EQUATIONS OF COUPLED CONDENSATE AND NON-CONDENSATE DYNAMICS IN A TRAPPED BOSE GAS

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PACS 67.85.De, 03.75. Kk

We constructed equations of the condensed Bose gas dynamics at nonzero temperatures on the bases of the first principles of statistical mechanics. We derived the equation of motion for the condensate wavefunction and the quantum kinetic equation for the distribution function of excited atoms. The obtained generalized Gross–Pitaevskii equation for a condensate includes the effect of collisions with thermal cloud atoms (non-condensate). The Boltzmann quantum kinetic equation for a non-condensate was obtained by means of Zubarev's method of nonequilibrium statistical operator.

1. Introduction

The Bose–Einstein condensation (BEC) – one of the most interesting phenomena foreseen by quantum statistical mechanics. It is unique, because it is the exclusively quantum-statistical phase transition and may occur without interaction between atoms. In 1995, BEC "in a clean kind" was got in trapped dilute ultracold clouds of alkaline metals [1–3].

The development of a microscopic theory that can describe the dynamics of a condensed Bose gas is a very important problem. The foundation of a theoretical description of such quantum degenerated systems with spontaneously broken symmetry was laid by Bogolyubov in the paper on a weakly-nonideal Bose gas [4]. The central place in his theory is occupied by the idea of separation of the condensate. In contrast to Bogolyubov's model of weakly nonideal Bose gas, the atomic Bose condensate is spatially inhomogeneous. At temperatures close to zero (0–0.4 T_{BEC}), when almost all atoms are in the condensate, the dynamics of the atomic Bose condensate is described by a nonlinear equation for the condensate wave function – a Gross–Pitaevskii (GP) equation. In this temperature range, a GP equation describes the experimental data concerned with the condensate (condensate oscillation, interferences of condensates, *etc.*) very precisely. At higher temperatures, when an appreciable fraction of atoms of the condensate is excited, the dynamics of a trapped gas becomes much more complicated, since it involves now the coupled variations in the degrees of freedom of the condensate and the non-condensate. Thus, the description of the atomic Bose condensate at nonzero temperatures in terms of the GP equation is inadequate.

In this paper, we obtain, from the first principles, a generalized GP equation which includes the non-condensate degrees of freedom. This equation is not closed, because the dynamics of non-condensate atoms is indefinite. For their description within Zubarev's method of nonequilibrium statistical operator (NSO), we derived the Boltzmann quantum kinetic equation for the distribution function of excited atoms. Thus, we obtained the complete dynamics of a condensed Bose gas at nonzero temperatures.

2. Dynamics of the Condensed Bose Gas at Nonzero Temperatures

At very low temperatures, the dynamics of a trapped Bose gas is described by the time-dependent GP equation [5] for the macroscopic wave function $\Phi(r,t)$ associated with the Bose condensate.

In terms of Bose quantum field operators, the many-body Hamiltonian describing the interacting bosons confined by an external potential $U_{\mathrm{ext}}(\mathbf{r})$ is given by

$$H=\int d\mathbf{r}\psi^{+}(\mathbf{r})\left(-\frac{\hbar^2\nabla^2}{2m}+U_{\mathrm{ext}}(\mathbf{r})\right)\psi(\mathbf{r})+$$

$$+\frac{1}{2}\int d\mathbf{r}d\mathbf{r}'V(\mathbf{r}-\mathbf{r}')\psi^{+}(\mathbf{r})\psi^{+}(\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}). \quad (1)$$

For a dilute cold gas, one can obtain, nevertheless, a proper expression for the interaction term by observing that, in this case, only binary collisions at low energies are relevant, and these collisions are characterized by a single parameter, the s-wave scattering length, independently of the details of the two-body potential. This allows one to replace $V(\mathbf{r}-\mathbf{r}')$ by an effective interaction

$$V(\mathbf{r}-\mathbf{r}') = g\delta(\mathbf{r}-\mathbf{r}'),$$

where the coupling constant g is related to the scattering length a of the true potential through the relation

$$g = \frac{4\pi\hbar^2 a}{m}.$$

The time-dependent GP equation has the following form:

$$i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(\mathbf{r}) + gn_c\right)\Phi(\mathbf{r},t). \qquad (2)$$

Here, $n_c(\mathbf{r},t) = |\Phi(\mathbf{r},t)|^2$ is the density of atoms in the condensate.

The GP equation describes the motion of the condensate moving in the Hartree dynamic mean field produced by other atoms in the condensate and yields a closed equation for the order parameter $\Phi(\mathbf{r},t)$.

