TORIC GEOMETRY AND CALABI–YAU COMPACTIFICATIONS

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These notes contain a brief introduction to the construction of toric Calabi–Yau hypersurfaces and complete intersections with a focus on issues relevant for string duality calculations. The last two sections can be read independently and report on recent results and work in progress, including torsion in cohomology, classification issues, and topological transitions

Introduction

Toric geometry is a beautiful part of mathematics that relates discrete and algebraic geometries and provides an elegant and intuitive construction of many nontrivial examples of complex manifolds [1–6]. Sparked by Batyrev's construction of toric Calabi–Yau hypersurfaces, which relates the mirror symmetry to a combinatorial duality of convex polytopes [7], toric geometry also became a pivotal tool in string theory. It provides efficient tools for the construction and analysis of large classes of models, and for computing quantum cohomology and symplectic invariants [8–11], fibration structures [12–15] for non-perturbative dualities [16–19], and Lagrangian submanifolds for open string and D-brane physics [20–22]. F-theory compactifications [23, 24], which are based on elliptic Calabi–Yau 4-folds, are maybe the most promising approach to realistic unified string models for particle physics [25], but also 3-folds with torsion in cohomology have been used for the successful model building [26].

In the present notes, we describe some tools that are provided by toric geometry and report on some recent results. Section 1 contains the basic definitions and constructions of toric varieties, working mainly with a homogeneous coordinate ring. In section 2, we recall the string theory context in which Calabi–Yau geometry becomes important for particle physics and describe the toric construction of hypersurfaces and complete intersections. Section 3 explains fibrations and torsion in cohomology in terms of the combinatorics of polytopes. In section 4, we summarize recent results and work in progress and conclude with a list of open problems.

1. Basics of Toric Geometry

The (algebraic) n-torus is a product $T = (\mathbb{C}^*)^n$ of n copies of the punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ we regard as a multiplicative Abelian group. It is, hence, a complexification of the real n-torus $U(1)^n$. A toric variety is defined as a (partial) compactification of T in the following sense: It is a (normal) variety X that contains an n-torus T as a dense open subset such that the natural action of the torus T on itself extends to an action of T on the variety X.

The beauty of toric geometry comes from the fact that the data is encoded in combinatorial terms and that this structure can be used to derive simple formulas for sophisticated topological and geometrical objects. More precisely, the data is given by a fan Σ, which is a finite collection of strongly convex rational polyhedral cones (i.e. cones generated by a finite number of lattice points and not containing a complete line) such that all faces of cones and all intersections of any two cones also belong to the fan.

The space, in which Σ lives, can be obtained as follows [6]: If we parametrize the torus by coordinates $(t_1, \ldots, t_n)$, the character group $M = \{\chi : T \to \mathbb{C}^*\}$ of T can be identified with a lattice $M \cong \mathbb{Z}^n$, where $m \in M$ corresponds to the character $\chi^m((t_1, \ldots, t_n)) = t_1^{m_1} \ldots t_n^{m_n} \equiv t^m$. Another natural lattice that comes with the torus T can be identified with the group of algebraic one-parameter subgroups, $N \cong \{\lambda : \mathbb{C}^* \to T\}$, where $u \in N$ corresponds to the group homomorphism $\lambda^u(\tau) = (\tau^{u_1}, \ldots, \tau^{u_n}) \in T$ for $\tau \in \mathbb{C}^*$. The composition $(\chi \circ \lambda)(\tau) = \chi(\lambda(\tau)) = \tau^{\langle \chi, \lambda \rangle}$ defines a canonical pairing $\langle \chi^m, \lambda^u \rangle = m \cdot u$ which makes N and $M \cong \mathrm{Hom}(N, \mathbb{Z})$ a dual pair of lattices (or free Abelian groups). The characters χ^m for $m \in M$ can be regarded as holomorphic functions on the torus T and hence as rational functions on the toric variety X. We will see that the lattice N, or rather its real extension $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, is where the cones $\sigma \in \Sigma$ of the fan live.

First, we construct the rays ρ_j with $j = 1, \ldots, r$, i.e. the one-dimensional cones $\rho_j \in \Sigma^{(1)}$ of the fan, where the n-skeleton $\Sigma^{(n)}$ contains the n-dimensional cones of Σ. Recall that divisors are formal linear combinations of subvarieties of X of (complex) codimension 1, i.e. of dimension $n-1$. It can be shown that normality of X implies that the divisors $\mathrm{div}(\chi^m)$, i.e. the hypersurfaces defined by the equations $\chi^m = 0$, are equal to sums $\sum_1^r a_j D_j$ for some finite set of irreducible divisors D_j. Like $\mathrm{div}(\chi^m)$, the D_j are T-invariant and hence unions of complete orbits of the torus action. The coefficients $a_j(m)$ are unique so that the decomposition $\mathrm{div}\,\chi^m = \sum a_j D_j$ defines linear maps $m \to a_j(m) = \langle m, v_j\rangle$. The irreducible divisors D_j define, hence, vectors $v_j \in N$ for $j \le r$ with $a_j(m) = \langle m, v_j\rangle$. These vectors are the primitive generators of the rays ρ_j that constitute the 1-skeleton $\Sigma^{(1)}$ of the fan Σ. If we locally write the equation of the divisor as $D_j = \{z_j = 0\}$ with z_j a section of some local line bundle, then we can write the torus coordinates, with appropriate choice of normalizations, as $t_i = \prod z_j^{\langle e_i, v_j\rangle}$ on some dense subset of $T \subseteq X$.

1.1. Homogeneous coordinates

We now regard $\{z_j\}$ as global homogeneous coordinates $(z_1 : \ldots : z_r)$ in a generalization of the construction of the projective space $\mathbb{P}^n$ as a $\mathbb{C}^*$-quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ via the identification $(z_0 : \ldots : z_n) \equiv (\lambda z_0 : \ldots : \lambda z_n)$. If all z_j are non-zero, then the coordinates

$$(\lambda^{q_1} z_1 : \ldots : \lambda^{q_r} z_r) \sim (z_1 : \ldots : z_r), \qquad \lambda \in \mathbb{C}^* \tag{1}$$

describe the same point of the torus T with coordinates

$$t_i = \prod z_j^{\langle e_i, v_j\rangle} \quad \in \quad T = X \setminus \bigcup D_j \tag{2}$$

if $\sum q_j v_j = 0$, where v_j are the generators of the rays $\rho_j \in \Sigma^{(1)}$ of the fan Σ. Since the vectors $v_j \in N$ belong to a lattice of dimension n, the scaling exponents $\{q_j\} \in \mathbb{Z}^{\Sigma^{(1)}} \cong \mathbb{Z}^r$ in identification (1) are restricted by n independent linear equations. Naively, we might expect therefore that the toric variety X can be written as a quotient $(\mathbb{C}^r \setminus Z)\,/\,(\mathbb{C}^*)^{r-n}$, where $\{(z_j)\} = \mathbb{C}^r \setminus Z$ is the set of allowed values for the homogeneous coordinates and the $(\mathbb{C}^*)^{r-n}$-action on $\{(z_j)\}$ implements identification (1) with $\sum q_j v_j = 0$. The identification group corresponds, however, to the kernel of the map $(z_j) \to (t_i = \prod z_j^{\langle e_i, v_j\rangle})$ for $(z_j) \in (\mathbb{C}^*)^r$. If v_j do not span the N-lattice, i.e. if the quotient

$$G \cong N/(\mathrm{span}_{\mathbb{Z}}\{v_1, \ldots, v_r\}) \tag{3}$$

is a finite Abelian group of order $|G| > 1$, then this kernel is $(\mathbb{C}^*)^r/(\mathbb{C}^*)^n \cong (\mathbb{C}^*)^{r-n} \times G$ and contains a discrete factor G. The toric variety X can be constructed, hence, in terms of homogeneous coordinates z_j, an exceptional set $Z \subset \mathbb{C}^r$, and the identification group $(\mathbb{C}^*)^{r-n} \times G$ as

$$X = (\mathbb{C}^r - Z)\,/\,((\mathbb{C}^*)^{r-n} \times G), \tag{4}$$

where the quotient can be shown to be "geometrical" if the fan Σ is simplicial [27] (cf., however, Example 2 below), For a given set of generators $v_j \in \mathbb{R}^n$ of the rays $\rho_j \in \Sigma^{(1)}$, we can construct different toric varieties by choosing different lattices $N \subset \mathbb{R}^n$ that contain v_j as primitive lattice vectors. These are all Abelian quotients of the variety, for which N is the integral span of $\{v_j\}$.

