QUANTUM UNIVERSE ON EXTREMELY SMALL SPACE-TIME SCALES

V.E. KUZMICHEV, V.V. KUZMICHEV

Bogolyubov Institute for Theoretical Physics, Nat. Acad. of Sci. of Ukraine
(14b, Metrolohichna Str., Kyiv 03680, Ukraine)

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The semiclassical approach to the quantum geometrodynamical model is used for the description of the properties of the Universe on extremely small space-time scales. Under this approach, the matter in the Universe has two components of the quantum nature which behave as antigravitating fluids. The first component does not vanish in the limit $\hbar \to 0$ and can be associated with dark energy. The second component is described by an extremely rigid equation of state and goes to zero after the transition to large space-time scales. On small space-time scales, this quantum correction turns out to be significant. It determines the geometry of the Universe near the initial cosmological singularity point. This geometry is conformal to a unit four-sphere embedded in a five-dimensional Euclidean flat space. During the consequent expansion of the Universe, when reaching the post-Planck era, the geometry of the Universe changes into that conformal to a unit four-hyperboloid in a five-dimensional Lorentz-signatured flat space. This agrees with the hypothesis about the possible change of geometry after the origin of the expanding Universe from the region near the initial singularity point. The origin of the Universe can be interpreted as a quantum transition of the system from a region in the phase space forbidden for the classical motion, but where a trajectory in imaginary time exists, into a region, where the equations of motion have the solution which describes the evolution of the Universe in real time. Near the boundary between two regions, from the side of real time, the Universe undergoes almost an exponential expansion which passes smoothly into the expansion under the action of radiation dominating over matter which is described by the standard cosmological model.

1. Introduction

It is accepted that the present-day Universe as a whole can be considered as a cosmological system described by the standard model based on general relativity [1, 2]. According to the Standard Big-Bang Model [2, 3], the early Universe was very hot and dense. In order to describe that era, one must take into account that the Universe has passed in the course of its evolution through a stage with quantum degrees of freedom of the gravitational and matter fields before turning into the cosmological system, whose properties are described well by general relativity. This means that a consistent description of the Universe as a nonstationary cosmological system should be based on quantum general relativity in the form admitting the passage to general relativity in semiclassical limit $\hbar \to 0$ [4, 5].

A consistent quantum theory of gravity, in principle, can be constructed on the basis of the Hamiltonian formalism with the application of the canonical quantization method. The first problem on this way is to choose generalized variables. Following, e.g., Refs. [6–8], it is convenient to choose metric tensor components and matter fields as such variables. But the functional equations obtained in this approach prove to be insufficiently suitable for specific problems of quantum theory of gravity and cosmology. These equations do not contain a time variable in an explicit form. This, in turn, gives rise to the problem of interpretation of the state vector of the Universe (see, e.g., the discussion in Ref. [3] and references therein). A cause of the failure can be easily understood with the help of Dirac's constraint system theory [9]. It is found that the structure of constraints in general relativity is such that the variables which correspond to true dynamical degrees of freedom cannot be singled out from canonical variables of geometrodynamics. This difficulty is caused by the absence of a predetermined way to identify space-time events in generally covariant theory [10].

One of the possible versions of a theory with a well-defined time variable is proposed in Refs. [11, 12] in the case of the homogeneous, isotropic, and closed Universe. The Universe is supposed to be filled with a homogeneous scalar field which stands for the primordial matter[1] and (macroscopic) relativistic matter associated with the *material* reference frame. As calculations have demonstrated [11, 12], the equations of the quantum model may be reduced to the form, in which the matter energy density in the Universe has a component which is a condensate of massive quanta of a scalar field.

[1] Since we deal with the quantum theory, we should describe the matter content of the Universe by some fundamental Lagrangians of fields.

Under the semiclassical description, this component behaves itself as an antigravitating fluid. Such a property has the quantum nature, and it is connected with the fact that the states with all possible masses of a condensate contribute to the state vector of the quantum Universe. If one discards the corresponding quantum corrections, the quantum fluid degenerates into a dust, i.e. the matter component of the energy density which is commonly believed to make a dominant contribution to the mass-energy of the ordinary matter in the present Universe in the standard cosmological model. Let us note that the presence of a condensate in the Universe, as well as the availability of a dust representing an extreme state of a condensate, is not presupposed in the initial Lagrangian of the theory. An antigravitating condensate arises out of the quantum description of fields. If one supposes that the properties of our Universe are described in an adequate manner by such a quantum theory, an antigravitating condensate being found out can be associated with dark energy [12]. Assuming that particles of a condensate can decay to baryons, leptons (or to their antiparticles), and particles of dark matter, one can describe the percentage of baryons, dark matter, and dark energy observed in the Universe [13].

In the semiclassical limit, the negative pressure fluid arises as a remnant of the early quantum era. This antigravitating component of the energy density does not vanish in the limit $\hbar \to 0$. In addition to this component, the stress-energy tensor contains the term vanishing after the transition to general relativity, i.e. to large space-time scales. However, on small space-time scales, quantum corrections $\sim \hbar$ turn out to be significant. As is shown in this paper, the effects caused by these corrections determine the equation of state of matter and the geometry near the initial cosmological singularity point. They define a boundary condition that should be imposed on the state vector at the origin, so that a nucleation of the Universe from the initial cosmological singularity point becomes possible.

In this paper, we use the modified Planck system of units: $l_P = \sqrt{2G\hbar/(3\pi c^3)}$ is taken as a unit of length, $\rho_P = 3c^4/(8\pi G l_P^2)$ is a unit of energy density, and so on. All relations are written for dimensionless quantities.

2. Equations of Motion

2.1. Classical model

Let us consider the homogeneous, isotropic, and closed Universe which is described by the Robertson–Walker metric

$$ds^2 = d\tau^2 - a^2 d\Omega_3^2, \tag{1}$$

where τ is the proper time, a is the cosmic scale factor, and $d\Omega_3^2$ is a line element on a unit three-sphere. It is convenient to pass to a new time variable η,

$$d\tau = aN d\eta, \tag{2}$$

where N is the lapse function that specifies the time reference scale.