The GP equation (2) provides a very accurate description of the static and dynamic properties of a trapped Bose gas at low temperatures $T \leq 0.4T_{\text{BEC}}$, as it is confirmed by many experiments in the last few years [5]. In superfluid ^{4}He, the non-condensate fraction at $T = 0$ is close to 90%. Thus, in superfluid ^{4}He, one always has to deal with both condensate and non-condensate atoms. It is clear that a closed GP equation for $\Phi(\mathbf{r},t)$ is never valid in superfluid ^{4}He.

At higher temperatures (say, $T > 0.4T_{\text{BEC}}$), when an appreciable fraction of atoms of the condensate is excited, the dynamics of the trapped gas becomes much more complicated, since it now involves the coupled variations of the condensate and non-condensate degrees of freedom.

To find a generalized GP equation, we start with the usual Heisenberg equation of motion for a quantum field operator with the many-body Hamiltonian (1)

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(r)\right)\psi(\mathbf{r},t)+$$

$$+g\psi^+(\mathbf{r},t)\psi(\mathbf{r},t)\psi(\mathbf{r},t).$$

The introduction of the usual definition of a non-condensate field operator $\tilde{\psi}(\mathbf{r},t)$, according to

$$\psi(\mathbf{r},t) = \Phi(\mathbf{r},t) + \tilde{\psi}(\mathbf{r},t),$$

yields

$$i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(\mathbf{r}) + gn_c(\mathbf{r},t) + 2g\tilde{n}(\mathbf{r},t)\right)\times$$

$$\times\Phi(\mathbf{r},t) + g\tilde{m}(\mathbf{r},t)\Phi^*(\mathbf{r},t) + g\langle\tilde{\psi}^+(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\rangle, \quad (3)$$

where $\tilde{n}(\mathbf{r},t) = \left\langle\tilde{\psi}^+(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\right\rangle$ is the non-equilibrium non-condensate density, $\tilde{m}(\mathbf{r},t) = \left\langle\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\right\rangle$.

Equation (3) is reduced to the GP equation (2) if all the atoms are in the condensate (i.e., $\tilde{n} = 0$) and the anomalous correlations ($\tilde{m}$ and $\langle\tilde{\psi}^+(\mathbf{r},t)\tilde{\psi}(\mathrm{r},t)\tilde{\psi}(\mathbf{r},t)\rangle$) are ignored. This is a very effective approximation for $T \ll T_{\text{BEC}}$; at $T = 0$, the non-condensate fraction in trapped atomic gases is estimated to be less than 1% [6].

To derive a quantum Boltzmann kinetic equation for the distribution function of excited atoms, we consider the equation of motion for $\tilde{\psi}(\mathbf{r},t)$

$$i\hbar\frac{\partial\tilde{\psi}}{\partial t} = \left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(\mathbf{r}) + 2gn\right)\tilde{\psi} - 2g\tilde{n}\tilde{\psi} + g\Phi^2\tilde{\psi}^+ +$$

$$+g\Phi^*(\tilde{\psi}\tilde{\psi}-\tilde{m}) + 2g\Phi(\tilde{\psi}^+\tilde{\psi}-\tilde{n}) + g(\tilde{\psi}^+\tilde{\psi}\tilde{\psi}-\langle\tilde{\psi}^+\tilde{\psi}\tilde{\psi}\rangle), \quad (4)$$

where $n = n_c + \tilde{n}$ is the total density. It allows us to derive a kinetic equation for excited atoms. Following Kirkpatrick and Dorfman [7], it is convenient to define the time evolution of $\tilde{\psi}$ by

$$\tilde{\psi}(\mathbf{r},t) = U^+(t,t_0)\tilde{\psi}(\mathbf{r},t_0)U(t,t_0),$$

where the unitary operator $U(t,t_0)$ evolves according to the equation of motion

$$i\hbar\frac{\partial U(t,t_0)}{\partial t} = H_{\text{eff}}(t)U(t,t_0).$$

Here, t_0 is the time, at which the initial nonequilibrium density matrix $\hat{\rho}(t_0)$ is specified. The effective Hamiltonian is given by