The last piece of information we need is the exceptional set Z. The limit points $(z_j) \in X \setminus T = \bigcup D_j$ that are added to T are determined by the conditions under which homogeneous coordinates are allowed to vanish. This is where the information of the fan Σ enters: A subset of the coordinates z_j is allowed to vanish simultaneously iff there is a cone $\sigma \in \Sigma$ containing all of the corresponding rays ρ_j. In geometrical terms, this means that the corresponding divisors D_j intersect in X. The exceptional set Z is, hence, the union of sets

$$Z_I = \{(z_1 : \ldots : z_r) \,|\, z_j = 0 \;\forall\, j \in I\}, \tag{5}$$

for which there is no cone $\sigma \in \Sigma$ such that $\rho_j \subseteq \sigma$ for all $j \in I$. Minimal index sets I with this property are called *primitive collections*. They correspond to the maximal irreducible components of $Z = \bigcup Z_I$.

1.2. Torus orbits

In terms of homogeneous coordinates, the torus action amounts to the effective part of the $(\mathbb{C}^*)^r$ action induced on the quotient (4) by independent scalings of z_j and thus extends from T to X. The torus orbits, into which X decomposes, are, hence, characterized by the index sets of the vanishing coordinates. It can be shown [3] that the *torus orbits* $\mathcal{O}_\sigma$ are in one-to-one correspondence with the cones $\sigma \in \Sigma$, where $\mathcal{O}_\sigma \cong (\mathbb{C}^*)^{n-\dim\sigma}$ is the intersection of all divisors D_j for $\rho_j \subset \sigma$ with all complements of the remaining divisors. The orbit $\mathcal{O}_\sigma$ is an open subvariety of the *orbit closure* V_σ, which is the intersection of all divisors D_j for $\rho_j \subset \sigma$ and which also has dimension $n - \dim\sigma$.

Further important sets are the *affine open sets* U_σ, which are the intersections of all complements $X \setminus D_j$

with $\rho_j \notin \sigma$ and which provide a covering of X. The relations between these sets can be summarized as

$$V_\sigma = \bigcup_{\tau\supseteq\sigma} \mathcal{O}_\tau, \quad U_\sigma = \bigcup_{\tau\subseteq\sigma} \mathcal{O}_\tau, \quad X = \bigcup_{\sigma\in\Sigma} U_\sigma, \tag{6}$$

where V_σ and U_σ are disjoint unions and the last union can be restricted to a covering of X by U_σ's for maximal cones σ. In the traditional approach [1–4], a toric variety X is constructed by gluing its open *affine patches* U_σ along their intersections

$$U_\sigma \cap U_\tau = U_{\sigma\cap\tau} \supseteq T = U_{\{0\}}. \tag{7}$$

The patches U_σ are constructed in terms of their rings of regular functions which are generated by the characters χ^m that are nonsingular on the relevant patch. Since

$$t_i = \prod z_j^{\langle e_i, v_j\rangle} \quad \Rightarrow \quad \chi^m = t^m = \prod z_j^{\langle m, v_j\rangle}, \tag{8}$$

the relevant exponent vectors $m \in M$ are the lattice points in the dual cone

$$\sigma^\vee = \{x \in M_\mathbb{R} \ : \ \langle x, v\rangle \geq 0 \ \ \forall\, v \in \sigma\}. \tag{9}$$

More abstractly, the algebra $A_\sigma = \mathbb{C}[\sigma^\vee \cap M]$ of the semigroup $\sigma^\vee \cap M$ is, by definition, the ring of regular functions on U_σ, so that the points of U_σ can be obtained as the spectrum $\mathrm{Specm}(A_\sigma)$ of maximal ideals. The Zariski topology of U_σ can be constructed in terms of the prime ideals and the gluing can be worked out by relating the characters in different patches (cf. Example 2 below). We now state two important theorems [1–4]:

Theorem 1. A toric variety is *compact* if and only if the fan is complete, i.e. if the support of the fan covers the N lattice $|\Sigma| = \bigcup_\Sigma \sigma = N_\mathbb{R}$.

We prove the *only if*: For an incomplete fan, we consider some $u \in N \setminus |\Sigma|$ and a one-parameter family of points $p_\lambda = (\lambda^{u_1} t_1, \ldots, \lambda^{u_n} t_n) \in T$. The evaluation of χ^m yields $\chi^m(p_\lambda) = \lambda^{\langle m,u\rangle}\chi^m(p_1)$. But the limit point $p_{\lambda\to 0}$ cannot be contained in any patch U_σ, because $\chi^m(p_\lambda)$ diverges as $\lambda \to 0$ for $m \in \sigma^\vee$ and $u \notin \sigma$, so that $\langle m, u\rangle < 0$.

Theorem 2. A toric variety is *non-singular* if and only if all cones are simplicial and basic, i.e. if all cones $\sigma \in \Sigma$ are generated by a subset of a lattice basis of N.

To illustrate these theorems, we work out two examples:

Example 1. The Hirzebruch surface

Hirzebruch surfaces $\mathbb{F}_n$ are $\mathbb{P}^1$ bundles over $\mathbb{P}^1$ that can be defined by $v_0 = (0, -1)$, $v_1 = (1, 0)$, $v_2 = (-1, n)$ and $v_3 = (0, 1)$, with linear relations $v_0 + v_3 = 0$, $nv_0 + v_1 + v_2 = 0$ and scaling parameters μ and λ, as shown in Fig. 1.

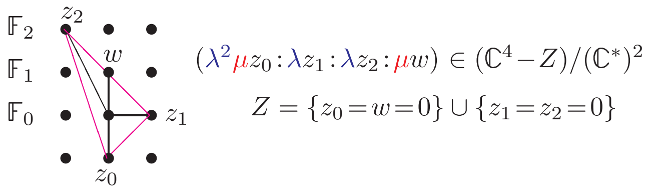

Fig. 1. Hirzebruch surface $\mathbb{F}_2$ as a blow-up of $W\mathbb{P}_{211}$

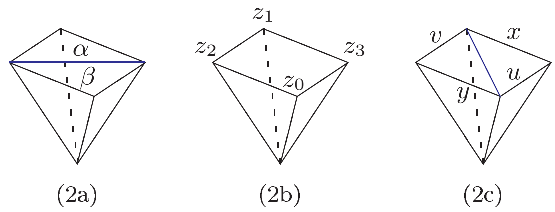

Fig. 2. Toric desingularizations of the conifold

We consider the case $n = 2$. If we drop the vertex v_3, the fan would consist of 3 cones, and we obtain the weighted projective space $W\mathbb{P}_{211}$ with scaling weights $(2, 1, 1)$. This space is singular because the cone spanned by (v_1, v_2) has volume 2. Indeed, if we drop the coordinate w and set $\mu = 1$, then $(1 : 0 : 0)$ is a fixed point of the $\mathbb{C}^*$ identification for $\lambda = -1$, i.e. we have an $\mathbb{Z}_2$ quotient singularity. The Hirzebruch surface $\mathbb{F}_2$ is a desingularization of this surface corresponding to a subdivision of the cone (v_1, v_2) into two basic cones (v_1, v_3) and (v_3, v_2). The exceptional set is accordingly modified to $Z = \{z_0 = w = 0\} \cup \{z_1 = z_2 = 0\}$. We can now consider two cases: If $w \neq 0$, then $\mu = 1/w$ scales w to $w = 1$. This yields all points $(z_0 : z_1 : z_2 : 1) \equiv (z_0 : z_1 : z_2)$ of $W\mathbb{P}_{211}$ except for its singular point $(1 : 0 : 0)$ which is excluded due to the subdivision of the cone (v_1, v_2) by v_3. If $w = 0$, we can scale $z_0 \neq 0$ to $z_0 = 1$ and find $(1 : z_1 : z_2 : 0) \in \mathbb{F}_2$. We thus observe that the singular point has been replaced by a $\mathbb{P}^1$ with homogeneous coordinates $(z_1 : z_2)$. This process of replacing a point by a projective space is called *blow-up*. In the present example, it desingularizes a weighted projective space. It can be shown [3] that all singularities of toric varieties can resolved by a sequence of blow-ups that correspond to subdivisions of the fan.

