 < \theta \leqslant s - \triangle s < s < 1 \ , \end{cases}$$

and, therefore, y possess the left and the right derivatives everywhere on I, $y'_+(s) = y^1_s(\cdot)$, $y'_-(s) = y^2_s(\cdot)$, where

$$y^1_s(\theta) = 1 \quad \text{for} \quad 0 < s < \theta < 1 \ , \qquad y^1_s(\theta) = 0 \quad \text{for} \quad 0 < \theta \leqslant s < 1 \ ,$$

and

$$y^2_s(\theta) = 1 \quad \text{for} \quad 0 < s \leqslant \theta < 1 \ , \qquad y^2_s(\theta) = 0 \quad \text{for} \quad 0 < \theta < s < 1 \ .$$

It is obvious that $y^1_s(\cdot) \neq y^2_s(\cdot)$ for each $s \in I$ and, therefore, the mapping y is nowhere differentiable on I. However, in fact, $y(s)$ is K-subdifferentiable at each $s \in I$ and

$$\partial_K y(s) = [y'_-(s);\ y'_+(s)] = [y^1_s(\cdot);\ y^2_s(\cdot)] \ . \tag{2.6}$$

(Here we consider that interval $[y;\ y]$ as the one-point set $\{y\}$).

Indeed, let $\widetilde{\partial}_K y(s)$ be a formal K-subdifferential (without the requirement of compactness). Taking into account separability of E_I, the equalities

$$(\ell \circ y)'_{\pm}(s) = \ell(y'_{\pm}(s)) \quad (\forall \ell \in E^*)$$

and

$$[(\ell \circ y)'_-(s);\ (\ell \circ y)'_+(s)] = \ell([y'_-(s);\ y'_+(s)]) = \ell(\widetilde{\partial}_K y(s))$$

imply (2.6) (here $\ell(A) = \{\ell(x) \mid x \in A\}$ for an arbitrary $A \subset E$). Obviously, the vector segment $\widetilde{\partial}_K y(s)$ is compact and, therefore, $\partial_K y(s) = \widetilde{\partial}_K y(s)$ exists everywhere on I.

2). Let us show that $y \in \mathrm{Lip}_K(I)$. First, we prove compactness of the set $D := \{y^i_s(\cdot) \mid i = \overline{1,2}\ ;\ s \in I\}$ in E_I. Consider an arbitrary sequence $\{y^1_{s_k}(\cdot)\}_{k=1}^{\infty}$ of functions from D. Denote by s_0 some partial limit of the sequence $\{s_k\}_{k=1}^{\infty}$ and choose a subsequence $\{s_{k_\ell}\}_{\ell=1}^{\infty}$ that converges to s_0 from the right, for definiteness. Then

$$y^1_{s_{k_\ell}}(\theta) - y^1_{s_0}(\theta) = \begin{cases} -1, & \text{for } \theta \in (s_0; s_{k_\ell})\ , \\ \\ 0, & \text{for } \theta \notin (s_0; s_{k_\ell})\ . \end{cases}$$

This means that

$$\|y^1_{s_{k_\ell}}(\theta) - y^1_{s_0}(\theta)\|_t = |(y^1_{s_{k_\ell}} - y^1_{s_0})(t)| \to 0$$

as $\ell \to \infty$, i.e., the sequence $\{y^1_{s_{k_\ell}}(\cdot)\}_{\ell=1}^{\infty}$ converges in the space E_I to the function $y^1_{s_0}(\cdot)$. Analogously, for the case $s_{k_\ell} \to s_0 - 0$, $y^1_{s_{k_\ell}}(\cdot)$ converges to $y^2_{s_0}(\cdot) \in D$.

Thus, an arbitrary sequence of functions from D has a partial limit contained in D. Therefore D is sequentially compact and moreover compact. Hence, the set $U := \overline{\mathrm{co}}\, D$ is convex compact in E, because E is a separable and complete space. Note that for each $s \in I$,

$$\partial_K y(s) = [y^1_s(\cdot);\ y^2_s(\cdot)] \subset U\ ,$$

whence $\partial_K y(I) \subset U$. Now, using the mean value theorem ([12]) for K-subdifferentials on an arbitrary segment $[\alpha; \beta] \subset (0; 1)$, we obtain

$$y(\beta) - y(\alpha) \in U \cdot (\beta - \alpha)\ ,$$

whence $y \in \mathrm{Lip}_K(I)$.