$$H_{\text{eff}}(t) = H_0(t) + H'(t), \qquad (5)$$

$$H'(t) = H_1'(t) + H_2'(t) + H_3'(t) + H_4'(t),$$

where the various contributions are defined as

$$\begin{aligned}
H_0(t) &= \int d\mathbf{r}\tilde{\psi}^+\left(-\frac{\hbar^2\nabla^2}{2m} + U_{\text{ext}}(\mathbf{r})\right)\tilde{\psi}, \\
H_1'(t) &= \int d\mathbf{r}\left(L_1(\mathbf{r},t)\tilde{\psi}^+ + L_1^*(r,t)\tilde{\psi}\right), \\
H_2'(t) &= \frac{g}{2}\int d\mathbf{r}\left(\Phi^2(\mathbf{r},t)\tilde{\psi}^+\tilde{\psi}^+ + \Phi^{*2}(\mathbf{r},t)\tilde{\psi}\tilde{\psi}\right), \\
H_3'(t) &= g\int d\mathbf{r}\left(\Phi^*(\mathbf{r},t)\tilde{\psi}^+\tilde{\psi}\tilde{\psi} + \Phi(\mathbf{r},t)\tilde{\psi}^+\tilde{\psi}^+\tilde{\psi}\right), \\
H_4'(t) &= \frac{g}{2}\int d\mathbf{r}\tilde{\psi}^+\tilde{\psi}^+\tilde{\psi}\tilde{\psi} - 2g\int dr\tilde{n}(\mathbf{r},t)\tilde{\psi}^+\tilde{\psi},
\end{aligned}$$

$$U(\mathbf{r}) = U_{\text{ext}}(\mathbf{r}) + 2g\tilde{n}_c(\mathbf{r},t),$$

$$L_1(\mathbf{r},t) = -g\Big(\Phi^*(\mathbf{r},t)\tilde{m}(\mathbf{r},t) + 2\Phi(\mathbf{r},t)\tilde{n}(\mathbf{r},t)+$$

$$+\langle\tilde{\psi}^+\tilde{\psi}\tilde{\psi}\rangle\Big).$$

The expectation value of an arbitrary operator $O(t)$, which is made up of some combination of non-condensate field operators, with respect to the initial density matrix $\hat{\rho}(t_0)$ can be expressed as

$$\langle O(t)\rangle = \langle O\rangle^t = \text{Sp}\hat{\rho}(t_0)O(t) = \text{Sp}\tilde{\rho}(t,t_0)O(t_0),$$

where

$$\tilde{\rho}(t,t_0) = U(t,t_0)\hat{\rho}(t_0)U^+(t,t_0)$$

satisfies the equation

$$\frac{\partial\tilde{\rho}(t,t_0)}{\partial t} + \frac{1}{i\hbar}[\tilde{\rho}(t,t_0), H_{\text{eff}}(t)] = 0.$$

Our ultimate objective is to obtain a quantum kinetic equation for non-condensate atoms. For this purpose, we define the Wigner operator

$$\hat{f}(\mathbf{p},\mathbf{r},t_0) \equiv \int d\mathbf{r}' e^{\frac{i\mathbf{p}\mathbf{r}'}{\hbar}}\tilde{\psi}^+(\mathbf{r}+\frac{\mathbf{r}'}{2},t_0)\tilde{\psi}(\mathbf{r}-\frac{\mathbf{r}'}{2},t_0).$$

Its expectation value yields the Wigner distribution function

$$f(\mathbf{p},\mathbf{r},t) = \text{Sp}\hat{f}(\mathbf{p},\mathbf{r},t_0)\tilde{\rho}(t,t_0).$$

Knowledge of this function allows one to calculate various nonequilibrium expectation values such as the non-condensate density

$$\tilde{n}(\mathbf{r},t) = \int\frac{d\mathbf{p}}{(2\pi\hbar)^3}f(\mathbf{p},\mathbf{r},t).$$

The equation of motion for f is

$$\frac{\partial f(\mathbf{p},\mathbf{r},t)}{\partial t} = \frac{1}{i\hbar}\text{Sp}\tilde{\rho}(t)[f(\mathbf{p},\mathbf{r},t), H_{\text{eff}}(t)] =$$

$$= \frac{1}{i\hbar}\text{Sp}\tilde{\rho}(t)[f(\mathbf{p},\mathbf{r},t), H^0(t)] + \frac{1}{i\hbar}\text{Sp}\tilde{\rho}(t)[f(\mathbf{p},\mathbf{r},t), H'(t)]. \tag{6}$$

The first term on the right-hand side of (6) defines the free-streaming operator in the kinetic equation. With the assumption that $U(\mathbf{r},t)$ varies slowly in space, we then have

$$\frac{\partial f(\mathbf{p},\mathbf{r},t)}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla f(\mathbf{p},\mathbf{r},t) - \nabla U\cdot\nabla_{\mathbf{p}}f(\mathbf{p},\mathbf{r},t) =$$

$$= \frac{1}{i\hbar}\text{Sp}\tilde{\rho}(t)[f(\mathbf{p},\mathbf{r},t), H'(t)].$$

The right-hand side of this equation clearly represents the effect of collisions between atoms.

The reduction of this term to the form of a binary collision integral is performed with the use of the method of NSO (see Appendix).