Example 2. The conifold singularity

According to Theorem 2, the second source of singularities is the nonsimplicity of a cone, which is only possible in at least 3 dimensions. We consider, hence, a quadratic cone σ as displayed in Fig. 2,*b* with generators $v_0 = (1, 0, 0)$, $v_1 = (0, 1, 0)$, $v_2 = (1, 0, 1)$, $v_3 = (0, 1, -1)$

and relation $\sum q_i v_i = 0$ with $q = (1, 1, -1, -1)$ so that

$$(z_0 : z_1 : z_2 : z_3) = (\lambda z_0 : \lambda z_1, \tfrac{1}{\lambda} z_2 : \tfrac{1}{\lambda} z_3), \qquad (10)$$

or $X = \mathbb{P}(1, 1, -1, -1)$. The dual cone $\sigma^\vee$ has generators $m_0 = (1, 0, 0), m_1 = (0, 1, 0), m_2 = (0, 1, 1), m_3 = (1, 0, -1)$. According to (8), the coordinate ring A_σ is generated by

$$x = \chi^{m_0} = z_0 z_2, \qquad y = \chi^{m_1} = z_1 z_3, \qquad (11)$$

$$u = \chi^{m_2} = z_1 z_2, \qquad v = \chi^{m_3} = z_0 z_3, \qquad (12)$$

which are regular, invariant under the scaling (10), and obey the relation $xy = uv$. Hence, $A_\sigma = \mathbb{C}[\sigma^\vee \cap M] \cong \mathbb{C}[x, y, u, v]/\langle xy - uv\rangle$ and $X = U_\sigma$ can be identified with the hypersurface $xy = uv$ in $\mathbb{C}^4$ which has a "conifold singularity" at the origin. As shown in Fig. 2, the cone σ can be triangulated in two different ways. In the first case, $\sigma_\alpha = \langle v_0, v_2, v_3\rangle$, $\sigma_\beta = \langle v_1, v_2, v_3\rangle$, and the dual cones are $\sigma_\alpha^\vee = (m_\alpha, m_0, m_3)$, $\sigma_\beta^\vee = (m_\beta, m_1, m_2)$ with $m_\alpha = -m_\beta = (-1, 1, 1)$, so that we obtain the algebras

$$A_\alpha = \mathbb{C}(m_\alpha, m_0, m_3) \ni (z_1/z_0,\ x = z_0 z_2,\ v = z_0 z_3), \quad (13)$$

$$A_\beta = \mathbb{C}(m_\beta, m_2, m_1) \ni (z_0/z_1,\ u = z_1 z_2,\ y = z_1 z_3). \quad (14)$$

With the exceptional set $Z = \{z_0 = z_1 = 0\}$, we observe that the singular point $x = y = u = v = 0$ has been replaced by a $\mathbb{P}^1 \ni (z_0 : z_1)$, and the transition functions show that $X_{\{\sigma_\alpha, \sigma_\beta\}}$ can be identified with the total space of the rank two bundle $O(-1) \oplus O(-1) \to \mathbb{P}^1$. The reader may verify that the homogeneous coordinates work straightforwardly (geometrical) in the simplicial cases (2,a) and (2,c). In the non-simplicial case (2,b), quotient (4) has to be taken in the "GIT–sense" [27], because all $\mathbb{C}^*$ orbits $(\lambda z_0 : \lambda z_1 : 0 : 0)$ and $(0 : 0 : \frac{1}{\lambda} z_2 : \frac{1}{\lambda} z_3)$ map to the tip $0 \in U_\sigma$ of the conifold (the *categorial quotient* of *geometric invariant theory* (GIT) involves the dropping of "bad" orbits).

The two toric resolutions correspond to blowups replacing the singular point by two different $\mathbb{P}^1$'s (which topologically are 2-spheres S^2). We can go, hence, from one "small resolution" to the other via the conifold by blowing down the $\mathbb{P}^1$ to a point and blowing up that point in a different way. This is called a *flop transition.*

There is also a non-toric possibility to resolve the singularity by deforming the hypersurface equation to $xy - uv = \varepsilon$. Topologically, this amounts to replacing the singularity by a 3-sphere S^3. This can be seen as follows: by a linear change of variables $\{\frac{x\mp y}{2}, \frac{u\pm v}{2}\} \leftrightarrow \{i^l w_l\}$, we can write the deformed conifold equation as $\sum_{l=1}^4 w_l^2 = \varepsilon$. With $w_l = a_l + ib_l$, its real and imaginary part become

$$\sum_{l\le 4} a_l^2 = \varepsilon + \sum_{l\le 4} b_l^2, \qquad \sum_{l\le 4} a_l b_l = 0. \qquad (15)$$

For $\varepsilon > 0$, the four real variables $a'_k = a_k/\sqrt{\varepsilon + \sum_l b_l^2}$ parametrize a 3-sphere, and b_k with $\sum_l a'_l b_l = 0$ parametrize the fibers of the cotangent bundle T^*S^3. The topology change between the small resolution and the deformation is called conifold transition.

1.3. Line bundles

The conifold is the standard example of a *non-compact* (local) Calabi–Yau geometry. A *compact* toric varieties, on the other hand, never have $c_1 = 0$. Hence, we will not only be interested in toric varieties themselves but also in hypersurfaces or complete intersections thereof, which are smooth Calabi–Yau spaces under appropriate conditions. Their defining equations will be sections of non-trivial line bundles. The relevant data of these bundles are the transition functions between different patches. These data are closely related to the topological data of *Cartier divisors* which are locally given, by definition, in terms of the rational equations $f_\alpha = 0$ with f_α/f_β regular and nonzero on the overlap of two patches. Since the multiplication by a rational function does not change the line bundle, we are interested in *the classes of divisors* with respect to *linear equivalence*, i.e. modulo addition of *principal divisors* div(f), which are the divisors of rational *functions* f. Hence, Cartier divisor *classes* determine the Picard group Pic(X) of holomorphic line bundles.

Finite formal sums of irreducible varieties of codimension one are called *Weil divisors* (which may not be Cartier, i.e. locally principal, on singular varieties). On a toric variety, it can be shown that the *Chow group* $A_{n-1}(X)$ of Weil divisors modulo linear equivalence is generated by the T-invariant irreducible divisors D_j modulo the principal divisors div(χ^m) with $m \in M$, i.e. there is an exact sequence

$$0 \to M \to \mathbb{Z}^{\Sigma^{(1)}} \to A_{n-1}(X) \to 0, \qquad (16)$$

where $M \ni m \to (\langle m, v_j\rangle) \in \mathbb{Z}^{\Sigma^{(1)}}$ and $\mathbb{Z}^{\Sigma^{(1)}} \ni (a_j) \to \sum a_j D_j$. Hence, the Chow group $A_{n-1}(X)$ has rank $r - n$. It contains the Picard group as a subgroup, which is torsion-free if X_Σ is compact [3].