Let us give, for completeness, a sketch of the proof of the "K-mean value theorem" [12] that was used above in Example 2.2.

Proposition 2.2. *Let E be a real LCS, a mapping $F : [a; b] \to E$ be continuous on $[a; b]$ and K-subdifferentiable on $(a; b)$. Then*

$$\frac{F(b) - F(a)}{b - a} \in \overline{\mathrm{co}}\ \partial_K F((a; b))\ . \tag{2.7}$$

Proof. We use the following result for scalar-valued functions (see [11], Theorem 9).

If a function $f : [a; b] \to \mathbb{R}$ is continuous on $[a; b]$ and K-subdifferentiable on $(a; b)$, then

$$\frac{f(b) - f(a)}{b - a} \leqslant \sup \partial_K f((a; b))\ . \tag{2.8}$$

Assume that inclusion (2.7) is not true. Then, in view of a known corollary from Hahn-Banach theorem ([21], Corollary 2.1.4), there is a functional $\ell \in E^*$ such that

$$\ell\left(\frac{F(b) - F(a)}{b - a}\right) > \sup \ell\left(\overline{\mathrm{co}}\ \partial_K F((a; b))\right)\ .$$

Denoting by $f = \ell \circ F$, the last inequality takes the form

$$\frac{f(b) - f(a)}{b - a} > \sup \partial_K f((a; b))\ ,$$

which contradicts (2.8). □

Next, in the monographs [2, 19], examples of Lipschitz mappings acting from an interval into Banach spaces, which are nowhere differentiable, are considered. Let us show now that the corresponding mappings are, in fact, nowhere K-subdifferentiable.

Example 2.3. Let $\mathcal{L}_1(I)$ be the space of the all Lebesgue integrable real-valued functions $x = \xi(t)$ on $I = [0;1]$ with the norm $\|x\| = \int_0^1 |\xi(t)|\,dt$. Define a mapping $y(\cdot) : I \to \mathcal{L}_1(I)$ by

$$y(s)(t) = 1 \quad \text{for} \quad 0 \leqslant t \leqslant s\ , \qquad y(s)(t) = 0 \quad \text{for} \quad s \leqslant t \leqslant 1\ .$$

Suppose that $y(\cdot)$ is K-subdifferentiable at some point $s_0 \in I$. Then, for some sequence $h_k \searrow 0$, the corresponding sequence

$$\widetilde{y}_k = \frac{y(s_0 + h_k) - y(s_0)}{h_k} \qquad (k \in \mathbb{N})$$

converges in $\mathcal{L}_1(I)$. Hence,

$$\forall \varepsilon > 0\ \exists k_0 \in \mathbb{N}\ \forall k \geqslant k_0 : \quad \|\widetilde{y}_k - \widetilde{y}_{k_0}\| < \varepsilon\ . \tag{2.9}$$

Since

$$\frac{y(s+h) - y(s)}{h}(t) = \begin{cases} \frac{1}{h}\ , \text{ for } s \leqslant t \leqslant s+h\ , \\ 0\ , \text{ otherwise}\ , \end{cases}$$

we obtain

$$\begin{aligned}
\|\widetilde{y}_k - \widetilde{y}_{k_0}\| &= \Big\| \frac{y(s_0 + h_k) - y(s_0)}{h_k} - \frac{y(s_0 + h_{k_0}) - y(s_0)}{h_{k_0}} \Big\| \\
&= \int_0^1 \Big| \Big(\frac{y(s_0 + h_k) - y(s_0)}{h_k} - \frac{y(s_0 + h_{k_0}) - y(s_0)}{h_{k_0}} \Big)(t) \Big| dt \\
&= \int_{s_0}^{s_0 + h_k} \Big| \frac{1}{h_k} - \frac{1}{h_{k_0}} \Big| dt\ + \int_{s_0}^{s_0 + h_{k_0}} \frac{dt}{h_{k_0}} = h_k \cdot \Big(\frac{1}{h_k} - \frac{1}{h_{k_0}} \Big) + \frac{h_{k_0} - h_k}{h_{k_0}} \\
&= 2 - 2\frac{h_k}{h_{k_0}} \to 2 \quad \text{as} \quad k \to \infty\ ,
\end{aligned}$$

which contradicts (2.9) for $\varepsilon < 2$. So, $y(\cdot)$ is nowhere K-subdifferentiable. At the same time, $V(y) < +\infty$ because $y(\cdot)$ is Lipschitz.