The collision integral is a sum of two contributions

$$\left.\frac{\partial f}{\partial t}\right|_{\text{coll}} = C_{12}[f] + C_{22}[f],$$

where

$$C_{12}[f] = \frac{2g^2 n_c}{(2\pi)^2\hbar^4}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\times$$

$$\times\delta(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3)\delta(\tilde{\varepsilon}_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3})\times$$

$$\times[\delta(\mathbf{p}-\mathbf{p}_1) - \delta(\mathbf{p}-\mathbf{p}_2) - \delta(\mathbf{p}-\mathbf{p}_3)]\times$$

$$\times[(1+f_1)f_2f_3 - f_1(1+f_2)(1+f_3)] \tag{7}$$

describes collisions between thermal atoms, and

$$C_{22}[f] = \frac{2g^2}{(2\pi)^5\hbar^7}\int d\mathbf{p}_2\int d\mathbf{p}_3\int d\mathbf{p}_4\times$$

$$\times\delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4)\delta(\tilde{\varepsilon}_p + \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3} - \tilde{\varepsilon}_{p_4})\times$$

$$\times[(1+f)(1+f_2)f_3f_4 - ff_2(1+f_3)(1+f_4)] \tag{8}$$

describes collisions between thermal atoms and condensate atoms.

$$f = f(\mathbf{p},\mathbf{r},t), \quad f_i = f(\mathbf{p}_i,\mathbf{r},t).$$

Therefore, the quantum kinetic equation for the distribution function for excited atoms reads

$$\frac{\partial f(\mathbf{p},\mathbf{r},t)}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla f(\mathbf{p},\mathbf{r},t) - \nabla U\cdot\nabla_{\mathbf{p}}f(\mathbf{p},\mathbf{r},t) =$$

$$= C_{22}[f] + C_{12}[f,\Phi]. \tag{9}$$

The generalized GP equation is as follows:

$$i\hbar\frac{\partial\Phi(\mathbf{r},t)}{\partial t}=\Big(-\frac{\hbar^2\nabla^2}{2m}+U_{\rm ext}(\mathbf{r})+gn_c(\mathbf{r},t)+$$

$$+2g\tilde{n}(\mathbf{r},t)-i\hbar R(\mathbf{r},t)\Big)\Phi(\mathbf{r},t), \qquad (10)$$

where

$$R(\mathbf{r},t)=\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{C_{12}[f,\Phi]}{2n_c(\mathbf{r},t)}.$$

The dissipative term R in (10) is associated with the exchange of atoms between the condensate and the non-condensate, as described by the collision integral $C_{12}[f]$ in (7). We see that (9) and (10) must be solved self-consistently.

3. Conclusions

In this paper, we derived a system of equations which describes the dynamics of a trapped Bose-condensed gas at finite temperatures. These equations consist of a generalized Gross–Pitaevskii equation for the condensate order parameter $\Phi(\mathbf{r},t)$ and a semiclassical kinetic equation for the distribution function $f(\mathbf{p},\mathbf{r},t)$ of excited atoms (non-condensate). By limiting ourselves to higher temperatures, we arrive at a simple intuitive picture, in which the excited atoms behave themselves as particles moving in a dynamic Hartree–Fock field. Collisions between all the atoms are included and, in particular, allow the transfer of atoms between the two components. A big part of our discussion is devoted to the hydrodynamic regime and the derivation of a closed system of generalized hydrodynamic equations for the two components. These equations are based on the assumption that collisions between excited atoms are sufficiently rapid to drive the distribution function $f(\mathbf{p},\mathbf{r},t)$ toward the local equilibrium Bose–Einstein distribution $\tilde{f}(\mathbf{p},\mathbf{r},t)$, with the consequence that the $C_{22}[f]$ collision integral for excited atoms vanishes. However, the $C_{12}[f]$ collision integral describing collisions between condensate atoms and non-condensate ones remains finite. As a result, our equations can be used to describe the situation, in which the condensate is not in diffusive local equilibrium with the non-condensate.

The work was supported by the State Foundation for Fundamental Research of Ukraine (Project No. F25.2/011).

APPENDIX

In the Appendix, we give detailed derivation of the expressions for the anomalous correlation functions $\langle\tilde{\psi}\tilde{\psi}\rangle$ and $\langle\tilde{\psi}^+\tilde{\psi}\tilde{\psi}\rangle$, as well as the collision integrals given by Eqs. (7) and (8).

For this goal, we use Zubarev's method of nonequilibrium statistical operator.

The NSO method allows one to generalize the Gibbs ensemble method to the nonequilibrium case and to construct a nonequilibrium statistical operator which enables one to obtain the transport equations and to calculate the kinetic coefficients in terms of correlation functions. This operator, in the case of equilibrium, goes over to the Gibbs distribution.