We assume that the Universe is originally filled with a homogeneous scalar field ϕ and a perfect fluid as a macroscopic medium which defines the so-called *material* reference frame [10, 11, 14]. In the model of the Universe under consideration, the action has the form

$$S = \int d\eta \left\{ \pi_a \frac{da}{d\eta} + \pi_\phi \frac{d\phi}{d\eta} + \pi_\Theta \frac{d\Theta}{d\eta} + \pi_{\tilde\lambda} \frac{d\tilde\lambda}{d\eta} - H \right\}, \tag{3}$$

where π_a, π_ϕ, π_Θ, $\pi_{\tilde\lambda}$ are the momenta canonically conjugate with the variables a, ϕ, Θ, $\tilde\lambda$,

$$H = \frac{N}{2}\left\{-\pi_a^2 - a^2 + a^4[\rho_\phi + \rho]\right\} +$$

$$+\lambda_1 \left\{\pi_\Theta - \frac{1}{2}\, a^3 \rho_0 s\right\} + \lambda_2 \left\{\pi_{\tilde\lambda} + \frac{1}{2}\, a^3 \rho_0\right\}, \tag{4}$$

is the Hamiltonian,

$$\rho_\phi = \frac{2}{a^6}\, \pi_\phi^2 + V(\phi) \tag{5}$$

is the energy density of a scalar field with the potential $V(\phi)$, and $\rho = \rho(\rho_0, s)$ is the energy density of a perfect fluid which is a function of the density of the rest mass ρ_0 and the specific entropy s. The quantity Θ is the thermasy (potential for the temperature, $T = \Theta_{,\nu} U^\nu$), $\tilde\lambda$ is the potential for the specific free energy f taken with an inverse sign, $f = -\tilde\lambda_{,\nu} U^\nu$, and U^ν is the four-velocity. The momenta π_{ρ_0} and π_s conjugate with the variables ρ_0 and s vanish identically,

$$\pi_{\rho_0} = 0, \qquad \pi_s = 0. \tag{6}$$

Hamiltonian (4) of such a system has the form of a linear combination of constraints and weakly vanishes (in Dirac's sense [9]),

$$H \approx 0, \tag{7}$$

where the sign $\approx$ means that the Poisson brackets must all be worked out before the use of the constraint equations. The quantities N, λ_1, and λ_2 are Lagrange multipliers. The variation of action (3) with respect to them leads to three constraint equations

$$-\pi_a^2 - a^2 + a^4[\rho_\phi + \rho] \approx 0, \quad \pi_\Theta - \frac{1}{2}\,a^3\rho_0 s \approx 0,$$

$$\pi_{\tilde{\lambda}} + \frac{1}{2}\,a^3\rho_0 \approx 0. \tag{8}$$

It follows from the conservation of these constraints in time that the conservation laws hold,

$$E_0 \equiv a^3\rho_0 = \text{const}, \qquad s = \text{const}, \tag{9}$$

where the first relation describes the conservation law of a macroscopic quantity which characterizes the number of particles of a perfect fluid, and the second equation represents the conservation of the specific entropy. With regard for these conservation laws and Eqs. (6), one can discard degrees of freedom corresponding to the variables ρ_0 and s and convert the second-class constraints into first-class constraints [11] in accordance with Dirac's proposal.

The equation of motion for the classical dynamical variable $\mathcal{O} = \mathcal{O}(a, \phi, \pi_a, \pi_\phi, \dots)$ has the form

$$\frac{d\mathcal{O}}{d\eta} \approx \{\mathcal{O}, H\}, \tag{10}$$

where H is Hamiltonian (4), and $\{.,.\}$ are Poisson brackets.

2.2. Quantum model

In quantum theory, the first-class constraint equations (8) become constraints on the state vector Ψ. Passing from the classical variables to corresponding operators and using the conservation laws (9), we obtain three equations

$$\left\{-\partial_a^2 + a^2 - a^4[\rho_\phi + \rho]\right\}\Psi = 0,$$

$$\left\{-i\partial_\Theta - \frac{1}{2}\,E_0 s\right\}\Psi = 0, \quad \left\{-i\partial_{\tilde{\lambda}} + \frac{1}{2}\,E_0\right\}\Psi = 0. \tag{11}$$

It is convenient to pass from the generalized variables Θ and $\tilde{\lambda}$ to the non-coordinate co-frame

$$h\,d\tau = s\,d\Theta\, -\, d\widetilde{\lambda}, \qquad h\,dy = s\,d\Theta\, +\, d\tilde{\lambda}, \tag{12}$$

where $h = \frac{\rho+p}{\rho_0}$ is the specific enthalpy which plays the role of inertial mass, p is the pressure, τ is proper time in every point of space, and y is a supplementary variable. The corresponding derivatives commute between themselves, $[\partial_\tau,\, \partial_y] = 0$.

It follows from the first equation in system (11) that it is convenient to choose a perfect fluid in the form of relativistic matter. Introducing the quantity

$$E \equiv a^4\rho = \text{const}, \tag{13}$$

we come to the equations which describe the quantum Universe [11]

$$\left\{-\,i\,\partial_{\tau_c} - \frac{1}{2}\,E_0\right\}\Psi = 0, \qquad \partial_y\Psi = 0, \tag{14}$$

$$\left\{-\,\partial_a^2 + a^2 - 2a\hat{H}_\phi - E\right\}\Psi = 0, \tag{15}$$

where τ_c is the time variable connected with the proper time τ by the differential relation $d\tau_c = h\,d\tau$,

$$\hat{H}_\phi = \frac{1}{2}\,a^3\left[-\frac{2}{a^6}\,\partial_\phi^2 + V(\phi)\right] \tag{16}$$

is the operator of mass-energy of a scalar field in a comoving volume $\frac{1}{2}\,a^3$. It follows from Eqs. (14) that Ψ does not depend on the variable y. The first equation of system (14) has a particular solution in the form

$$\Psi = e^{\,\frac{i}{2}\,E\bar{\tau}}|\psi\rangle, \tag{17}$$

where $\bar{\tau} = \frac{E_0}{E}\tau_c$ is the rescaled time variable. The state vector $|\psi\rangle$ is defined in the space of two variables a and ϕ and is determined by the equation

$$\left(-\,\partial_a^2 + a^2 - 2a\hat{H}_\phi - E\right)|\psi\rangle = 0. \tag{18}$$