A Weil divisor of the form $D = \sum a_j D_j$ is Cartier and, hence, defines a line bundle $\mathcal{O}(D) \in$ Pic(X), if there exists an $m_\sigma \in M$ for each maximal cone $\sigma \in \Sigma$ such that $\langle m_\sigma, v_j\rangle = -a_j$ for all $\rho_j \in \sigma$. The transition functions of $\mathcal{O}(D)$ between the patches U_σ and U_τ are then given by $\chi^{m_\sigma - m_\tau}$. If X is smooth, then all Weil divisors are Cartier. For a simplicial fan, kD is Cartier for some positive integer k. For Cartier divisors, the

Σ-piecewise linear real function ψ_D on $N_{\mathbb{R}}$ defined by

$$\psi_D(v) = \langle m_\sigma, v\rangle \qquad \text{for} \qquad v \in \sigma \tag{17}$$

is called support function. If X is compact and $D = \sum a_j D_j$ is Cartier, then $\mathcal{O}(D)$ is generated by global sections iff the support function ψ_D is convex, and D is ample iff ψ_D is strictly convex, i.e. if $\langle m_\sigma, v_j\rangle > -a_j$ for $\dim\sigma = n$ and $\rho_j \not\subset \sigma$. For convex support functions,

$$\Delta_D = \{m \in M_{\mathbb{R}} : \langle m, v_j\rangle \geq -a_j \quad \forall\ j \leq r\} \tag{18}$$

$$= \{m \in M_{\mathbb{R}} : \langle m, u\rangle \geq \psi_D(u) \ \ \forall\ u \in N\} \tag{19}$$

defines a convex lattice polytope $\Delta_D \subset M_{\mathbb{R}}$, whose lattice points provide the global sections of the line bundle $\mathcal{O}(D)$ corresponding to a divisor D (the first equality defines Δ_D also if D is not Cartier). In particular, $\Delta_{kD} = k\Delta_D$ and $\Delta_{D+\mathrm{div}(\chi^m)} = \Delta_D - m$ so that the polytope can be translated in the M lattice without changing the divisor class and the transition functions.

In terms of polytope (18), D is generated by global sections iff Δ_D is the convex hull of $\{m_\sigma\}$, and D is ample iff Δ_D is n-dimensional with vertices m_σ for $\sigma \in \Sigma^{(n)}$ and with $m_\sigma \neq m_\tau$ for $\sigma \neq \tau \in \Sigma^{(n)}$. In the latter case, there is a bijection between faces of Δ_D and cones in Σ or, more pricisely, Σ is the *normal fan* of Δ_D: By definition, the cones σ_τ of the normal fan Σ_Δ of a polytope Δ are the dual cones of the cones over $\Delta - x$, where $x \in M_{\mathbb{R}}$ is any point in the relative interior of a face $\tau \subset \Delta$. If $0 \in M$ is in the interior of Δ, as can always be achieved by a rational translation of Δ by $\delta x \in M_{\mathbb{Q}}$, then the normal fan Σ_Δ coincides with the fan of cones over the faces of the *polar* polytope $\Delta^\circ \subseteq N_{\mathbb{R}}$ defined by

$$\Delta^\circ = \{y \in N_{\mathbb{R}} \ :\ \ \langle x, y\rangle \geq -1 \ \ \forall\, x \in \Delta\}. \tag{20}$$

On smooth compact toric varieties, it can be shown that every ample T-invariant divisor is very ample. The sections χ^m of such $\mathcal{O}(D)$ provide an embedding of X_Σ into $\mathbb{P}^{K-1}$ via $(\chi^{m_1} : \ldots : \chi^{m_K})$, where $K = |\Delta_D \cap M|$ is the dimension of the space of global sections of $\mathcal{O}(D)$. Therefore, a toric variety X_Σ is *projective* iff Σ is the normal fan of a lattice polytope $\Delta \subset M_{\mathbb{R}}$ [2, 3].

Summarizing, the equations defining Calabi–Yau hypersurfaces or complete intersections will be sections of the line bundles $\mathcal{O}(D)$ given by Laurent polynomials

$$f = \sum_{m\in\Delta_D\cap M} c_m\chi^m = \sum_{m\in\Delta_D\cap M} c_m \prod_j z_j^{\langle m,v_j\rangle}, \tag{21}$$

whose exponent vectors m span the convex lattice polytopes $\Delta_D \subseteq M_{\mathbb{R}}$ defined in eq. (18). In an affine patch U_σ, the local section $f_\sigma = f/\chi^{m_\sigma}$ is a regular function $f_\sigma \in A_\sigma$ because $\Delta_D - m_\sigma \subset \sigma^\vee$.

1.4. Intersection ring and Chern classes

If a collection $\rho_{j_1}, \ldots, \rho_{j_k}$ of rays is not contained in a single cone, then the corresponding homogeneous coordinates z_{j_l} are not allowed to vanish simultaneously, and the corresponding divisors D_{j_l} have no common intersection. For the intersection ring, we expect the *non-linear relations* $R_I = D_{j_1} \cdot \ldots \cdot D_{j_k} = 0$, where it is sufficient to consider the *primitive collections* $I = \{j_1 \ldots j_k\}$ as defined by Batyrev, i.e. the *minimal* index sets such that the corresponding rays do not all belong to the same cone (cf. the definition of the exceptional set $Z = \bigcup Z_I$ in Section 1 1.1). The ideal in $\mathbb{Z}[D_1, \ldots, D_r]$ generated by these R_I is called Stanley–Reisner ideal J, and $\mathbb{Z}[D_1, \ldots, D_r]/J$ is a Stanley–Reisner ring.

The Chow groups $A_k(X)$ of a variety X are generated by k-dimensional irreducible closed subvarieties of X modulo rational equivalence by divisors of rational functions on subvarieties of dimension $k+1$. For an arbitrary toric variety X_Σ, it can be shown that $A_k(X)$ is generated by the equivalence classes of orbit closures V_σ for cones $\sigma \in \Sigma^{(n-k)}$. The intersection ring of a *non-singular compact* toric variety X_Σ is [1]

$$A_*(X_\Sigma) = \mathbb{Z}[D_1, \ldots, D_r]\,/\big\langle R_I, \textstyle\sum_j \langle m, v_j\rangle D_j\big\rangle \tag{22}$$

(for a definition of the intersection product see [3]). The intersection ring can be obtained from the Stanley–Reisner ring by adding the *linear relations* $\sum_j \langle m, v_j\rangle D_j \simeq 0$, where it is sufficient to take a set of basis vectors of the M-lattice for m. The Chow ring also determines the homology groups $H_{2k}(X_\Sigma, \mathbb{Z}) = A_k(X, \mathbb{Z})$. These results actually generalize to the *simplicial projective* case with the exception that one needs to admit rational coefficients [2, 3]. In particular, for a maximal-dimensional simplicial cone σ spanned by $v_{j_1}, \ldots, v_{j_n}$, the intersection number of the corresponding divisors is

$$D_{j_1} \cdot \ldots \cdot D_{j_n} = 1/\,\mathrm{Vol}(\sigma), \tag{23}$$

where $\mathrm{Vol}(\sigma)$ is the lattice-volume (i.e. the geometrical volume divided by the volume $1/n!$ of a basic simplex).