Example 2.4. Let $E(I)$ be the space of the all measurable bounded real-valued functions $x = \xi(t)$ on $I = \left[\frac{1}{3}; \frac{2}{3}\right]$ with the norm $\|\xi\| = \sup_{t \in I} |\xi(t)|$. Define a mapping $y(\cdot) : I \to E(I)$ by

$$y(s)(t) = \frac{s}{t} \quad \text{for} \quad s \leqslant t \leqslant \frac{2}{3}\ , \qquad y(s)(t) = \frac{s-1}{t-1} \quad \text{for} \quad \frac{1}{3} \leqslant t \leqslant s\ .$$

Repeating the arguments from the preceding example, choose a sequence $\{\widetilde{y}_k\}_{k=1}^{\infty}$ in $E(I)$ satisfying (2.9). Since, in this case,

$$\frac{y(s+h) - y(s)}{h}(t) = \begin{cases} \frac{1}{t-1} + \frac{1}{h} \cdot \frac{s-t}{t(t-1)}\ , \text{ for } s \leqslant t \leqslant s+h\ , \\ \frac{1}{t-1}\ , \text{ for } t \leqslant s\ , \end{cases}$$

we obtain

$$\|\widetilde{y}_k - \widetilde{y}_{k_0}\| \geqslant \sup_{t \in I} \Big| \frac{s-t}{t(t-1)} \Big| \cdot \Big| \frac{1}{h_k} - \frac{1}{h_{k_0}} \Big| \to +\infty \quad \text{as} \quad k \to \infty\ ,$$

which contradicts (2.9). So, like in the preceding example, $y(\cdot)$ is nowhere K-subdifferentiable.

Note, in the conclusion of this section, that a compact variation mapping also can be nowhere K-subdifferentiable. The corresponding example will be considered in the following section.

3. Description of the indefinite Bochner integral in terms of compact variation and compact subdifferentiability

The main results obtained in this section are representation of each (strongly) absolutely continuous and K-subdifferentiable a.e. mapping $F : I = [a; b] \to E$, where E is an arbitrary real LCS, as an indefinite Bochner integral,

$$F(x) = F(a) + (B)\int_a^x f(t)\,dt \quad (a \leqslant x \leqslant b)$$

(Theorem 3.1) and that the class of the all indefinite Bochner integrals in Frechet spaces coincides with the class of all mappings absolutely continuous and differentiable a.e. on I (Theorem 3.2). First, we need several auxiliary statements.

Lemma 3.1. *Let E be a real LCS, $F : I \to E$ be K-subdifferentiable at a point $x \in I$. If a sequence $[\alpha_n; \beta_n]$ contracts to x, then $\partial_K F(x)$ contains all possible partial limits of the sequence*

$$\frac{F(\beta_n) - F(\alpha_n)}{\beta_n - \alpha_n} \quad (n \in \mathbb{N}) \ . \tag{3.1}$$

Moreover, in the case of where E is a Frechet space, such partial limits do exist.

Proof. The identity

$$\frac{F(\beta_n) - F(\alpha_n)}{\beta_n - \alpha_n} = \frac{F(\beta_n) - F(x)}{\beta_n - x} \cdot \frac{\beta_n - x}{\beta_n - \alpha_n} + \frac{F(\alpha_n) - F(x)}{\alpha_n - x} \cdot \frac{\alpha_n - x}{\alpha_n - \beta_n}$$

implies

$$\frac{F(\beta_n) - F(\alpha_n)}{\beta_n - \alpha_n} \in \operatorname{co}\left\{\frac{F(\beta_n) - F(x)}{\beta_n - x}, \frac{F(\alpha_n) - F(x)}{\alpha_n - x}\right\} \subset \partial_K F(x, \delta)$$

for $|\beta_n - x| \leqslant \delta$, $|\alpha_n - x| \leqslant \delta$. Hence, each possible partial limit of the sequence (3.1) is contained in each partial K-subdifferential $\partial_K F(x, \delta > 0)$ and therefore is contained in $\partial_K F(x)$.