This equation describes the state of the Universe with a definite value of the parameter E. The vector $|\psi\rangle$ represents the dynamical state of the Universe at some instant of time η_0 which is connected with time $\bar{\tau}$ by the relation $\bar{\tau} = \frac{4}{3}\int^{\eta_0} N d\eta$. Considering the vector $|\psi\rangle$ as an immovable vector of the Heisenberg representation, one can describe the motion of the quantum Universe by the equation

$$\langle\psi|\frac{1}{N}\,\frac{d}{d\eta}\hat{\mathcal{O}}|\psi\rangle = \frac{1}{N}\,\frac{d}{d\eta}\langle\psi|\hat{\mathcal{O}}|\psi\rangle = \frac{1}{i}\,\langle\psi|[\hat{\mathcal{O}},\,\frac{1}{N}\hat{H}]|\psi\rangle, \tag{19}$$

where $[.,.]$ is a commutator, and $\hat{H}$ is determined by expression (4), in which all dynamical variables are substituted with operators. The observable $\hat{\mathcal{O}}$ corresponds to the classical dynamical variable $\mathcal{O}$. For $\hat{\mathcal{O}} = a$, we obtain

$$\langle\psi| - i\partial_a|\psi\rangle = -\langle\psi|a\frac{da}{d\tau}|\psi\rangle. \tag{20}$$

In the classical theory, the corresponding momentum has the form

$$\pi_a = \partial_a S = -a\frac{da}{d\tau} \equiv -a\dot{a}, \tag{21}$$

where S is the action. For $\hat{\mathcal{O}} = -i\partial_a$, we find

$$\langle\psi| - i\frac{1}{N}\frac{d}{d\eta}\partial_a|\psi\rangle = \langle\psi|a - \frac{2}{a^3}\,\partial_\phi^2 - 2a^3V(\phi)|\psi\rangle. \tag{22}$$

3. Scalar Field Model

Equation (18) can be integrated with respect to ϕ, if one determines the form of the potential $V(\phi)$. As in Ref. [12], we consider the solution of Eq. (18) in the era when the field ϕ oscillates with a small amplitude near the minimum of its potential at the point $\phi = \sigma$. Then $V(\phi)$ can be approximated by the expression

$$V(\phi) = \rho_\sigma + \frac{m_\sigma^2}{2}\,(\phi - \sigma)^2, \tag{23}$$

where $\rho_\sigma = V(\sigma)$, $m_\sigma^2 = [d^2V(\phi)/d\phi^2]_\sigma > 0$. If $\phi = \sigma$ is the point of absolute minimum, then $\rho_\sigma = 0$, and the state σ corresponds to the true vacuum of a primordial scalar field, while the state with $\rho_\sigma \neq 0$ matches with the false vacuum [15].

Introducing a new variable

$$x = \left(\frac{m_\sigma a^3}{2}\right)^{1/2}(\phi - \sigma), \tag{24}$$

which describes a deviation of the field ϕ from its equilibrium state, and defining the harmonic oscillator functions $\langle x|u_k\rangle$ as solutions of the equation

$$\left(-\partial_x^2 + x^2\right)|u_k\rangle = (2k+1)|u_k\rangle, \tag{25}$$

where $k = 0, 1, 2, ...$ is the number of a state of the oscillator, we find

$$\hat{H}_\phi|u_k\rangle = \left(M_k + \frac{1}{2}\,a^3\rho_\sigma\right)|u_k\rangle, \tag{26}$$

where the quantity

$$M_k = m_\sigma\left(k + \frac{1}{2}\right) \tag{27}$$

can be interpreted as an amount of matter-energy (or mass) in the Universe related to a scalar field. This energy is represented in the form of a sum of the excitation quanta of spatially coherent oscillations of the field ϕ about the equilibrium state σ, k is the number of these excitation quanta. The mentioned oscillations correspond to a condensate of zero-momentum ϕ quanta with the mass m_σ.

We look for a solution of Eq. (18) in the form of a superposition of the states with different masses M_k,

$$|\psi\rangle = \sum_k |f_k\rangle|u_k\rangle. \tag{28}$$

Using the orthonormality of $|u_k\rangle$, we obtain the equation for the vector $|f_k\rangle$:

$$\left(-\partial_a^2 + U_k - E\right)|f_k\rangle = 0, \tag{29}$$

where

$$U_k = a^2 - 2aM_k - a^4\rho_\sigma \tag{30}$$

is the effective potential. In the case $\rho_\sigma = 0$, this equation is exactly integrable [11]. The corresponding eigenvalue is equal to

$$E \equiv E_{n,k} = 2n + 1 - M_k^2, \tag{31}$$

where $n = 0, 1, 2, ...$ is the number of a state of the quantum Universe with the mass M_k in the potential well (30). The vectors $|f_k\rangle$ and $|f_{k'}\rangle$ at $k \neq k'$ are, generally speaking, nonorthogonal between themselves. So that, the transition probability $w(n, k \to n', k') = |\langle f_{k'}|f_k\rangle|^2$ is nonzero. For example, the probability of the transition of the Universe from the ground (vacuum) state $n = 0$ to any other state obeys the Poisson distribution

$$w(0, k \to n', k') = \frac{\langle n'\rangle^{n'}}{n'!}\,e^{-\langle n'\rangle}, \tag{32}$$

where $\langle n'\rangle = \frac{1}{2}\,(M_{k'} - M_k)^2$ is the mean value of the quantum number n'.