In order to obtain a closed kinetic equation for the distribution function f, we must express higher order correlation functions in terms of f. For this purpose, we treat $H'(t)$ as a perturbation to the zeroth-order Hamiltonian $H_0(t)$, as defined in (5). We will effectively calculate collision integrals to the second order in g.

Let us consider the Liouville equation with an infinitesimally small source on the right-hand side (we set $\hbar=1$ throughout this Appendix)

$$\frac{\partial\rho(t)}{\partial t}+\frac{1}{i}[\rho(t),H(t)]=-\epsilon(\rho(t)-\rho_q(t)), \qquad (11)$$

where $\epsilon\to 0$ after the thermodynamic limit ($V\to\infty,\ N\to\infty$). The quasiequilibrium distribution is defined as

$$\rho_q=\exp\left\{-\Phi(t)-\sum_n F_n(t)P_n\right\}.$$

Here, $\Phi(t)$ is a Massieu–Planck functional. The Lagrange multipliers F_m should be found from the self-consistency conditions

$$\langle P_m\rangle^t=\langle P_m\rangle_q^t\,,$$

where P_m are the basic dynamic variables that describes the nonequilibrium state.

In our case, the Hamiltonian has the form (5)

$$H=H_0(t)+H'(t),$$

where $H'(t)$ is treated as a perturbation. In such a situation, it is useful to pass to the interaction representation.

Let us now rewrite the Liouville equation (11) in the form suitable to integration:

$$\left(\frac{\partial}{\partial t'}+\epsilon\right)\rho(t')+\frac{1}{i}[\rho(t'),H_0(t')]=\epsilon\rho_q(t')-\frac{1}{i}[\rho(t'),H'(t')].$$

Multiplying this equation by $\exp(\epsilon t')$, as well as by $U_0(t')=\exp\{-iH_0t'\}$ and $U_0^+(t')=\exp\{iH_0t'\}$ (from left and right, respectively), integrating over the time (from $-\infty$ to t), and using the property of the evolution operators, we find

$$\rho(t)=\epsilon\int\limits_{-\infty}^{t}dt'\exp(-\epsilon(t-t'))U_0(t,t')\rho_q(t')U_0^+(t,t')-$$

$$-\int\limits_{-\infty}^{t}dt'\exp(-\epsilon(t-t'))U_0(t,t')\frac{1}{i}[\rho(t'),H'(t')]U_0^+(t,t').$$

Here, $U_0(t,t')=U_0(t)U_0^+(t')$.

By integrating the term proportional to ϵ by parts, we obtain

$$\rho(t) = \rho_q(t) - \int\limits_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} U_0(t,t') \left\{ \frac{\partial \rho_q(t')}{\partial t'} + \right.$$

$$\left. + \frac{1}{i}[\rho_q(t'), H_0(t')] \right\} U_0^+(t,t') - \int\limits_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} \times$$

$$\times U_0(t,t') \frac{1}{i} [\rho(t'), H'] U_0^+(t,t'). \qquad (12)$$

We now find how $\dfrac{\partial \rho_q(t)}{\partial t}$ depends on the interaction. We have

$$\frac{\partial \rho_q(t)}{\partial t} = \sum_n \frac{\delta \rho_q(t)}{\delta \langle P_n \rangle^t} \frac{\partial \langle P_n \rangle^t}{\partial t} = \sum_n \frac{\delta \rho_q(t)}{\delta \langle P_n \rangle^t} \times$$

$$\times \left[i \sum_l \Omega_{nl}(t) \langle P_n \rangle^t + J_n(t) \right], \qquad (13)$$

where Ω_{nl} satisfies the relation

$$i \sum_l \Omega_{nl}(t) P_l = \frac{1}{i} [\rho_q(t), H_0(t)].$$

The quantity $J_n(t)$ is defined as

$$J_n(t) = -i \langle [P_n, H'] \rangle^t = -i \mathrm{Sp}\Big([P_n, H'] \rho(t) \Big). \qquad (14)$$

Taking relations (13) and (14) into account, we obtain

$$\frac{\partial \rho_q(t)}{\partial t} - i[\rho_q(t), H_0(t)] = \sum_n \frac{\delta \rho_q(t)}{\delta \langle P_n \rangle^t} J_n(t). \qquad (15)$$

Substituting (15) in (12), we find

$$\rho(t) = \rho_q(t) - \int\limits_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} U_0(t,t') \times$$

$$\times \left\{ \sum_m \frac{\delta \rho_q(t')}{\delta \langle P_m \rangle^{t'}} J_m(t') - i[\rho(t'), H'] \right\} U_0^+(t,t'). \qquad (16)$$

Such a form of the Liouville equation is useful for using the perturbation theory.