Next, let E be a Frechet space. Since, for some sequence of ε_n-neighborhoods $U_{\varepsilon_n}(0)$ of zero in E, $\varepsilon_n \to 0$,

$$\frac{F(\beta_n) - F(\alpha_n)}{\beta_n - \alpha_n} \subset \partial_K F(x, \beta_n - \alpha_n) \subset \partial_K F(x) + U_{\varepsilon_n}(0) \ ,$$

we have

$$\Big(\frac{F(\beta_n) - F(\alpha_n)}{\beta_n - \alpha_n} + U_{\varepsilon_n}(0)\Big) \bigcap \partial_K F(x) \neq \varnothing \ .$$

Let us choose a sequence of the points x_n from the intersections above. Since $\partial_K F(x)$ is sequentially compact set, $\{x_n\}$ contains a convergent subsequence $x_{n_i} \to x_0$, $x_0 \in \partial_K F(x)$. Then

$$\frac{F(\beta_{n_i}) - F(\alpha_{n_i})}{\beta_{n_i} - \alpha_{n_i}} \to x_0 \ , \quad \text{too.} \quad \square$$

Lemma 3.2. *Let E be a real LCS, $F : I \to E$ be absolutely continuous and K-subdifferentiable a.e. on I. Then, for all $\ell \in E^*$ and a selector $\widehat{\partial}_K F$ of $\partial_K F$,*

$$\ell(F(x))' = \ell(\widehat{\partial}_K F(x)) \quad \textit{for} \quad \textit{a.e.} \quad x \in I \ . \tag{3.2}$$

Proof. By the properties of the K-subdifferential ([12]), $\partial_K \ell(F(x)) = \ell(\partial_K F(x))$. But in view of absolute continuity, the real-valued function $\ell(F(x))$ is differentiable a.e. on I. Hence, $\ell(\partial_K F(x))$ is one-point a.e. on I and

$$\ell(F(x))' = \ell(\partial_K F(x)) \quad \text{a.e. on} \quad \mathrm{I}\ ,$$

whence (3.2) follows. □

Lemma 3.3. *Let E be a real Banach space, $F : I = [a;b] \to E$ be continuous on I and K-subdifferentiable a.e. on I. Then each selector $\widehat{\partial}_K F$ is almost separable-valued.*

Proof. Similarly to [2], note that F is continuous and hence $F(I)$ is contained in a closed separable subspace E_0 of E. Since limits of the elements from E_0 are contained in E_0, and so almost all values of $\widehat{\partial}_K F$ will lie in E_0. □

Lemma 3.4. ([20], Ch. III, § 6). *Let (S, μ) be a finite measure space, $\{f_n\}_{n=1}^{\infty}$ be a sequence of the real-valued μ-integrable functions on S such that $f_n \geqslant 0$ (mod μ) and $\int_S f_n d\mu \leqslant M < \infty$, $n \in \mathbb{N}$. Then the function $h(t) = \varliminf_{n\to\infty} f_n(t)$ is also μ-integrable on S.*

Let us now pass to the main theorems. Recall that Bochner integrability in LCS implies the same for all the factor spaces defined by kernels of the defining seminorms and then completed in factor norms.

Theorem 3.1. *Let E be a real LCS, $F : I = [a;b] \to E$ be a mapping (strongly) absolutely continuous and K-subdifferentiable a.e. on I. Then an arbitrary selector $\widehat{\partial}_K F$ of $\partial_K F$ is Bochner integrable over I and*

$$F(x) = F(a) + (B)\int_a^x \widehat{\partial}_K F(t)\, dt \quad (a \leqslant x \leqslant b)\ . \tag{3.3}$$