Substituting Eq. (28) into Eq. (22), we obtain

$$\langle f_k| - \frac{i}{N}\frac{d}{d\eta}\partial_a|f_k\rangle = \langle f_k|a - 2a^3\rho_\sigma - 4M_k|f_k\rangle + \Delta_k, \tag{33}$$

where

$$\Delta_k = -3m_\sigma\langle f_k|\sum_{k'}\langle u_k|\partial_x^2|u_{k'}\rangle|f_{k'}\rangle. \tag{34}$$

Here, k' takes the values k and $k\pm2$. This term describes the component of the pressure of the condensate caused by the motion of ϕ quanta in the phase space with the momentum $-i\partial_x$. Using Eq. (25), we find

$$\Delta_k = 3M_k\langle f_k|f_k\rangle -$$

$$-\frac{3}{2}\sqrt{\left(M_k+\frac{3}{2}\,m_\sigma\right)\left(M_k+\frac{1}{2}\,m_\sigma\right)}\langle f_k|f_{k+2}\rangle$$

$$-\frac{3}{2}\sqrt{\left(M_k-\frac{3}{2}\,m_\sigma\right)\left(M_k-\frac{1}{2}\,m_\sigma\right)}\langle f_k|f_{k-2}\rangle. \qquad (35)$$

In the case $k\gg1$, the masses $M_{k\pm2}\simeq M_k\gg\frac{1}{2}\,m_\sigma$ and, according to Eq. (29), the vectors $|f_{k\pm2}\rangle\simeq|f_k\rangle$. Then

$$\Delta_k = 0 \qquad \text{at} \qquad k\gg1. \qquad (36)$$

This means that the contributions to the sum with respect to k' in Eq. (34) from the different k-states of the Universe are mutually cancelled. As a result, the pressure of a condensate is determined only by its quantum properties (see Eq. (44) below). We note that if one discards the contributions from the transition amplitudes $\langle f_k|f_{k\pm2}\rangle$, a condensate turns into an aggregate of separate macroscopic bodies with zero pressure (dust) [12]. The existence of this limit argues in favor of the reliability of this quantum model.

4. Semiclassical Approach

4.1. Einstein-type equations

In order to give the physical meaning to the different quantities emergent in this theory, we reduce Eqs. (29) and (33) to the form of the Einstein equations. To this end, we choose the vector $|f_k\rangle$ in the form

$$\langle a|f_k\rangle = \frac{\text{const}}{\sqrt{\partial_a S(a)}}\,e^{iS(a)}, \qquad (37)$$

where S is the unknown function of a (we omit the index k here and below). Substituting Eq. (37) into (29) and (33) and taking (36) into account, we obtain the equations which can be represented in the "standard" form for the perfect fluid source

$$\frac{1}{a^4}\left(\partial_a S\right)^2-\rho_m-\rho_u+\frac{1}{a^2}=0, \qquad (38)$$

$$\frac{1}{a^2}\,\frac{d}{d\tau}\left(\partial_a S\right)+\frac{1}{2}\left(\rho_m-3p_m\right)+\widetilde{\rho}_u-\frac{1}{a^2}=0, \qquad (39)$$

where

$$\rho_u = \frac{1}{a^4}\left\{\frac{3}{4}\left(\frac{\partial_a^2 S}{\partial_a S}\right)^2-\frac{1}{2}\,\frac{\partial_a^3 S}{\partial_a S}\right\} \qquad (40)$$

and

$$\widetilde{\rho}_u = \frac{i}{2a^2}\,\frac{d}{d\tau}\left(\frac{\partial_a^2 S}{\partial_a S}\right) \qquad (41)$$

are the quantum corrections to the stress-energy tensor, $\rho_u\sim\hbar^2$, and $\widetilde{\rho}_u\sim\hbar$ (in ordinary units [12]),

$$\rho_m=\rho_k+\rho_\sigma+\rho, \qquad p_m=p_k+p_\sigma+p \qquad (42)$$

can be interpreted as the (effective) energy density and the isotropic pressure having the form of the sum of the components,

$$\rho_k=\frac{2M_k}{a^3}, \qquad \rho_\sigma\equiv V(\sigma)\equiv\frac{\Lambda}{3}, \qquad \rho=\frac{E}{a^4}, \qquad (43)$$

Λ is the cosmological constant. The equations

$$p_k=-\rho_k, \qquad p_\sigma=-\rho_\sigma, \qquad p=\frac{1}{3}\,\rho, \qquad (44)$$

stand for the equations of state. The equations of state for the vacuum component $\rho_\sigma=$ const and the relativistic matter ρ are dictated by the formulation of the problem. The vacuum-type equation of state of a condensate with the density ρ_k, which does not remain constant throughout the evolution of the Universe but decreases according to a power law with increase of a, follows from the condition of consistency of Eqs. (38) and (39).

From (42)–(44), we can conclude that a condensate behaves itself as an antigravitating medium. Its antigravitating effect has a purely quantum nature. Its appearance is determined by the fact that the Universe's state vector (28) is a superposition of quantum states with all possible values of the quantum number k.

In the classical limit ($\hbar=0$), the terms ρ_u and $\widetilde{\rho}_u$ can be discarded, and Eqs. (38) and (39) reduce to the Einstein equations which predict an accelerating expansion of the Universe in the era with $\rho_k>\frac{2}{3}\,\rho$, even if $\Lambda=0$. Since $\rho\sim a^{-4}$ decreases with a more rapidly than $\rho_k\sim a^{-3}$ (or even $\sim a^{-2}$ [12]), the era of accelerating expansion should begin with increasing a, even if the state with $\rho_k<\frac{2}{3}\,\rho$ and $\Lambda\sim0$ existed in the past, when the expansion was decelerating. The condensate of a quantized primordial scalar field can be identified with the dark energy [12, 13].