Having discussed the cycles, we did not turn to differential forms. The canonical bundle of a non-singular toric variety can be obtained by considering the rational form

$$\omega = \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n} \tag{24}$$

which, by an appropriate choice of the orientation (i.e. the order of the local coordinates x_i in the affine patches), is a rational section of Ω^n_X. This implies

$$\Omega^n_X = \mathcal{O}_X(-\sum_{j=1}^{r} D_j) \tag{25}$$

and for the canonical divisor $-D = -\sum D_j$. The computation of the total Chern class requires an expression for the (co)tangent bundle, for which there is an exact sequence

$$0 \;\to\; \Omega^1_X \;\to\; \Omega^1_X(\log D) \;\overset{\text{res}}{\to}\; \bigoplus_j \mathcal{O}(D_j), \tag{26}$$

where $\Omega^1_X(\log D)$ turns out to be trivial, and the residue map takes $\omega = \sum f_j\, dz_j/z_j \overset{\text{res}}{\to} \oplus f_{j|D_j}$ [3]. A calculation yields the total Chern class of the tangent bundle

$$c(T_X) = \prod_1^r (1 + D_j) = \sum_{\sigma\in\Delta}[V_\sigma] \tag{27}$$

and the Todd class

$$\mathrm{td}(T_X) = \prod_1^r \frac{D_j}{1-\exp(-D_j)} = 1 + \frac{1}{2}\,c_1 + \frac{1}{12}\,c_1^2 c_2 + \dots \tag{28}$$

The first Chern class $c_1 = \sum D_j$ is positive for compact toric varieties, but it vanishes for the conifold, because of the linear relations $D_0 + D_2 \sim 0$, $D_1 + D_3 \sim 0$ and $D_1 \sim D_3$. (Implications for volumes and numbers of lattice points can now be derived by applying the Hirzebruch–Riemann–Roch formula $\chi(X, E) = \int \mathrm{ch}(E)\,\mathrm{Td}(X)$ to the case of line bundles of Cartier divisors as described, for example, in the last chapter of [3].)

1.5. Symplectic reduction

There is another approach to toric geometry in terms of symplectic instead of complex geometry, which is important because, in addition to the complex structure, we will also need the Kähler metric. Moreover, the symplectic approach can be given a direct physical interpretation in terms of supersymmetric gauged linear sigma models [28]. The idea is that the $\mathbb{C}^*$-quotient can be performed in two steps: we first divide out the phase parts which amount to compact $U(1)$ quotients and then – instead of a radial identification – fix the values of appropriate "radial" variables to certain sizes t_a that will parametrize the Kähler metric.

In order that the quotient inherits a Kähler form (and, hence, a symplectic structure) from the natural Kähler form on $\mathbb{C}^r$

$$\omega = i\sum dz_j \wedge d\bar z_j = 2\sum dx_j \wedge dy_j = \sum dr_j^2 \wedge d\varphi_j, \tag{29}$$

with $z = x + iy = re^{i\varphi}$, we use the symplectic reduction formalism. This requires that the G-action is Hamiltonian, i.e. given by a moment map $\mu : \mathbb{C}^r \to \mathfrak{g}^*$ to the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of $G = U(1)^{r-n}$ such that the Hamiltonian flows defined by μ generate the infinitesimal G-transformations. Then the symplectic reduction theorem guarantees that the restriction of the image of a moment map to fixed values t_a for $a = 1, \dots, n-r$ induces a symplectic structure on the quotient of the preimage $\mu^{-1}(t_a)/G$ for regular values of t_a.

In toric geometry, we consider the moment maps

$$\mu_a = \sum_j q_j^{(a)}|z_j|^2 \qquad \text{with} \qquad \sum_j q_j^{(a)} v_j = 0. \tag{30}$$

With $\omega^{-1} \sim \sum_j \frac{\partial}{\partial\varphi_j} \wedge \frac{\partial}{\partial r_j^2}$, the corresponding Hamiltonian flows $\omega^{-1}(\mu_a) \sim \sum_j q_j^{(a)} \frac{\partial}{\partial\varphi_j}$ generate the compact subgroups of the $\mathbb{C}^*$ actions of the homomorphic quotients (4). In the gauged linear sigma model [28], the quantities $q_j^{(a)}$ are the charges of r chiral superfields z_j under a $U(1)^{r-n}$ gauge group, and the moment maps μ_a are D-terms in the superpotential. The imaginary parts of the complexified radii t_a thus correspond to θ-angles.

Under the symplectic reduction, the holomorphic r-form $\Omega = \prod dz^j$ on $\mathbb{C}^r$ descends to a holomorphic n-form iff it is invariant under the group action, i.e. if $\sum_j q_j^{(a)} = 0$ for all $a \le r-n$. These equations can be interpreted as $U(1)$ gauge anomaly cancellation conditions in the linear sigma model [28]. Since the existence of a holomorphic n-from on X_Σ is equivalent to $c_1 = 0$, we thus obtain a simple form of the Calabi–Yau condition (cf. the vanishing of $\sum_j q_j$ for a conifold).

Instead of the linear combinations (30), we can consider all moment maps $\mu_j = |z_j|^2$, whose flows are phase rotations of the homogeneous coordinates z_j. After the symplectic reduction, the effective part $\mathcal{G} \cong U(1)^n$ of this $U(1)^r$-action yields the compact part of the torus action $\mathcal{G} \subset T$. The image of the corresponding momentum map is a convex polytope, the *Delzant polytope* $\Delta(t_a)$, whose corners correspond to fixed points of $\mathcal{G}$. Since the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ can be identified with the the real extension of the N-lattice, $\Delta(t_a)$ is a polytope in $\mathfrak{g}^* \equiv M_\mathbb{R}$.

Example 3. For the projective space $\mathbb{P}^n$, all $q_j = 1$, and, with $t_a = r^2$, we obtain the symplectic quotient as $\{|z_0|^2 + \ldots + |z_n|^2 = r^2\}/U(1)$. Hence, the value $t_a = r^2$ of the moment map of the symplectic reduction parametrizes the size of the projective space. The image of the moment map for $\mathcal{G} \subset T$ on the resulting toric variety is the simplex $\{t_i \ge 0, r^2 - \sum_1^n t_i = t_0 \ge 0\} \subset M_\mathbb{R}$, whose faces of codimension k correspond to the vanishing of k moment maps t_i and, hence, to fixed points of a $U(1)^k$ subgroup of $\mathcal{G}$. The fan of the toric variety $\mathbb{P}^n$ is the normal fan of Δ. We thus can construct $\mathbb{P}^n$ as a (compact) torus fibration over a polytope $\Delta \in \mathbb{R}^n$, whose fibers degenerate to lower-dimensional tori over the faces of Δ. This fibration structure has been used by Strominger, Yau, and Zaslow [20] for an interpretation of the mirror symmetry as the T-duality on a torus-fibered Calabi–Yau manifold.

Example 4. As a non-compact example, we consider the conifold, whose Kähler metric is parametrized by $t = |x_0|^2 + |x_1|^2 - |x_2|^2 - |x_3|^2$. Obviously, $t = 0$ is a singular value, while, for $t = \pm\varepsilon^2 \to 0$, the size ε of one of the blown-up $\mathbb{P}^1$'s shrinks to 0. Regular values of the moment maps t_a lead to a smooth symplectic quotient and, in particular, to a (projective) triangulation of the fan. The corresponding smooth Kähler metric is parametrized by the $r - n$ values t_a which can be interpreted as sizes of certain two-cycles, in accord with the dimension $r - n$ of $H_2(X_\Sigma)$. The regular values correspond to open cones of the secondary fan which parametrizes the Kähler moduli spaces and whose chambers are separated by walls that correspond to *flop transitions* [29] between different smooth *phases* (in the physicist's language [28]). At the transition, a cycle shrinks to a point that is blown up according to a different triangulation $\Sigma_\Delta(t)$ on the other side of the wall [29].

2. Strings, Geometry, and Reflexive Polytopes

At observable energy scales, string theory leads to an effective theory that corresponds to a 10-dimensional supergravity compactified on a 6-dimensional manifold K. At small distances, space-time looks, hence, like $M_4 \times K$, where M_4 is our 4-dimensional Minkowski space, as long as quantum fluctuations of the metric are sufficiently small to allow for a semiclassical geometrical interpretation.