Let us assume that

$$\rho(t) = \rho_q(t) + \sum_{k=1}^{\infty} \rho^{(k)}(t), \quad J_n(t) = \sum_{k=1}^{\infty} J_n^{(k)}(t).$$

Here, $\rho^{(k)}$ and $J_n^{(k)}$ are proportional to g^k (g is a constant of interaction).

Notice that, due the space homogeneity,

$$J_n^{(1)}(t) = -i \langle [P_n, H'] \rangle_q^t = 0.$$

Iterating (16) to the first order in $H'(t)$, we have

$$\rho(t) = \rho_q(t) - i \int\limits_{-\infty}^{t} dt' \exp(-\epsilon(t-t')) U_0(t,t') \times$$

$$\times [H'(t), \rho_q(t)] U_0^+(t,t'). \qquad (17)$$

In the first approximation for an arbitrary dynamic operator $O(t)$, the expectation value can be expressed as

$$\langle O(t) \rangle^t = \langle O(t) \rangle_q^t - i \int\limits_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} \times$$

$$\times \Big\langle [U_0(t,t') O(t) U_0^+(t,t'), H'(t)] \Big\rangle_q^{t'}.$$

First, we consider the three-field correlation function $\langle \tilde\psi^+ \tilde\psi \tilde\psi \rangle$. To the lowest order in the interaction g, the three-field correlation function is given by

$$\Big\langle \tilde\psi^+ \tilde\psi \tilde\psi \Big\rangle^t = \Big\langle \tilde\psi^+ \tilde\psi \tilde\psi \Big\rangle_q^t - i \int\limits_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} \times$$

$$\times \Big\langle [U_0(t,t') \tilde\psi^+ \tilde\psi \tilde\psi U_0^+(t,t'), H'(t)] \Big\rangle_q^{t'}. \qquad (18)$$

The first term on the right-hand side of (18) gives the initial correlation. Since the initial anomalous correlations regarding the broken symmetry in the quasiequilibrium statistical operator are absent, the term $\Big\langle \tilde\psi^+ \tilde\psi \tilde\psi \Big\rangle_q^t$ vanishes. In addition, we kept only those parts of H_1' and H_3' which include the equal numbers of creation and annihilation operators.

Thus,

$$\Big\langle \tilde\psi^+ \tilde\psi \tilde\psi \Big\rangle^t = -i \int\limits_{-\infty}^{t} dt' \exp\{-\epsilon(t-t')\} \times$$

$$\times \Big\langle [U_0(t,t') \tilde\psi^+ \tilde\psi \tilde\psi U_0^+(t,t'), H_1'(t) + H_3'(t)] \Big\rangle_q^{t'}. \qquad (19)$$

Suppose that the hydrodynamic variables vary slowly in space and time. Thus, the effective Hamiltonian contains the terms arising from $\mathbf{r}'$ and t' close to $\mathbf{r}$ and t.

Let us assume that

$$n_c(\mathbf{r}', t') \simeq n_c(\mathbf{r}, t), \ \tilde n(\mathbf{r}', t') \simeq \tilde n(\mathbf{r}, t), \ U(\mathbf{r}', t') \simeq U(\mathbf{r}, t),$$

$$\theta(\mathbf{r}', t') \simeq \theta(\mathbf{r}, t) + \frac{\partial \theta}{\partial t}(t' - t) + \nabla\theta \cdot (\mathbf{r}' - \mathbf{r}) =$$

$$= \theta(\mathbf{r}, t) + \varepsilon_c (t' - t) + m \mathbf{v}_c \cdot (\mathbf{r}' - \mathbf{r}), \qquad (20)$$

$$U_0(t,t') = T_\tau \exp\Big\{ -i \int\limits_{t}^{t'} H_0(\tau) d\tau \Big\} \simeq \exp\{-i H_0 (t - t')\}.$$

Here, $n_c(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)|^2$ is the condensate density, θ is a phase of the macroscopic wave function ($\Phi(\mathbf{r}, t) = |\Phi(\mathbf{r}, t)| e^{i\theta(\mathbf{r}, t)}$), $\mathbf{v}_c(\mathbf{r}, t) = 1/m \nabla\theta(\mathbf{r}, t)$ is the condensate velocity, and $\varepsilon_c = \dfrac{\partial \theta}{\partial t} = \mu_c + \dfrac{m v_c^2}{2}$ is the local energy of a condensate atom.