In this connection, it should be emphasized that the energy density dynamics of the condensate ρ_k has to be considered in accordance with quantum theory. A dust component, as such, is a limiting case of the quantum description, when the transition amplitudes $\langle f_k | f_{k\pm 2}\rangle$ in Eq. (35) are not taken into account. According to quantum theory, the quantum fluctuations of density which depend on time and the scale factor in an explicit form are imposed on a classical background. Equation (35) implies that it is convenient to choose the energy density of a dust as such a background. Then the energy density of the condensate can be written in the form $\rho_k = \rho_k^{\rm dust} + \delta\rho_k$, where $\delta\rho_k$ describes quantum fluctuations. At such a semiclassical description, the condensate is a source of ordinary matter and dark matter [12] which together form $\rho_k^{\rm dust}$. Quantum fluctuations $\delta\rho_k$ contribute to the vacuum component, so that the latter appears to be non-zero, even if $\rho_\sigma = 0$. Thus, taking quantum fluctuations into account guarantees the satisfaction of the energy conservation law for the total stress-energy tensor.

Let us calculate the corrections ρ_u and $\widetilde{\rho}_u$. These terms are essential in the very early Universe at $a < 1$ [2]. This quantum theory predicts the quantum origin (nucleation) of the Universe from the region $a \sim 0$ [11]. This means that the state vector in this region is constant, $\langle a \sim 0 | f_k\rangle = \text{const}$. For such a state,

$$S = \frac{i}{2}\,\ln a + \text{const} \tag{45}$$

and the quantum corrections (40) and (41) are equal to

$$\rho_u = -\frac{1}{4a^6}, \qquad \widetilde{\rho}_u = \frac{i\dot{a}}{2a^4} = -\frac{i}{2a^5}\,\partial_a S = \frac{1}{4a^6}, \tag{46}$$

where we used representation (21) for the calculation of $\widetilde{\rho}_u$. It can be done in the semiclassical approach under consideration[3].

With regard for Eqs. (46), Eqs. (38) and (39) can be reduced to the form of the standard Einstein equations for the homogeneous, isotropic, and closed Universe:

$$\left(\frac{\dot{a}}{a}\right)^2 = \rho_{\rm tot} - \frac{1}{a^2}, \qquad \frac{\ddot{a}}{a} = -\frac{1}{2}\left[\rho_{\rm tot} + 3p_{\rm tot}\right], \tag{47}$$

where the quantities

$$\rho_{\rm tot} = \rho_m + \rho_u, \qquad p_{\rm tot} = p_m + p_u \tag{48}$$

describe the total energy density and the pressure of the matter in the Universe which take its quantum nature into account in the semiclassical approximation. The quantum correction ρ_u may be identified with the ultrastiff matter with the equation of state

$$p_u = \rho_u, \tag{49}$$

where p_u is the pressure. This 'matter' has quantum origin.

Let us estimate the ratio of the energy density $|\rho_u|$ to ρ_m. Passing to the ordinary units, we have

$$\mathcal{R} \equiv \left[\left(\frac{2G\hbar}{3\pi c^3}\right)^2 \frac{1}{4a^6}\right] \Big/ \left[\frac{8\pi G}{3c^4}\,\rho_m\right]. \tag{50}$$

where ρ_m and a are measured, respectively, in GeV/cm^3 and cm. For our Universe today, $\rho_m \sim 10^{-5}$ GeV/cm^3, $a \sim 10^{28}$ cm, and

$$\mathcal{R}_{\rm today} \sim 10^{-244}, \tag{51}$$

i.e. the quantum correction may be neglected to an accuracy of $\sim O(10^{-244})$. In the Planck era, $\rho_m \sim 10^{117}$ GeV/cm^3, $a \sim 10^{-33}$ cm, and the relation

$$\mathcal{R}_{\rm Planck} \sim 1 \tag{52}$$

shows that the densities ρ_m and ρ_u are of the same order of magnitude.

4.2. Quantum effects on sub-Planck scales

On sub-Planck scales, $a < 1$, the contributions from the condensate, cosmological constant, and curvature may be neglected. As a result, the equations of the model take the form

$$\frac{1}{2}\,\dot{a}^2 + U(a) = 0, \qquad \ddot{a} = -\frac{dU}{da}, \tag{53}$$

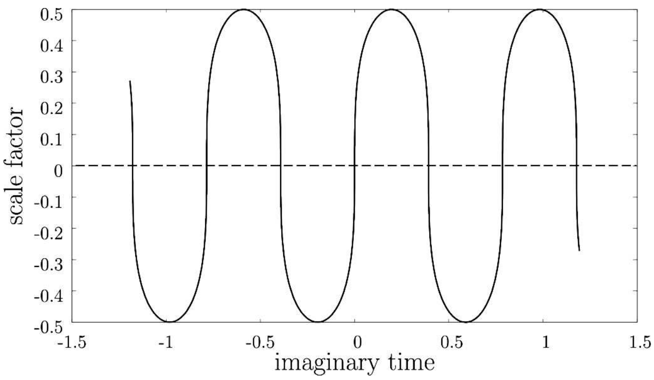
where

$$U(a) \equiv \frac{1}{2}\left[\frac{1}{4a^4} - \frac{E}{a^2}\right]. \tag{54}$$

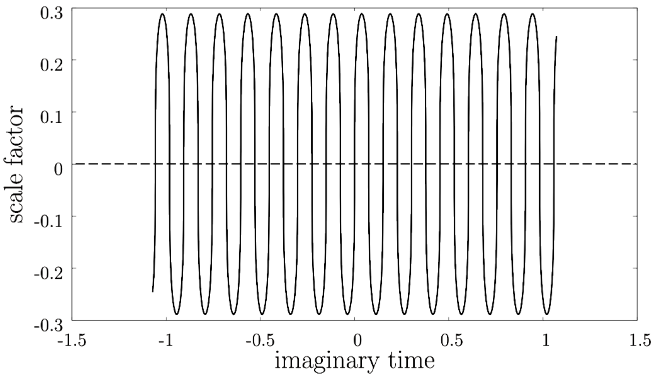
These equations are similar to ones of Newtonian mechanics. Using this analogy, they can be considered as equations which describe the motion of a 'particle' with unit mass and zero total energy under the action of the force $-\frac{dU}{da}$, $U(a)$ is the potential energy, and $a(\tau)$ is a generalized variable. A point $a_c = \frac{1}{2\sqrt{E}}$, where $U(a_c) = 0$, divides the region of motion of a 'particle' into the subregion $a < a_c$, where the classical motion of

[2] For the present-day Universe, we have $a \sim 10^{61}$ in accepted dimensionless units.