For phenomenological reasons, we require usually that supersymmetry survives the compactification, which implies the existence of a covariantly constant spinor $\nabla\eta = 0$ on the internal manifold K. In the simplest situation, RR background fields and the B field vanish. Candelas, Horowitz, Strominger, and E. Witten [30] showed that this implies that K is a complex Kähler manifold with the vanishing first Chern class. (The inclusion of B fields was already discussed in a beautiful paper by Strominger [31], but RR fluxes were largely omitted for a long time, until their importance for moduli stabilization in type II theories was recognized [32]. The investigation of their geometry leads to the important new concept of generalized complex structures [33–35].) Explicitly, the Kähler form ω and the holomorphic 3-form Ω of the Calabi–Yau hypersurface can be constructed in terms of η as

$$\omega_{ij} = i\eta^\dagger\gamma_{[i}\gamma_{j]}\eta, \qquad \Omega \sim \eta^\dagger\gamma_{[i}\gamma_j\gamma_{k]}\eta, \tag{31}$$

and the integrability condition ${N_{ij}}^k = 0$ for the complex structure ${J_i}^l = \omega_{ij}g^{jl}$ with the Nijenhuis tensor ${N_{ij}}^k = {J_i}^l\partial_l {J_j}^k - {J_j}^l\partial_l {J_i}^k - \partial_i {J_j}^l {J_l}^k + \partial_j {J_i}^l {J_l}^k$ is a trivially satisfied for the torsion-free metric-compatible connection that stabilizes $\eta = 0$.

The condition $c_1 = 0$, which is equivalent to the existence of a holomorphic 3-form Ω, has been conjectured by Calabi and proven by Yau to be also equivalent to the existence of a Ricchi-flat Kähler metric, so that the vacuum Einstein equations are satisfied.

In the standard construction of anomaly-free heterotic strings with the gauge group E_6, it turns out that charged particles and anti-particles show up in conjunction with elements of the Dolbeault cohomology groups H^{11} and H^{21}, respectively. While H^{11} parametrizes the Kähler metric, H^{21} can be related to complex structure deformations via contraction with the holomorphic 3-form, $\Omega_{\mu\nu\lambda}\delta {J^\lambda}_{\bar\rho} \in H^{21}$. Since the exchange of particles and anti-particles, as well as the corresponding sign of a $U(1)$ charge in the sigma model description of the Calabi–Yau compactifications, are mere conventions, physicists came up with the idea of mirror symmetry [36] which was used by Candelas *et al.* [37] to construct a mirror map between the Kähler and complex structure moduli spaces of a Calabi–Yau manifold X and its mirror dual X^*, whose topologies are related by

$$h_{11}(X) = h_{21}(X^*), \qquad h_{21}(X) = h_{11}(X^*). \tag{32}$$

The power series expansions of this map could be interpreted as instanton corrections in the quantum theory and thus lead to a prediction of numbers of rational curves [8, 9].

2.1. Toric hypersurfaces

The beauty of the toric construction of Calabi–Yau spaces is based on the fact that it relates the mirror symmetry to a combinatorial duality of lattice polytopes, as was discovered by Batyrev [7]. He showed that the Calabi–Yau condition for a hypersurface, i.e. the vanishing of the first Chern class, requires as a necessary and sufficient condition that the polytope $\Delta_D \subseteq M_\mathbb{R}$ of the line bundle $\mathcal{O}(D)$, whose section defines the hypersurface, is polar to the lattice polytope $\Delta^* = \Delta_D^\circ \subseteq N_\mathbb{R}$, where Δ^* is the convex hull of the generators v_j of rays $\rho_j \in \Sigma^{(1)}$ of the fan of the ambient toric variety X_Σ. A lattice polytope, whose polar polytope (20) is again a

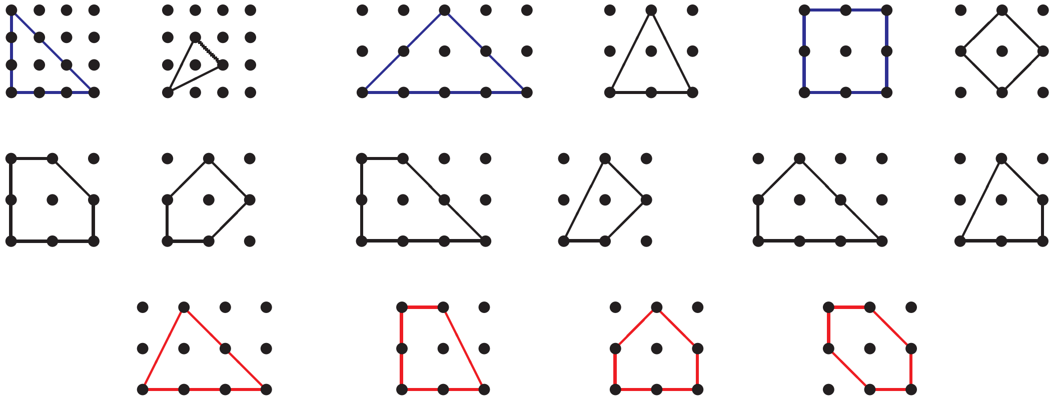

Fig. 3. All 16 reflexive polygons in 2D: the first 3 dual pairs are maximal/minimal and contain all others as subpolygons, while the last 4 polygons are self-dual

lattice polytope, is called *reflexive*. Batyrev also derived a combinatorial formula for the Hodge numbers

$$h_{11}(X_\Delta) = h_{2,1}(X_{\Delta^\circ}) = l(\Delta^\circ) - 1 - \dim\Delta - \\ -sum_{\mathrm{codim}(\theta^\circ)=1} l^*(\theta^\circ) + \sum_{\mathrm{codim}(\theta^\circ)=2} l^*(\theta^\circ) l^*(\theta) \qquad (33)$$

where θ and θ° is a dual pair of faces of Δ and Δ°, respectively. The quantity $l(\theta)$ is the number of lattice points of a face θ, and $l^*(\theta)$ is the number of its interior lattice points. Mirror symmetry now amounts to the exchange of Δ and Δ°, and the formula for the Hodge data makes the topological duality (32) manifest.

Formula (33) has a simple interpretation: the principal contributions to h_{11} come from the toric divisors D_j that correspond to lattice points in Δ° different from the origin. There are $\dim(\Delta)$ linear relations among these divisors. The first sum corresponds to the subtraction of interior points of facets. The corresponding divisors of the ambient space do not intersect a generic Calabi–Yau hypersurface. Lastly, the bilinear terms in the second sum can be understood as multiplicities of toric divisors, and their presence indicates that only a subspace of the Kähler moduli is accessible to toric methods.

The enumeration of all reflexive polygons has been achieved by Batyrev many years ago (see Fig. 3). In dimensions 3 and 4, which are relevant for K3 surfaces and Calabi–Yau 3-folds, respectively, the enumeration required the extensive use of computers and was achieved in [38, 39] and [40, 41]. The code was later included into the software package PALP [42] which can be used for the reconstruction of the data, as well as for many other purposes like the analysis of fibrations and integral cohomology (see Section 3) and the construction of higher-dimensional examples. All results can be accessed on the web page [43] and we just note that the numbers of reflexive polytopes in 3d and 4d are 4319 and 473 800 776, respectively. The resulting 30108 different Hodge data of 3-folds amount to 15122 mirror pairs as shown in Fig. 4.