Proof. (i). Choose an arbitrary selector $\widehat{\partial}_K F : I \to E$ (a.e.) and $\ell \in E^*$. Taking into account absolute continuity of $\ell(F)$ and equality (3.2), we obtain

$$\ell(F(x) - F(a)) = \ell(F(x)) - \ell(F(a)) = \int_a^x [\ell(F(t))]' dt = \int_a^x \ell(\widehat{\partial}_K F(t))\, dt \quad (a \leqslant x \leqslant b)$$

whence $\widehat{\partial}_K F$ is Pettis integrable over I and

$$F(x) = F(a) + (P)\int_a^x \widehat{\partial}_K F(t)\, dt\ . \tag{3.4}$$

(ii). Let us now show that an arbitrary selector $\widehat{\partial}_K F$ in (3.4) is Bochner integrable. Let $\{\|\cdot\|_j\}_{j\in J}$ be a defining system of seminorms in E, $\widehat{E}_j$ be completions of the factor spaces $E_j = E/\ker\|\cdot\|_j$ in the factor norms $\|\cdot\widehat{\|}_j$, $\varphi_j : E \to E_j$ be the canonical embeddings, $F_j = \varphi_j \circ F : I \to \widehat{E}_j$.

First, applying φ_j to the both sides of (3.4), we obtain

$$F_j(x) - F_j(a) = (P)\int_a^x \widehat{\partial}_K F_j(t)\, dt \quad (j \in J)\ .$$

It follows from here and Lemma 3.3 that each selector $\widehat{\partial}_K F_j = \varphi_j(\widehat{\partial}_K F)$ is almost separable-valued and, hence ([2], Th. 3.5.2), strongly measurable.

Next, introduce the following function sequence on I:

$$\begin{aligned} f_n(t) &= \frac{2^n}{b-a}\Big[F\Big(a + k\frac{b-a}{2^n}\Big) - F\Big(a + (k-1)\frac{b-a}{2^n}\Big)\Big] \\ \text{for} \quad t &\in \Big[a + (k-1)\frac{b-a}{2^n};\ a + k\frac{b-a}{2^n}\Big]\ , \quad k = \overline{1, 2^n}, \quad n \in \mathbb{N}\ . \end{aligned} \tag{3.5}$$

In view of absolute continuity of F it is easy to see that for all $n \in \mathbb{N}$, $j \in J$,

$$\int_a^b \|f_n(t)\|_j \, dt \leqslant V_j(F) < +\infty \ . \tag{3.6}$$

Further, for an arbitrary point $t \in I$ of K-subdifferentiability of F (and, hence of F_j; $\partial_K F_j(t) = \varphi_j(\partial_K F(t))$) there exists a sequence of the segments of type (3.5) that contracts to t. Then, by Lemma 3.1, the set of partial limits of the sequence $\{f_n(t)\}_{n=1}^{\infty}$ is nonempty and is contained in $\partial_K F_j(t)$. Denote by

$$\ell_j(t) = \varliminf_{n\to\infty} \|f_n(t)\|_j \quad (j \in J)$$

and choose $f^j_{n_i}(t)$ for each $i \in \mathbb{N}$ such that

$$\|f^j_{n_i}(t)\|_j < \ell_j(t) + \frac{1}{i} \ .$$

Choosing, if necessary, without loss of generality, a subsequence from $\{f^j_{n_i}(t)\}$, we obtain $f^j_{n_i}(t) \to \widehat{\partial}_K F(t) \in \partial_K F(t)$. It follows that

$$\|\widehat{\partial}_K F(t)\|_j \leqslant \varliminf_{n\to\infty} \|f_n(t)\|_j \quad \text{(a.e. on I)}$$

for all $j \in J$, whence $\forall j \in J$ we obtain

$$\|\widehat{\partial}_K F_j(t)\|\widehat{\ }_j \leqslant \varliminf_{n\to\infty} \|f^j_n(t)\|\widehat{\ }_j \quad \text{(a.e. on I)} \ . \tag{3.7}$$

Since $\widehat{\partial}_K F_j$ is strongly measurable, from (3.6), (3.7), Lemma 3.4, and Fatou's theorem it follows that

$$\int_a^b \|\widehat{\partial}_K F_j(t)\|\widehat{\ }_j dt \leqslant \varliminf_{n\to\infty} \int_a^b \|f^j_n(t)\|\widehat{\ }_j dt \leqslant V_j(F) < +\infty \ ,$$

whence Bochner integrability of $\widehat{\partial}_K F_j$ over I immediately follows ([2], Th. 3.7.4). So, the equality (3.4) implies

$$F_j(x) = F_j(a) + (B)\int_a^x \widehat{\partial}_K F_j(t) \, dt \ . \tag{3.8}$$