Performing the Fourier transformation of the non-condensate field operators according to

$$\tilde\psi(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} a_{\mathbf{p}} e^{i\mathbf{p}\mathbf{r}}, \quad \tilde\psi^+(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} a_{\mathbf{p}}^+ e^{-i\mathbf{p}\mathbf{r}}$$

and using (20) and the definition of H_1', we obtain

$$\Big\langle [U_0^+(t,t') \tilde\psi^+(\mathbf{r}, t_0) \tilde\psi(\mathbf{r}, t_0) \tilde\psi(\mathbf{r}, t_0) U_0(t,t'), H_1'(t)] \Big\rangle_q^{t'} \simeq$$

$$\simeq -2g \tilde n(\mathbf{r}, t) n_c^{1/2}(\mathbf{r}, t) \exp\{i\theta(\mathbf{r}, t)\} \times$$

$$\times \frac{1}{V} \sum_{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4} e^{-i(\mathbf{p}_{1c} + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \cdot \mathbf{r}} e^{i(\varepsilon_c + \tilde\varepsilon_1 - \tilde\varepsilon_2 - \tilde\varepsilon_3)(t - t')} \times$$

$$\times \delta_{\mathbf{p}_4, \mathbf{p}_c} \Big(\langle a_{\mathbf{p}_1}^+ a_{\mathbf{p}_2} \rangle_q^{t'} \delta_{\mathbf{p}_3, \mathbf{p}_4} + \langle a_{\mathbf{p}_2}^+ a_{\mathbf{p}_3} \rangle_q^{t'} \delta_{\mathbf{p}_2, \mathbf{p}_4} \Big), \qquad (21)$$

where $\mathbf{p}_c = m \mathbf{v}_c$, $\tilde\varepsilon_{p_i}(\mathbf{r}) = \frac{p_i^2}{2m} + U(\mathbf{r})$.

By analogy, using Wick's theorem for the higher order correlation functions, we obtain

$$\left\langle [U_0^+(t,t')\tilde{\psi}^+(\mathbf{r},t_0)\tilde{\psi}(\mathbf{r},t_0)\tilde{\psi}(\mathbf{r},t_0)U_0(t,t'),H_3'(t)]\right\rangle_q^{t'} \simeq$$

$$\simeq 2g\tilde{n}(\mathbf{r},t)n_c^{1/2}(\mathbf{r},t)\exp\{i\theta(\mathbf{r},t)\}\times$$
$$\times\frac{1}{V^2}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3} e^{-i(\mathbf{p}_{1c}+\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)\cdot\mathbf{r}}e^{i(\varepsilon_c+\tilde{\varepsilon}_1-\tilde{\varepsilon}_2-\tilde{\varepsilon}_3)(t-t')}\times$$
$$\times\sum_{\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'}\delta_{\mathbf{p}_c+\mathbf{p}_1',\mathbf{p}_2'+\mathbf{p}_3'}\Big(\delta_{\mathbf{p}_2,\mathbf{p}_2'}\delta_{\mathbf{p}_3,\mathbf{p}_3'}\langle a_{\mathbf{p}_1}^+a_{\mathbf{p}_1'}\rangle_q^{t'}+\delta_{\mathbf{p}_2,\mathbf{p}_2'}\langle a_{\mathbf{p}_1}^+a_{\mathbf{p}_1'}\rangle_q^{t'}\times$$
$$\times\langle a_{\mathbf{p}_3}^+a_{\mathbf{p}_3'}\rangle_q^{t'}+\delta_{\mathbf{p}_3,\mathbf{p}_3'}\langle a_{\mathbf{p}_1}^+a_{\mathbf{p}_1'}\rangle_q^{t'}\langle a_{\mathbf{p}_2'}^+a_{\mathbf{p}_2}\rangle_q^{t'}-\delta_{\mathbf{p}_1,\mathbf{p}_1'}\left\langle a_{\mathbf{p}_2'}^+a_{\mathbf{p}_2}\right\rangle_q^{t'}\times$$
$$\times\left\langle a_{\mathbf{p}_3'}^+a_{\mathbf{p}_3}\right\rangle_q^{t'}+\delta_{\mathbf{p}_2,\mathbf{p}_2'}\langle a_{\mathbf{p}_1}^+a_{\mathbf{p}_3}\rangle_q^{t'}\left\langle a_{\mathbf{p}_3'}^+a_{\mathbf{p}_1'}\right\rangle_q^{t'}+\delta_{\mathbf{p}_3,\mathbf{p}_3'}\times$$
$$\times\langle a_{\mathbf{p}_1}^+a_{\mathbf{p}_2}\rangle_q^{t'}\left\langle a_{\mathbf{p}_2'}^+a_{\mathbf{p}_2'}\right\rangle_q^{t'}\Big). \qquad (22)$$