2.2. Complete intersections

Soon after the hypersurface case, Batyrev and Borisov discovered another beautiful combinatorial duality that corresponds to the mirror symmetry of toric complete intersections [44]. In this generalization, two polar pairs of reflexive lattice polytopes are involved with the defining conditions summarized in the following equations,

$$\Delta = \Delta_1 + \ldots + \Delta_r \qquad \Delta^\circ = \langle \nabla_1, \ldots, \nabla_r \rangle_{\mathrm{conv}} \\ (\Delta_l, \nabla_m) \geq -\delta_{lm} \qquad (34) \\ \nabla^\circ = \langle \Delta_1, \ldots, \Delta_r \rangle_{\mathrm{conv}} \qquad \nabla = \nabla_1 + \ldots + \nabla_r,$$

where r is the codimension of the Calabi–Yau variety, and the defining equations $f_i = 0$ are sections of $\mathcal{O}(\Delta_i)$. The decomposition of the M-lattice polytope $\Delta \subset M_\mathbb{R}$ into a Minkowski sum $\Delta = \Delta_1 + \ldots + \Delta_r$ is now dual to a nef (numerically effective) partition of the vertices of $\Delta^\circ \subset N_\mathbb{R}$ such that the convex hulls ∇_i of the respective vertices and $0 \in N$ only intersect at the origin [5, 44]. $\nabla = \nabla_1 + \ldots + \nabla_r$ is another reflexive polytope, whose dual ∇° has a nef partition in terms of the vertices of Δ_i.

The Hodge numbers h_{pq} of the corresponding complete intersections have been computed and shown to obey (32) in [45]. They are summarized for arbitrary dimension $n - r$ of the Calabi–Yau variety in a generating polynomial $E(t, \bar t)$ as

$$E(t, \bar t) = \sum (-1)^{p+q}\, h_{pq}\, t^p\, \bar t^q = \\ = \sum_{I=[x,y]} \frac{(-)^{\rho_x} t^{\rho_y}}{(t\bar t)^r} S(C_x, \frac{\bar t}{t}) S(C_y^*, t\bar t) B_I(t^{-1}, \bar t) \qquad (35)$$

in terms of the combinatorial data of the $n+r$ dimensional Gorenstein cone $\Gamma(\{\Delta_i\})$ spanned by vectors of the form (e_i, v), where e_i is a unit vector in $\mathbb{R}^r$ and $v \in \Delta_i$. In this formula, x, y label faces C_x of dimension ρ_x of $\Gamma(\{\nabla_i\})$, and $C_x^\vee$ denotes the dual face of the dual cone of $\Gamma(\{\Delta_i\})$. The interval $I = [x, y]$ labels all cones that are faces of C_y containing C_x. The *Batyrev–Borisov polynomials* $B_I(t, \bar{t})$ encode certain combinatorial data of the face lattice [45]. The polynomials $S(C_x, t) = (1-t)^{\rho_x} \sum_{n\geq 0} t^n l_n(C_x)$ of degree $\rho_x - 1$ are related to the numbers $l_n(C_x)$ of lattice points at degree n in C_x and, hence, to the Ehrhart polynomial of the Gorenstein polytope generating C_x. (The Gorenstein polytope Δ_C consists of the degree 1 points of a Gorenstein cone C. In the hypersurface case, $r = 1$ and $\Delta_C = \Delta = \Delta_1$.)

Without going into details, let us emphasize that formula (35) contains positive and negative contributions, whose interpretation in terms of individual contributions from toric divisors D_j is, in contrast to the hypersurface formula (33), unfortunately unknown. In additional to efficiency problems in specific calculations (the formula is implemented in the nef-part of PALP [42], but becomes quite slow for codimension $r > 2$), this entails important theoretical problems which will be comment below.

3. Fibrations and Torsion in Cohomology

Fibration structures play an important role in string theory, like, e.g., in heterotic-type II duality [10, 16–18], F-theory [23–25], but also for the construction of vector bundles in heterotic compactifications, where they are often combined with non-trivial fundamental groups [26]. We now discuss how these topological properties manifest themselves in combinatorial properties of the polytopes that define toric Calabi–Yau varieties.

3.1. Torsion in cohomology

We begin with a discussion of the fundamental group which is trivial for every compact toric variety [1] but may become non-trivial for hypersurfaces and complete intersections. First, we need to discuss smoothness conditions and to focus on the hypersurface case. If we consider the normal fan of a reflexive polytope $\Delta \subseteq M_\mathbb{R}$, then X_Σ will generically have singularities which have to be resolved if they have positive dimension, while point-like singularities can be avoided by a generic choice of the hypersurface equation. The resolution can be performed by the choice of a convex (or *coherent*) triangulation of

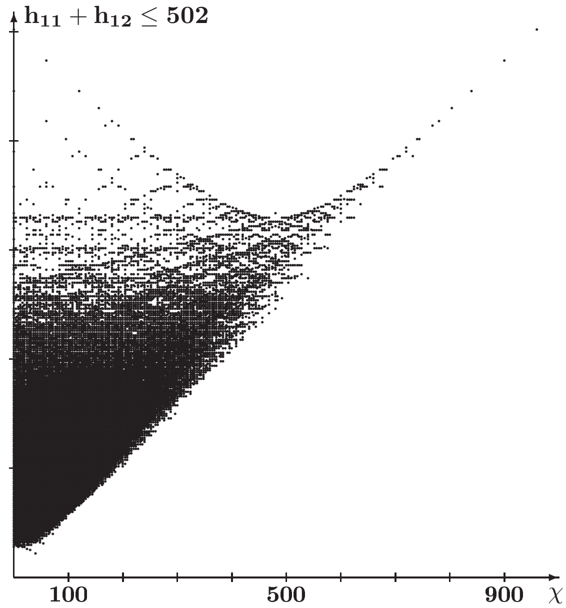

Fig. 4. Hypersurface spectra for $h_{11} \leq h_{12}$. The maximal $h_{11}+h_{12}$ comes from (251,251) and (491,11)

the fan Σ_Δ [10, 29, 46], whose rays should consist of all rays over lattice points of Δ° (this amounts to a maximal star triangulation of Δ°; rays of lattice points in $N \setminus \Delta^\circ$ would contribute to c_1 and, hence, destroy the Calabi–Yau condition if the corresponding divisors intersect the hypersurface). For K3-surfaces, such a triangulation already leads to a smooth toric ambient space, because reflexivity implies that the facets are at distance one from the origin, and every maximal triangulation of a polygon consists of basic simplices. For Calabi–Yau 3-folds, only the codimension-two cones of the triangulation are basic, while maximal-dimensional cones may contain point-like singularities. This is still o.k., because point-like singularities can be avoided by a generic hypersurface. Toric 4-fold hypersurfaces, on the other hand, may have terminal singularities that cannot be avoided, so that many 5-dimensional reflexive polytopes cannot be used for the construction of smooth Calabi–Yau hypersurfaces. For complete intersections, the situation is analogous: 3-folds are generically smooth, because the codimension $\dim \Delta - 3$ of the Calabi–Yau hypersurface is larger than the dimension $\dim \Delta - 4$ of the singular locus of X_Σ.

If we now consider a fixed polytope $\Delta \subset \mathbb{R}^4$ without specification of the lattice, then the reflexivity, i.e. the integrality of the vertices of Δ and of Δ°, implies that N is a sublattice of the dual of the lattice M_V generated by the vertices of Δ and that N contains the lattice N_V

generated by the vertices of Δ°

$$N_V \subseteq N \subseteq M_V^\circ. \qquad (36)$$

A refinement of the N lattice amounts to a geometrical quotient by a group action $G \subset T$ we call *toric*, because it acts diagonally on the homogeneous coordinates. Such a refinement always entails additional quotient singularities in the ambient space but no contributions to its fundamental group [1]. If, however, a Calabi–Yau hypersurface does not intersect the singular locus of that quotient, then the group acts freely on that variety and contributes to π_1. This is the case if the refinement of the lattice does not lead to additional lattice points of Δ° (more precisely, lattice points in the interior of facets can be ignored, because the corresponding divisors do not intersect the hypersurface, according to Eq. (33).

For a given pair of reflexive polytopes, there are only a finite number of lattices N that obey (36) and, hence, only a finite number of possible toric free quotients. In [47], we have shown that the fundamental group of a toric Calabi–Yau hypersurface is isomorphic to the lattice quotient of the N-lattice divided by its sublattice $N^{(3)}$ generated by the lattice points on 3-faces of Σ. All fundamental groups of toric hypersurfaces thus come from toric quotients and are Abelian, so that π_1 is isomorphic to the torsion in H^2. We also found a combinatorial formula for the Brauer group B, which is the torsion in the third cohomology H^3, in terms of the sublattice $N^{(2)}$ generated by lattice points on 2-faces of Σ. Here, however, $B \times B$ must be a subgroup of $N/N^{(2)}$.