Since $\widehat{E}_j$ is a Banach space, in view of (3.8) and property of differentiability a.e. of the Bochner integral, we have

$$\partial_K F_j(t) = \frac{d}{dt} F_j(t) \quad \text{a.e. on} \quad I \ ,$$

i.e., $\partial_K F_j(t) = \varphi_j(\partial_K F(t))$ is one-point a.e. on I. Therefore all selectors $\widehat{\partial}_K F_j(t)$ are coincide a.e. on I and an arbitrary selector $\widehat{\partial}_K F_j(t) = \varphi_j(\partial_K F(t))$ is Bochner integrable on I. This implies Bochner integrability on I of an arbitrary selector $\widehat{\partial}_K F(t)$ of K-subdifferential $\partial_K F(t)$. So, the equality (3.4) takes form of (3.3). ☐

Lemma 3.5. *Let E be a real Frechet space, $f : I = [a;b] \to E$ be Bochner integrable. Then the mapping $F(x) = (B)\int_a^x f(t)dt$ is a.e. differentiable on I; moreover, $F'(x) = f(x)$ a.e. on I.*

Proof. Let $\{\|\cdot\|_j\}_{j\in\mathbb{N}}$ be a countable system of defining seminorms in E. Since, in the notation from the preceding proof, $\widehat{E}_j$ are Banach spaces, for all $j \in \mathbb{N}$ we have

$$\lim_{h\to 0} \left\| \frac{1}{h}(B)\int_x^{x+h} f(t)\,dt - f(x) \right\|_j = 0 \quad \forall x \in [a;b] \backslash e_j \ ,$$

where e_j has measure zero. Then for all $x \in [a;b] \backslash \bigcup_{j\in\mathbb{N}} e_j$, that is a.e. on $[a;b]$, the last equality holds for all $j \in \mathbb{N}$ simultaneously. This precisely means that $F'(x) = f(x)$ a.e. on $[a;b]$. ☐

The next result of this section easily follows from Theorem 3.1, Remark 2.2, (strong) absolute continuity and differentiability a.e. of the indefinite Bochner integral for the case of real Frechet spaces (Lemma 3.5).

Theorem 3.2. *Let E be a real Frechet space, $F : I = [a; b] \to E$. Then the following assertions are equivalent:*

(i) *F is an indefinite Bochner integral, i.e.,*

$$F(x) = F(a) + (B)\int_a^x f(t)\,dt \quad (a \leqslant x \leqslant b\ )\ ;$$

(ii) *F is (strongly) absolutely continuous and differentiable a.e. on I ;*

(iii) *F is absolutely continuous and K-subdifferentiable a.e. on I .*

Remark 3.1. Note that, as it immediately follows from Example 2.2, the result of Theorem 3.2 is not valid in general if E is not a Frechet space and therefore condition of K-subdifferentiability a.e. in Theorem 3.1 cannot be replaced with that of differentiability a.e. on I. This means that the K-subdifferential describes the differential properties of the Bochner integral of LCS-valued mappings better then the usual derivative.

To prove another main result, we need preliminary some equivalent form of Definition 1.3.

Definition 3.1. Let E be a real LCS, $F : I \to E$. For every $P \in \mathcal{P}(I)$ let us introduce a *partial C^0-variation*,

$$C^0V(F, P) = \sum_{k=1}^{n} \triangle F(I_k)\ .$$

The set

$$C^0V(F) = \overline{\bigcup_{P\in\mathcal{P}(I)} C^0V(F, P)}$$

is called the (*total*) *C^0-variation* of F on I. If $C^0V(F)$ is a compact set, we say that F is *compact C^0-variation mapping* ($F \in C^0V(I)$).

It is obvious that $C^0V(F) \subset CV(F)$, whence $(F \in CV(I)) \Rightarrow (F \in C^0V(I))$. Let us establish the converse implication.

Proposition 3.1. *Let E be a real separable LCS, $F : I \to E$. Then*

$$CV(F) = \overline{\mathrm{co}}\ (C^0V(F))\ . \tag{3.9}$$

If, in addition, E is quasicomplete, then the conditions $F \in CV(I)$ and $F \in C^0V(I)$ are equivalent.