Relation (20) yields

$$\langle a_{\mathbf{p}_i}^+a_{\mathbf{p}_j}\rangle_q^{t'} = \mathrm{Sp}\Big(\rho_q(t_0)U_0^+(t',t_0)a_{\mathbf{p}_i}^+a_{\mathbf{p}_j}U_0(t',t_0)\Big)\simeq$$
$$\simeq e^{i(\tilde{\varepsilon}_i-\tilde{\varepsilon}_j)}\langle a_{\mathbf{p}_i}^+a_{\mathbf{p}_j}\rangle_q^{t_0}. \qquad (23)$$

In view of the properties of the quasiequilibrium statistical operator, we have

$$\langle a_{\mathbf{p}_i}^+a_{\mathbf{p}_j}\rangle_q^{t_0} = \delta_{\mathbf{p}_i,\mathbf{p}_j}f(\mathbf{p}_i,\mathbf{r},t_0). \qquad (24)$$

Substituting (21) and (22) in (19) and using (23) and (24), we obtain

$$\left\langle\tilde{\psi}^+(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\right\rangle = -i2\pi g n_c^{1/2}e^{i\theta}\frac{1}{V^2}\times$$
$$\times\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3}\left[\delta(\tilde{\varepsilon}_c+\tilde{\varepsilon}_1-\tilde{\varepsilon}_2-\tilde{\varepsilon}_3)+\frac{i}{\pi}P\frac{1}{\tilde{\varepsilon}_c+\tilde{\varepsilon}_1-\tilde{\varepsilon}_2-\tilde{\varepsilon}_3}\right]\times$$
$$\times\delta_{\mathbf{p}_c+\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3}\Big[f_1(1+f_2)(1+f_3)-(1+f_1)f_3f_3\Big], \qquad (25)$$

where $f_i = f(\mathbf{p}_i,\mathbf{r},t)$. We emphasize that, by deriving (25), we used the Sokhotskii's formula

$$\int_{-\infty}^{t} dt' e^{i(x-i\epsilon)(t-t')} = \pi\delta(x)+iP\left(\frac{1}{x}\right), \text{ if } \epsilon\to 0.$$

We note that the three-field correlation function in (25) is explicitly proportional to $\sqrt{n_c}$ and the interaction strength g. Thus, it vanishes above T_{BEC} as well as in a non-interacting Bose gas.

Using similar techniques and approximations, we evaluate the rest quantities:

$$\tilde{m}(\mathbf{r},t) = \left\langle\tilde{\psi}(\mathbf{r},t)\tilde{\psi}(\mathbf{r},t)\right\rangle = -i\pi g\Phi\frac{1}{V}\sum_{\mathbf{p}_1,\mathbf{p}_2}\delta_{\mathbf{p}_1+\mathbf{p}_2,2\mathbf{p}_c}\times$$
$$\times[1+f_1+f_2]\left[\delta(\tilde{\varepsilon}_1+\tilde{\varepsilon}_2-2\tilde{\varepsilon}_c)+\frac{i}{\pi}P\frac{1}{\tilde{\varepsilon}_1+\tilde{\varepsilon}_2-2\tilde{\varepsilon}_c}\right].$$

For the collision integrals, we find results (7) and (8).

1. M. Anderson, J. Ensher, M. Matthews, C. Wieman, and E. Cornell, Science **269**, 198 (1995).
2. K. Davis, M. Mewes, M. Andrews, N. van Druten, D. Durfee, D. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).
3. C. Bradley, C. Sackett, J. Tollet, and R. Hulet, Phys. Rev. Lett. **75**, 1687 (1995).
4. N. Bogolyubov, J. Phys. (USSR) **11**, 23 (1947).
5. F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
6. D.A.W. Hutchinson, E. Zaremba, and A. Griffin, Phys. Rev. Lett. **78**, 1842 (1997).
7. T.R. Kirkpatrick and J.R. Dorfman, Phys. Rev. A **28**, 2576 (1983).
8. D. Zubarev, V. Morozov, G. Röpke, *Statistical Mechanics of Nonequilibrium Processes: Basic Concepts, Kinetic Theory* (Akademie Verlag, Berlin, 1997).

Received 16.09.09

РІВНЯННЯ ОБ'ЄДНАНОЇ ДИНАМІКИ КОНДЕНСАТУ ТА НАДКОНДЕНСАТУ В ЗАХОПЛЕНОМУ БОЗЕ-ГАЗІ

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Резюме

Виходячи з перших принципів статистичної механіки, побудовано рівняння динаміки конденсованого бозе-газу при температурах, відмінних від нуля. Зокрема, виведено рівняння руху для хвильової функції конденсату та квантове кінетичне рівняння для функції розподілу збуджених атомів. Одержане узагальнене рівняння Гросс–Пітаєвського для конденсату враховує зіткнення з надконденсатними атомами. Квантове кінетичне рівняння Больцмана для надконденсату отримано за допомогою методу Зубарєва нерівноважного статистичного оператора.