Various dualities imply that the complete torsion in the cohomology groups is determined in terms of π_1 and B, and we conjectured, based on some K-theory arguments, that these groups are exchanged with the duals of each other under the mirror duality [47]. This conjecture could be verified for all toric hypersurfaces by explicit calculation. Two well-known examples are the free $\mathbb{Z}_5$ quotient of the quintic and the free $\mathbb{Z}_3$ quotient of the Calabi–Yau hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$. For the complete list of 473 800 776 reflexive polytopes, one finds 14 more examples of toric free quotients [42]: the elliptically fibered $\mathbb{Z}_3$ quotient of the degree 9 surface in $\mathbb{P}^4_{11133}$, whose group action on the homogeneous coordinates is given by the phases $(1,2,1,2,0)/3$, and 13 elliptic K3 fibrations, for which the lattice quotient has index 2. The 16 non-trivial Brauer groups showed up, as expected, exactly for the 16 polytopes that are polar to the ones that lead to a non-trivial fundamental group. For complete intersections, it is possible to have both a non-trivial fundamental group and a non-trivial Brauer group at the same time, and our conjecture was verified for a codimension-2 Calabi–Yau hypersurface, for which both groups are $\mathbb{Z}_3 \times \mathbb{Z}_3$ [48].

3.2. Fibrations

For general K3 surfaces and Calabi–Yau 3-folds, there exists a criterion by Oguiso for the existence of elliptic and K3 fibrations in terms of intersection numbers [18, 49]. In the toric context, the data of a given reflexive polytope $\Delta^\circ \subseteq N_\mathbb{R}$ have to be supplemented by a triangulation of the fan, as discussed above, and fibration properties, as well as intersection numbers, depend on the chosen triangulation.

Computation of all intersection numbers for all triangulations is computationally quite expensive, but, for toric Calabi–Yau spaces, there is, fortunately, a more direct way to search for fibrations that manifest themselves in the geometry of the polytope and to single out the appropriate triangulations [12–15, 41]. These fibrations descend from toric morphisms of the ambient space [3, 4], which correspond to a map $\phi : \Sigma \to \Sigma_b$ of fans in N and N_b, respectively, where $\phi : N \to N_b$ is a lattice homomorphism such that, for each cone $\sigma \in \Sigma$, there is a cone $\sigma_b \in \Sigma_b$ that contains the image of σ. The lattice N_f for the fiber is the kernel of ϕ in N.

If we are interested in fibrations, whose fibers are Calabi–Yau varieties of lower dimension, then the restriction of the defining equations to the fan Σ_f in N_f needs to satisfy Batyrev's criterion. We hence require that the intersection $\Delta_f^\circ = \Delta^\circ \cap N_f$ is reflexive, like the intersection with the horizontal plane in the example of Fig. 5. The search for toric fibrations amounts to a search for reflexive sections of Δ° with appropriate dimension (or, equivalently, for reflexive projections in the M-lattice, which was used in a search for K3 fibrations in [12]). In order to guarantee the existence of the projection, we choose a triangulation of Δ_f° and then extend it to a triangulation of Δ° (this may not always be possible if the codimension is larger that 1, as was pointed out and analyzed by Rohsiepe [15]). For each such choice, we can interpret the homogeneous coordinates that correspond to rays in Δ_f° as coordinates of the fiber and the others as parameters of the equations and, hence, as moduli of the fiber space.

For hypersurfaces, the geometry of the resulting fibration has been worked out in detail in [14]. Even in the case of complete intersections, the reflexivity of the fiber polytope Δ_f° ensures that the fiber also is a complete intersection of the Calabi–Yau hypersurface, because a nef partition of Δ° automatically induces a nef partition of Δ_f° [10]. The codimension r_f of the fiber generically

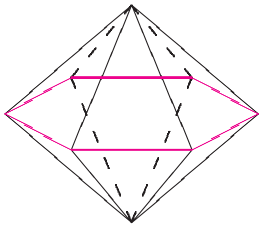

Fig. 5. Calabi–Yau fibration from a reflexive section of $\Delta^\circ \subseteq N_\mathbb{R}$

coincides with the codimension r of the fibered space. But, for $r > 1$, it may happen that Σ_f^1 does not intersect one (or more) of the Δ_i of the nef partition, in which case the codimension decreases [10]. In [10], we performed an extensive search for K3-fibrations in complete intersections which, due to modular properties, could be used for all-genus calculations of topological string amplitudes. In that search, we also encountered an example where the fibration does not extend to a morphism of the ambient spaces, because some exceptional points do not intersect the Calabi–Yau hypersurface. Even in that example, however, the K3 fiber is realized by a fan on a sublattice.

4. Work in Progress and Open Problems

For toric Calabi–Yau hypersurfaces in 3 dimensions, the enumeration and the computation of the integral cohomology has been completed. But, for the case of complete intersections, only the surface has been scratched [10, 50]. While the number of reflexive polytopes in 5 dimensions, which would be relevant for 4-folds as used in F-theory, is simply too large (maybe something like 10^{18}), there is some hope for that a classification of complete intersection 3-folds may be feasible, at least for small codimensions, via an enumeration of reflexive Gorenstein cones [51, 52]. On the theoretical side, it would be important to find a better formula for the Hodge data that allows a direct interpretation of the Picard number in terms of toric divisors (for codimension $r > 1$, even divisors that correspond to vertices of Δ° may not intersect the Calabi–Yau hypersurface [10]). A related issue is the search for a combinatorial formula for the torsion in cohomology, which would also be very useful for model building.

In spite of the fact that the toric construction yields by far the largest class of known Calabi–Yau spaces, it is unclear how generic these spaces are, and it is not even known whether the total number of topological types is finite [53]. A first step beyond the toric realm along the lines of [54] has been taken recently when we studied

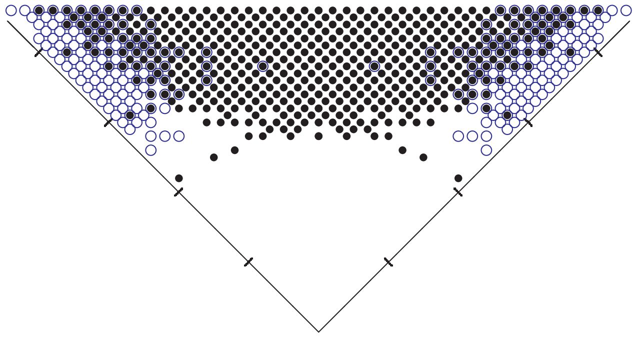

Fig. 6. Deformed conifold Hodge data (circles) and toric Calabi–Yau hypersurfaces (dots) with $h_{11} + h_{12} \leq 46$

the conifold transition to non-toric Calabi–Yau spaces [22]. As shown in Fig. 6, this construction, which still uses toric tools, yields a surprisingly rich class of new Calabi–Yau spaces with small Picard number h_{11}. The realm with small $h_{11} + h_{21}$, on the other hand, seems to be populated by varieties with non-trivial fundamental group [55]. Systematic studies of free quotients, however, so far have only been performed in special cases.

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ТОРІЧНА ГЕОМЕТРІЯ І КАЛАБІ–ЯУ КОМПАКТИФІКАЦІЇ

М. Кройцер

Р е з ю м е

Ці нотатки містять короткий вступ до побудови торічних Калабі–Яу гіперповерхней та повних перетинів з акцентом на розрахунках, що стосуються дуальності струн. Останні два розділи можуть бути прочитані незалежно від інших і присвячені недавнім результатам та роботам, які ще не закінчено, включаючи кручення в когомології, питання класифікації та топологічних переходів.