Proof. For all $P \in \mathcal{P}(I)$ and $\ell \in E^*$, the identities

$$\begin{aligned}\sup \ell(CV(F, P)) &= \sup \ell\Big(\sum_{k=1}^{n} \overline{w}F(I_k)\Big) = \sup \sum_{k=1}^{n} \ell(\overline{w}F(I_k)) = \sum_{k=1}^{n} \sup \ell(\overline{w}F(I_k))\\ &= \sup\Big(\sum_{k=1}^{n} \ell(\overline{\triangle F(I_k)})\Big) = \sup \ell\Big(\sum_{k=1}^{n} \triangle F(I_k)\Big) = \sup \ell(C^0V(F, P))\end{aligned}$$

hold true. Hence, passing to the closures of unions by P, we obtain

$$\sup \ell(CV(F)) = \sup \ell(C^0V(F)) \quad (\forall \ell \in E^*)\ ,$$

and so

$$\sup \ell(CV(F)) = \sup \ell(\overline{\mathrm{co}}\ C^0V(F)) \quad (\forall \ell \in E^*)\ .$$

Using functional separability of E ([21], 2.1.4), this implies (3.9).

Finally, in the case of quasicomplete E, compactness of $C^0V(F)$ implies compactness for its closed convex hull ([18], II.4.3) and $(F \in C^0V(I)) \Rightarrow (F \in CV(I))$ by virtue of (3.9). $\square$

Theorem 3.3. *Let E be a real complete LCS, $f : I = [a;b] \to E$ be Bochner integrable over I. Then, for an arbitrary $C \in E$, the mapping*

$$F(x) = C + (B)\int_a^x f(t)\,dt \tag{3.10}$$

has compact variation on I.

Proof. First, suppose that E is a Banach space. Since, for each partition $P \in \mathcal{P}(I)$,

$$\begin{aligned} C^0V(F,P) &= \sum_{k=1}^{n} \overline{\left\{(B)\int_{x_1^k}^{x_2^k} f(t)\,dt \;\middle|\; x_1^k,\ x_2^k \in I_k\right\}} \\ &= \overline{\left\{(B)\int_{\bigcup_{k=1}^n [x_1^k, x_2^k]} f(t)\,dt \;\middle|\; [x_1^k, x_2^k] \subset I_k\right\}}\,, \end{aligned}$$

it follows from Definition 3.1 that

$$C^0V(F) \subset \left\{ \pm (B)\int_A f(t)\,dt \;\middle|\; A \in \mathcal{B}(I)\right\}\,, \tag{3.11}$$

where $\mathcal{B}(I)$ is the Borel σ-algebra on I.

Let us now check that $C^0V(F)$ is totally bounded in E.

Given $\varepsilon > 0$, choose, by virtue of the definition of Bochner integrability ([21], X.3), a simple mapping

$$f_\varepsilon(t) = \sum_{k=1}^{n} c_k \cdot \chi_{T_k}(t) \quad \Bigl(c_k \in E\ ,\ I = \bigcup_{k=1}^{n} T_k\ ,\quad T_k \quad \text{are disjoint}\Bigr)$$

such that

$$\int_I \|f(t) - f_\varepsilon(t)\| dt < \frac{\varepsilon}{2(b-a)}\ .$$

Hence, for each $A \in \mathcal{B}(I)$,

$$\left\|(B)\int_A f(t)\,dt - (B)\int_A f_\varepsilon(t)\,dt\right\| \leqslant \int_A \|f(t) - f_\varepsilon(t)\,|dt < \frac{\varepsilon}{2(b-a)}\ . \tag{3.12}$$

In addition,

$$\begin{aligned} (B)\int_A f_\varepsilon(t)\,dt &= \sum_{k=1}^{n} c_k \cdot \mathrm{mes}(T_k \bigcap A) = \mathrm{mes}(A) \cdot \sum_{k=1}^{n} c_k \cdot \frac{\mathrm{mes}(T_k \bigcap A)}{\mathrm{mes}(A)} \\ &\in \mathrm{mes}(I) \cdot \overline{\mathrm{abs.co}}\{c_k\}_{k=1}^n =: K\ . \end{aligned}$$

The set K is compact in E, not depending on the choice of A. Let us choose a finite $\varepsilon/2$-net $N = \{x_1, \ldots, x_m\}$ for K. Then, in view of (3.12), N is ε-net for the set in the right-hand side of (3.11) and, therefore, N is an ε-net for $C^0V(F)$. This implies, since $C^0V(F)$ is closed, compactness of this set and thus, by Proposition 3.1, compactness of $CV(F)$.