[3] Let us note that the presence of a minus sign in ρ_u (46) is not extraordinary. According to quantum field theory, for instance, vacuum fluctuations make a negative contribution to the field energy per unit area (the Casimir effect).



(a) $E = 1$



(b) $E = 3$

Fig. 1. Scale factor $\tilde{a}$ vs imaginary time t for the cases $E = 1$, which corresponds to the ground state $n = 0$ (a), and $E = 3$ for the state with $n = 1$ (b)

a 'particle' is forbidden, and the subregion $a > a_c$, where the classical trajectory of a 'particle' moving in real time τ exists.

In the subregion $a < a_c$, there exists the classical trajectory of a 'particle' moving in imaginary time $t = -i\tau + \text{const}$ in the potential $-U(a)$. Denoting the corresponding solution as $\tilde{a}$, we find

$$\tilde{a} = a_c \sin z, \qquad t = \frac{a_c^3}{2}\,[2z - \sin 2z]. \tag{55}$$

At small z, i.e. in the region $\tilde{a} \sim 0$, we have

$$\tilde{a} = \left(\frac{3}{2}\,t\right)^{1/3}. \tag{56}$$

Comparing Eq. (56) with the standard model solution (see, e.g., [3, 16]), we conclude that it agrees with the fact that the 'matter' near the point $\tilde{a} = 0$ is described by the equation of state of the ultrastiff matter (49).

In Fig. 1, the scale factor $\tilde{a}$ is shown as a function of imaginary time t. It demonstrates that the scale factor oscillates in imaginary time near its zero value. The amplitude and the frequency of these oscillations depend on the parameter E. In the subregion $a \le a_c$, where the contributions from the condensate and the cosmological constant can be neglected, the eigenvalue E of Eq. (29) is quantized according to expression (31) with $M_k = 0$. The case $E = 1$ corresponds to the Universe which is in the ground state $n = 0$ (see Fig. 1,a). The value $E = 3$ refers to the quantum state $n = 1$ (see Fig. 1,b).

The amplitude of oscillations which determines the size of the region forbidden for the classical motion decreases as $n^{-1/2}$, while the frequency increases with n. The case of very high values of n describes the classical motion of the system. In the limit $n \to \infty$, the region forbidden for the classical motion shrinks to the point producing the initial singularity, in which the Universe is characterized by the infinitely large frequency of oscillations of the space curvature.

In the subregion $a > a_c$, the solution of Eqs. (53) can be written as

$$a = a_c \cosh\zeta, \qquad \tau = \frac{a_c^3}{2}\,[2\zeta + \sinh 2\zeta]. \tag{57}$$

At $\zeta \ll 1$, this implies that the scale factor at $\tau \ll 2a_c^3$ increases almost exponentially

$$a = a_c\left[1 + \left(\frac{1}{2a_c^2}\right)^3 \tau^2 + \ldots\right] \approx a_c \exp\left\{\frac{\tau^2}{8a_c^6}\right\}. \tag{58}$$

The almost exponential expansion of the early Universe in that era is related to the action of quantum effects which, according to Eqs. (46) and (49), cause the negative pressure, $p_u < 0$, i.e. produce an antigravitating effect on the cosmological system under consideration.

At $\zeta \gg 1$, solution (57) takes the form

$$a = \left(\frac{\tau}{a_c}\right)^{1/2}. \tag{59}$$

It describes the radiation-dominated era and corresponds to time $\tau \gg 2a_c^3$.

In Fig. 2, the scale factor a is shown as a function of real time τ. With increase of τ, it increases at first by law (58) and then in accordance with Eq. (59). The initial value $a(\tau = 0)$ depends on the quantum number n. Figure 2,a demonstrates the case where $n = 0$, while Fig. 2,b shows the case $n = 1$. In the limit $n \to \infty$, the initial singularity $a(\tau = 0) = 0$ will be the reference point of the scale factor a as a function of τ as in general relativity.

Solutions (55) and (57) are related to each other through the analytic continuation into the region of complex values of the time variable,

$$t = -i\tau + \frac{\pi}{2}\, a_c^3, \qquad z = \frac{\pi}{2} - i\zeta. \tag{60}$$

The scale factors $\tilde{a}$ (55) and a (57) are connected through the condition

$$a(\tau) = \tilde{a}\left(\frac{\pi}{2}\, a_c^3 - i\tau\right), \tag{61}$$

which describes the analytic continuation of the time variable τ into the region of complex values of Euclidean time t. This analytic continuation can be interpreted as a quantum tunneling of the Lorentzian space-time from the Euclidean one.