Secondly, let E be a real complete LCS with a defining system of seminorms $\{\|\cdot\|_j\}_{j\in J}$. In the notation from the preceding, we see that separation of E implies

$$\bigcap_{j\in J} \varphi_j^{-1}(\varphi_j(y)) = \{y\} \quad \text{for each} \quad y \in E\ . \tag{3.13}$$

Applying $\varphi_j (j \in J)$ to (3.10) we obtain

$$\widehat{F}_j(x) = \widehat{F}_j(a) + \varphi_j\Big((B)\int_a^x f(t)\,dt\Big) = \widehat{F}_j(a) + (B)\int_a^x \widehat{f}_j(t)\,dt\ ,$$

where $\widehat{f}_j(t) = \varphi_j(f(t))$. Hence, according to the first part of the proof, the sets $CV(\widehat{F}_j)$ are compact in $\widehat{E}_j$. But E is a projective limit of the Banach spaces $\widehat{E}_j$ [18] and, therefore, can be closely and continuously embedded into the product $\Pi_{j\in J}\widehat{E}_j$ ([18], II.5.3). Hence, $CV(F)$ can be injectively (in view of (3.13)) continuously and closely embedded into the product

$$\prod_{j\in J} CV(\widehat{F}_j)\ .$$

But the last set is compact by Tychonoff theorem ([22], I.9.5), whence compactness of $CV(F)$ follows. □

From the definition of Radon-Nikodym property (see Introduction) and Theorem 3.3, we immediately get the following.

Corollary 3.1. *Let E be a real Radon-Nikodym space* [7]*. Then each absolutely continuous mapping $F : I \to E$ has a compact variation on I. This is true, in particular, for all reflexive Banach spaces E.*

In the conclusion, let us give an example of a nowhere K-subdifferentiable mapping having compact variation. First, recall the definition of a strongly Pettis integrable mapping [23].

Definition 3.2. Let E be a real LCS, $f : I = [a; b] \to E$, $\mathfrak{B}(I)$ be the Borel σ-algebra over I. We say that f is *strongly Pettis integrable over I* if f is Pettis integrable over I and the set

$$m_f(\mathfrak{B}(I)) = \Big\{(P)\int_B f(x)\,dx \mid B \in \mathfrak{B}(I)\Big\}$$

is relatively compact in E.

The following proposition can be proved quite similarly to Theorem 3.3.

Theorem 3.4. *Let E be a real complete LCS, $f : I = [a; b] \to E$ be strongly Pettis integrable over I. Then for an arbitrary $C \in E$, the mapping*

$$F(x) = C + (P)\int_a^x f(t)\,dt \tag{3.14}$$

has compact variation on I.

Example 3.1. At first note that, according to ([23], Theorem 7), every mapping into a Banach space, which is strongly measurable and Pettis integrable over I, is strongly Pettis integrable and, therefore, has compact variation.

Secondly (see [24], remark to Theorem 1), for an arbitrary infinite-dimensional Banach space E, there exists a strongly measurable and Pettis integrable mapping $f : I \to E$ such that

$$\lim_{h\to 0}\Big\|\frac{1}{h}(P)\int_t^{t+h} f(x)\,dx\Big\| = \infty \quad (\forall t \in I).$$

This immediately implies that the corresponding mapping (3.14) is nowhere K-subdifferentiable on I.

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Taurida National V. Vernadsky University, 4 Vernadsky ave., Simpheropol, 95007, Ukraine
E-mail address: `old@crimea.edu`
E-mail address: `oiv@crimea.edu`

Taurida National V. Vernadsky University, 4 Vernadsky ave., Simpheropol, 95007, Ukraine
E-mail address: `fedyor@mail.ru`

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