4.3. Transition amplitude

The model determined by Eqs. (53) allows us to describe the origin (nucleation) of the Universe as the transition from the state in the subregion $a < a_c$ to the state in the subregion $a > a_c$. The corresponding transition amplitude can be written as follows [15]:

$$T \sim e^{-S_t}, \tag{62}$$

where S_t is the action on a trajectory in imaginary time t,

$$S_t = 2 \int\limits_{-\infty}^{\infty} dt U(\tilde{a}). \tag{63}$$

Let us proceed to the integration with respect to the time variable z. According to Eq. (55), the scale factor $\tilde{a}$ is a periodic function of z. At first, we consider the oscillations of a 'particle' on the finite time interval $[-z_0, z_0]$ with the boundary conditions $\tilde{a}(z_0) = \pm a_c$ and $\tilde{a}(-z_0) = \mp a_c$. Supposing that $z_0 = \frac{\pi}{2}\nu$, where $\nu = 1, 3, 5, \ldots$ numbers the quantity of half-waves of the function $\tilde{a}(z)$, centered at the points $z = \pm\pi q$, $q = 0, 1, 2, \ldots$, which cover the interval $[-z_0, z_0]$. Then the action S_t takes the form

$$S_t = 2 \int\limits_{-\frac{\pi}{2}\nu}^{\frac{\pi}{2}\nu} dz \frac{dt}{dz} U(\tilde{a}(z)). \tag{64}$$

Using the explicit form of solution (55), we find

$$S_t = -\sqrt{E}\pi\nu, \tag{65}$$

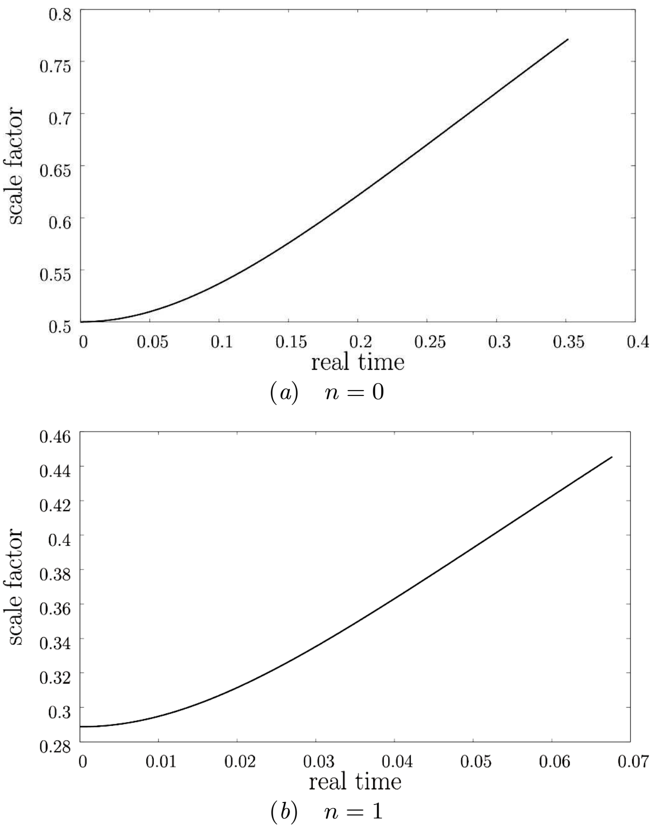


Fig. 2. Scale factor a vs real time τ for the values $n = 0$ (a) and $n = 1$ (b)

and amplitude (62) becomes

$$T \sim e^{\sqrt{E}\pi\nu}, \tag{66}$$

i.e. a 'particle' which is the equivalent of the Universe leaves the subregion forbidden for the classical motion with an exponential probability density. It is pushed out of the forbidden subregion into the subregion of very small values of a in real time τ by the antigravitating forces stipulated by the negative pressure which cause quantum processes at $a \sim 0$ (see Eqs. (46) and (49)). This phenomenon can be interpreted as the origin of the Universe from the region $a < a_c$. It is possible only if the probability density that the Universe is in the state with $a \sim 0$ is nonzero.

In the limit $\nu \to \infty$, the transition amplitude $T \to e^{\infty}$. This result can be interpreted so that the origin of the Universe occurs with probability 1 during the infinite imaginary time interval.

4.4. Geometry

Let us consider how the geometry of the Universe changes as a result of its transition from the region $a < a_c$ into $a > a_c$. In the model under consideration, the metric has the form (1). According to solutions (55) and (57), metric (1) takes the form

$$ds_E^2 = -a_c^2 \sin^2 z \left\{4a_c^4 \sin^2 z\, dz^2 + d\Omega_3^2\right\} \quad \text{at} \quad a < a_c \tag{67}$$

and

$$ds_L^2 = a_c^2 \cosh^2 \zeta \left\{4a_c^4 \cosh^2 \zeta\, d\zeta^2 - d\Omega_3^2\right\} \quad \text{at} \quad a > a_c, \tag{68}$$

where the interval with the Euclidean signature is denoted by the index E, and the one with the Lorentzian signature is marked by L. Introducing the new time variables ξ and ς according to

$$d\xi = 2a_c^2 \sin z dz, \qquad d\varsigma = 2a_c^2 \cosh \zeta d\zeta, \tag{69}$$

metrics (67) and (68) can be reduced to the conformally flat form

$$ds_E^2 = -a_c^2 \left[1 - \left(\frac{\xi}{2a_c^2}\right)^2\right] \left\{d\xi^2 + d\Omega_3^2\right\}, \tag{70}$$

$$ds_L^2 = a_c^2 \left[1 + \left(\frac{\varsigma}{2a_c^2}\right)^2\right] \left\{d\varsigma^2 - d\Omega_3^2\right\}. \tag{71}$$

Both metrics are related to each other through the analytic continuation into the region of complex values of the time variable $\varsigma = i\xi$. The conformal factor in metric (70) varies from zero value at $\xi = -2a_c^2$ to the maximum value a_c^2 at $\xi = 0$ and then vanishes again at $\xi = 2a_c^2$. The conformal factor in metric (71) ranges from its minimum value a_c^2 at $\varsigma = 0$ to infinity increasing with ς.

Metric (70) is conformal to the metric of a unit four-sphere in a five-dimensional Euclidean flat space. With increasing a, the Universe transits from the region $a < a_c$ into the region $a > a_c$, where the geometry is conformal to a unit hyperboloid embedded in a five-dimensional Lorentz-signatured flat space. Such a picture of changes in the space-time geometry during the transition of the Universe from the region near the initial singularity into the region of real physical scales agrees with the hypothesis [17, 18] widely discussed in the literature (see, e.g., the reviews [19, 20]) for the de Sitter space about a possible change in the four-space geometry after the spontaneous nucleation of the expanding Universe from the initial singularity point.

5. Conclusions

In this paper, we study the properties of the quantum Universe on extremely small space-time scales in the semiclassical approach to the well-defined quantum model. We show that quantum gravity effects $\sim \hbar$ exhibit themselves near the initial cosmological singularity point in the form of an additional matter source with the negative pressure and the equation of state as for the ultrastiff matter. The analytical solution of the equations of the theory of gravity, in which matter is represented by radiation and the additional matter source of the quantum nature, is found. It is shown that the geometry of the Universe is described at the stage of the evolution of the Universe, when quantum corrections $\sim \hbar$ dominate over radiation, by the metric which is conformal to a metric of a unit four-sphere in a five-dimensional Euclidean flat space. In the radiation-dominated era, the metric is found to be conformal to a unit hyperboloid embedded in a five-dimensional Lorentz-signatured flat space. One solution can be continued analytically into another one.

The origin of the Universe can be interpreted as a quantum transition of the system from the region in the phase space forbidden for the classical motion, but where a trajectory in imaginary time exists, into the region where the equations of motion have the solution which describes the evolution of the Universe in real time. Near the boundary between two regions, from the side of real time, the Universe undergoes almost an exponential expansion which passes smoothly into the expansion under the action of radiation dominating over matter which is described by the standard cosmological model. As a result of such a quantum transition, the geometry of the Universe changes. This agrees with the hypothesis about a possible change of the geometry after the origin of an expanding Universe from the region near the initial singularity point. In this paper, this phenomenon is demonstrated in the case of the early Universe filled with radiation and the ultrastiff matter which effectively takes into account quantum effects on extremely small space-time scales.

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1. R.M. Wald, *General Relativity* (Chicago University Press, Chicago and London, 1984).

2. K.A. Olive and T.A. Peacock, Phys. Lett. B **667**, 217 (2008).
3. E.W. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, 1990).
4. C.J. Isham, Proc. GR14 Conference (Florence, 1995); gr-qc/9510063.
5. C. Kiefer, in *Lecture Notes in Physics 541: Towards Quantum Gravity*, edited by J. Kowalski-Glikman (Springer, Heidelberg, 2000); gr-qc/9906100.
6. J.L. Anderson, in *Gravitation and Relativity*, edited by Hong-Yee Chin and W.F. Hoffman (Benjamin, New York, 1964).
7. J.A. Wheeler, in *Battelle Rencontres*, edited by C. DeWitt snd J.A. Wheeler (Benjamin, New York, 1968).
8. B.S. DeWitt, Phys. Rev. **160**, 1113 (1967).
9. P.A.M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964).
10. K.V. Kuchař and C.G. Torre, Phys. Rev. D **43**, 419 (1991).
11. V.E. Kuzmichev and V.V. Kuzmichev, Acta Phys. Pol. B **39**, 979 (2008); gr-qc/0712.0464.
12. V.E. Kuzmichev and V.V. Kuzmichev, Acta Phys. Pol. B **39**, 2003 (2008); gr-qc/0712.0465.
13. V.E. Kuzmichev and V.V. Kuzmichev, in *Trends in Dark Matter Research*, edited by J.V. Blain (Nova Science, Hauppage, 2005); astro-ph/0405455.
14. J.D. Brown and D. Marolf, Phys. Rev. D **53**, 1835 (1996); gr-qc/9509026.
15. S. Coleman, Phys. Rev. D **15**, 2929 (1977).
16. I. Dymnikova and M. Fil'chenkov, Phys. Lett. B **545**, 214 (2002).
17. J.B. Hartle and S.W. Hawking, Phys. Rev. D **28**, 2960 (1983).
18. S. Hawking and R. Penrose, *The Nature of Space and Time* (Princeton University Press, Princeton, 2000).
19. B.L. Al'tshuler and A.O. Barvinsky, Usp. Fiz. Nauk **166**, 459 (1996).
20. C. Kiefer and B. Sandhöfer, in *Beyond the Big Bang*, edited by R. Vaas (Springer, Berlin, 2008); gr-qc/0804.0672.

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КВАНТОВИЙ ВСЕСВІТ НА ЕКСТРЕМАЛЬНО МАЛИХ ПРОСТОРОВО-ЧАСОВИХ МАСШТАБАХ

В.Є. Кузьмичов, В.В. Кузьмичов

Р е з ю м е

Квазікласичний підхід до квантово-геометродинамічної моделі застосовано для опису властивостей всесвіту на екстремально малих просторово-часових масштабах. У цьому підході матерія у всесвіті має дві компоненти квантової природи, які поводять себе як антигравітуючі рідини. Перша компонента не набуває нульового значення в границі $\hbar \to 0$ та може бути асоційована з темною енергією. Друга компонента описується екстремально жорстким рівнянням стану і прямує до нуля після переходу до великих просторово-часових масштабів. На малих просторово-часових масштабах ця квантова поправка відіграє значну роль. Вона визначає геометрію всесвіту біля точки початкової космологічної сингулярності. Ця геометрія є конформною до одиничної 4-сфери, зануреної у 5-вимірний евклідовий плоский простір. Під час наступного розширення всесвіту, після досягнення пост-планківської ери, геометрія всесвіту перетворюється на геометрію, конформну до одиничного 4-гіперболоїда у 5-вимірному плоскому просторі з лоренцівською сигнатурою. Це узгоджується з гіпотезою про можливу зміну геометрії після виникнення всесвіту, що розширюється з області поблизу точки початкової сингулярності. Виникнення всесвіту може бути інтерпретовано як квантовий перехід системи з області у фазовому просторі, забороненої для класичного руху, але де існує траєкторія в уявному часі, в область, де рівняння руху мають розв'язок, що описує еволюцію всесвіту у реальному часі. Поблизу межі між двома областями, з боку реального часу, всесвіт зазнає майже експоненціального розширення, яке гладко переходить у розширення під дією випромінювання, що домінує над матерією, у відповідності із стандартною космологічною моделлю.